they were still bouncing off of the molecules and producing a similar decrease in the coherence length. This decoherence effect was the reason that the experiment was limited to molecules of the size they used. Even though the molecules took only about 400 nanoseconds to fly through the apparatus, there was a significant amount of decoherence. A larger molecule would have been a bigger target for photons and would have undergone decoherence more quickly, making interference unobservable.

As in the example of spying on one slit of a double-slit experiment, the question arises of what has happened to the phase information that appears to have been erased by decoherence, violating unitarity. The resolution is the same (p. 999): the information has flowed out into the environment, but is no longer in a form in which it is practical to recover it.

14.10 Quantum computing and the no-cloning theorem

Computers and information transmission systems such as the internet are currently implemented as classical devices. For example, the wavelengths of the electrons that carry signals in a computer chip are currently orders of magnitude shorter than the size of the logic gates, so that wave effects such as diffraction and interference are not important (problem 22, p. 944). Even if the current devices such as silicon chips and fiber-optic cables could simply be scaled down to sizes comparable to the electrons’ wavelengths, quantum effects would at some point simply make them start breaking down or behaving unreliably.

It is possible, however, to design qualitatively different devices in which information and signals are intentionally manipulated in an explicitly quantum-mechanical fashion. This is the frontier known as quantum computing. In a quantum computer, the basic unit of information is not the classical bit but the quantum bit or qubit. A qubit can exist in a superposition of the 0 and 1 states, with a well defined phase, e.g., $\Psi_0 + \Psi_1$ is a different state than $\Psi_0 - \Psi_1$ or $\Psi_0 + i\Psi_1$. Furthermore, one qubit can have its state entangled with another’s. For example, $\Psi_{01} + \Psi_{10}$ describes a state in which we have two bits, neither in a definite 0 or 1 state, but which are guaranteed to add up to 1. That is, if one is true, then the other is guaranteed to be false. It has been shown that some problems that are hard for classical computers are more tractable for a quantum computer. For example, there is a known quantum computing algorithm that is capable of efficiently factoring large integers, and when this is eventually implemented in a practical device, it will have the effect of breaking the cryptographic algorithms that you currently use for online privacy and security, since the security of those algorithms is predicated on the assumption that factorization is hard. This would
be a disaster for online economic activity and could have effects such as unmasking political dissidents.

A different application, and one that is easier to explain, is that quantum computing makes it possible in theory to make copy-proof information. This would not be useful to Hollywood studios trying to prevent copying of their movies, since the images have to pass through classical devices anyway in order to be displayed, but it means that one might be able to send private information through a quantum internet in such a way that it could not be copied by snoops, even in theory. In contrast, current classical methods of encryption are designed to allow eavesdropping on an information packet as it hops across the internet, but to make the copy useless to prying eyes because it cannot be decoded.

The theoretical key to this application of quantum computing is the counterintuitive no-cloning theorem, which states that it is not possible to make a copy of an unknown quantum state. To see why this works, suppose that we implement a qubit using the spin 1/2 of a silver atom, with the convention that the 0 state is represented by $s_x = -1/2$ and 1 by $s_x = +1/2$. If you provide me with an atom that you have prepared, then it might seem straightforward, at least in principle, for me to copy its state. I can shoot it through a magnetic spectrometer, as in the Stern-Gerlach experiment, and measure its $s_x$. Then I prepare another silver atom in the same state. What’s the problem?

The problem is that if the state of the atom is truly unknown to me, then I have no way of knowing that it is actually in one of the two states $s_x = -1/2$ and $s_x = +1/2$. It could instead be in some superposition of these, such as $\Psi_{s_x=-1/2} = \frac{1}{\sqrt{2}} + i\Psi_{s_x=+1/2}$, with a 90-degree phase angle between the two components. Then when I send your atom through the spectrometer, the world becomes one in which both the spectrometer and my brain are in a superposition of the two states. In one of these worlds, I then go ahead and prepare my copy-atom in the $s_x = -1/2$ state, and in the other one I set it up as a $+1/2$. You could say that my copy-atom is, like the original, in a superposition of the two $s_x$ states, but there is no reason to think that it will be the same superposition, with the same 90-degree phase angle. In fact, by the argument on p. 997 we know that it is not possible by any measurement to extract this phase information and convert it into classical information. Furthermore, our final result is not really as simple as a copy-atom in some unknown superposition of the two $s_x$ states. It is a silver atom whose spin is correlated with the state of the original, but also correlated with the state of the spectrometer and the state of my brain.

The impossibility of copying an unknown quantum state is en-

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11 The prohibition actually only applies to making a copy that can be separated from the original. For the complete statement of this, see p. 1006.
forced by nature in full generality, not just by the specific mechanisms described in the artificial scenario described above. To see why, consider what would happen to the state of the “blank” atom on which we had hoped to impose the copied state. Its state would have been overwritten, but this would imply a loss of information, which is forbidden by the unitarity postulate of quantum mechanics (p. 969).

The no-cloning theorem would seem to severely limit the practicability of quantum computing. When you run a program on a classical computer, the very first step to be performed by the operating system is to copy the program’s code and data from storage into random-access memory. If a quantum computer can’t copy anything, then how do we perform this initial step? But the no-cloning theorem doesn’t actually forbid copying any quantum state — it forbids copying an unknown state. Going back to the example of the silver atom, imagine that rather than presenting me with a silver atom in a completely unknown quantum state, you give me a solemn promise that it will be either in the state \( s_x = -1/2 \) or the state \( s_x = +1/2 \) — not some superposition of these. Then if you trace back through the logic of the scenario, you will find that there is absolutely nothing preventing me from making an accurate copy.

Once the software on a quantum computer starts running, its qubits will certainly start going into superpositions of the 0 state and the 1 state. By the no-cloning theorem, these cannot be copied from one memory location to another, overwriting the previous contents of the target location. But that simply isn’t how quantum computing works. Rather than attempting to copy, erase, and overwrite bits as in a classical computer, the software is designed to create complicated correlations between the different bits. This model of computing is not necessarily better or worse over all than classical digital computing, but it differs from it as much as an iPhone’s model of computing differs from that of a slide rule.

When a classical computer such as a cash register or phone is done with its computation, we have to find out the result through an output such as a paper tape or LCD screen. These are classical devices. If a quantum computer is to produce a result for use by humans, then it will also need to send its output through a classical device. We might hope to be able to convert the quantum information faithfully into classical information. But we can prove based on the no-cloning theorem that such a conversion will always be “lossy” — will always involve a degradation of the information. A lossless conversion, such as a unit conversion, is one that can be done as a round-trip, e.g., \( 1\text{ m} \rightarrow 100\text{ cm} \rightarrow 1\text{ m} \), with the final result being identical to the original. If we could completely encode qbits into bits, then we could make a second copy of the bits and violate no-cloning by converting back to qbits. This is a contradiction, so we conclude that lossless conversion of classical information
to quantum information is impossible.

14.11 More about entanglement

A basic difference between classical computing and quantum computing is that qubits can be entangled with each other. We’ve only discussed entanglement briefly in sec. 13.2.4, p. 882, where the basic idea was that either Alice or Bob could detect a certain photon, but not both. Alice and Bob’s states were entangled, as were the macroscopic diamonds in the 2012 real-world experiment described on p. 886. More generally, what is entanglement?

Entanglement is the opposite of separability (sec. 14.5.2). To see what is meant by this statement, consider figure a. In a/1, we have the function \( \Psi_1 = \sin x \sin 4y \). This could be a two-dimensional particle in a box, with a certain amount of momentum in the \( x \) direction, and four times that momentum in the \( y \) direction. It is because the Schrödinger equation for the particle in a box is separable in \( x \) and \( y \) that we can write down this wavefunction by multiplying two different one-dimensional wavefunctions. In figure a/2, \( \Psi_2 \) is like \( \Psi_1 \) but with \( x \) and \( y \) interchanged, while a/3 shows the superposition \( \Psi_3 = (\Psi_1 + \Psi_2)/\sqrt{2} \).

From a fancier theoretical point of view, we could say that this system, which seems like a single thing (the particle), is actually built out of two subsystems. One subsystem is the motion in the \( x \) direction, and the other is the \( y \). The fact that the Schrödinger equation is separable can be interpreted as being because the \( x \) and \( y \) motion are independent of one another. Exactly the same thing would happen if this were a classical pool ball on a square table. Its \( x \) and \( y \) motion don’t affect each other, and, e.g., if the ball hits the right-hand cushion and has its \( x \) momentum reflected, that doesn’t change its \( y \) momentum. It’s as if the pool ball in two dimensions were really two different beads, one sliding along a wire parallel to the \( x \) axis and the other sliding up and down. In either the classical case or the quantum-mechanical case, we have built a composite system out of two independent subsystems.

In an example like \( \Psi_1 \), it is possible to assign a definite state to
the subsystems: continuing to ignore units, we can write $p_x = \pm 1$ and $p_y = \pm 4$. The state with wavefunction $\Psi_2$ has the same energy as $\Psi_1$, and again the subsystems have a definite state, $p_x = \pm 4$ and $p_y = \pm 1$.

But for the superposition $\Psi_3$, this is no longer true. If we measure either $p_x$ or $p_y$ for this state, we may get either $\pm 1$ or $\pm 4$, with equal probability. We say that this state is entangled in the same way that Alice and Bob were entangled on p. 883. Neither Bob nor Alice is in a definite state of I-saw-a-photon or I-never-saw-a-photon. However, if we ask Bob whether he saw a photon, and he says yes, then we gain information about Alice: that she didn’t see a photon. Similarly, if we measure $p_x$ for the particle in state $\Psi_3$ and get $-4$, then we gain information about $p_y$: we know that it is $\pm 1$.

Because separable states are the simplest things we can make by putting together subsystems like legos, it’s convenient to have a notation for them. In the angle-bracket notation, all of the following are possible ways that people might notate a state like $\Psi_1$:

$|1, 4\rangle$ or $|1\rangle|4\rangle$ or $|1\rangle \otimes |4\rangle$

The cross with a circle around it, $\otimes$, doesn’t really indicate multiplication. It’s more like a punctuation mark or a conjunction, meaning “and also,” as in, “I’ll have the eggplant, and also a beer.” It’s called a tensor product, which makes it sound scary.

To show the generality of the idea of entanglement, let’s consider an example from particle physics. The $\pi^0$ is a particle that participates in strong nuclear interactions, and therefore can be created in nuclear reactions. It’s known as a pion. There are other types of pions. The $\pi^0$ is the only electrically neutral one, hence the superscript 0. All pions are unstable, which is why we need to create them in reactions rather than looking for them in rocks and trees. The $\pi^0$ has a half-life of only $10^{-16}$ s, and one of the ways in which it can decay is into an electron and a positron (antielectron),

$$\pi^0 \rightarrow e^- + e^+.$$

You can verify that charge is conserved in this reaction. In the frame of reference where the pion is initially at rest, the speeds of the electron and positron are fixed by conservation of energy and momentum, so there is not much that is interesting to measure about them other than their spins. The pion has zero spin, which makes it somewhat unusual in the world of particle physics. If we assume as well, for simplicity, that the electron and positron don’t have any orbital angular momentum, then by conservation of angular momentum, the spin-1/2 of the electron must be in the opposite direction compared to that of the positron.

The electron and positron fly off in opposite directions due to conservation of momentum, and they could be detected by two different particle detectors lying at macroscopic distances from the
The decay of a neutral pion is detected through its decay products. Although separated, they are entangled. Suppose each of the detectors is capable of detecting the component of the spin along a $z$ axis that is defined by the orientation of the detector itself. For example, the detector could in principle be a Stern-Gerlach spectrometer (sec. 14.1, p. 957), although in practice some other, more efficient method would be used. If one detector measures $s_z = +1/2$, then the other is guaranteed to see $s_z = -1/2$, because anything else would violate conservation of angular momentum. That is, the wavefunction of the system is of the form

$$\Psi = c | \uparrow \downarrow \rangle + c' | \downarrow \uparrow \rangle,$$

where normalization requires that $c^2 + c'^2 = 1$. If we had some way to point the pion in a certain direction before it decayed, or produce it so that it was pointed in a certain direction, then perhaps we could have arranged things so that one of the two possibilities, say $| \uparrow \downarrow \rangle$, was more likely. But the pion has spin 0, and a spinless particle is like a perfectly smooth and featureless ping-pong ball; there is no way to impose, define, or measure an orientation for it. Therefore by symmetry we have $c^2 = c'^2$. For example, we could have $c$ and $c'$ both equal to $1/\sqrt{2}$, or $c = i/\sqrt{2}$ and $c' = -1/\sqrt{2}$. The states $| \uparrow \downarrow \rangle$ and $| \downarrow \uparrow \rangle$ are separable in terms of the two spins, but $\Psi$ is entangled.

In the state $\Psi$, neither spin has a definite value, but measuring one spin determines the other spin.

In quantum computing, once a quantum computer has started running, all of its qubits will in general be entangled with one another. That means that if we read out one qubit, then later readouts of other qubits will have results that are correlated with what we got when we read out the first one. With classical information, we can always do things like splitting a book up into chapters, or distributing a long movie on two DVDs. That doesn’t always work for a quantum computer. It might work if part of the data was separable from another part, but we would need a computer program to scan through the data and figure out whether this was in fact possible. This is called the separability problem, and unfortunately it is known to be intractable.

The no-cloning theorem described on p. 1002 is only a prohibition on making a separable copy of an unknown state. To see why, consider an experiment like the one in figure c, in which we set up the detectors so that their spin-detecting axes are in perpendicular orientations. Say one detector measures the spin of the electron along the $x$ axis, while the other measures the positron’s $z$ spin. Now it seems that we can infer simultaneous values of both $L_x$ and $L_z$ for each particle, but that is impossible because $L_x$ and $L_z$ are incompatible observables (p. 922). Well, suppose that we measure the electron’s $L_x$ first, and then the positron’s $L_z$. This is actually equivalent to measuring $-L_x$ for the the positron, and then $L_z$ for the positron. No paradox arises, because one of the mea-
surements will inevitably have changed the positron’s spin. Going back to the version of the experiment using the entangled electron and positron, the same thing happens. For example, measuring the electron’s spin has the ability to change the positron’s spin, because they’re entangled. The no-cloning theorem cannot possibly prohibit making entangled copies, because then it would forbid entanglement itself. Only making separable copies inevitably leads to paradoxes.
Problems

The symbols √, □, etc. are explained on page 1014.

1 Nearly all naturally occurring oxygen nuclei are the isotope $^{16}\text{O}$. The extremely neutron-rich isotope $^{22}\text{O}$ has been produced in accelerator experiments, but only with great difficulty, and little is known about its properties. The only states that have been observed and assigned reliable spins are the ground state, with spin 0, and an excited state with spin 2 and an excitation energy of 3.2 MeV. The excited state was detected by observing gamma rays for the $2 \rightarrow 0$ transition. On the hypothesis that the spin-2 excited state is a rotation, predict the gamma-ray energy that experimentalists should expect from the $4 \rightarrow 2$ transition in the same rotational band.

2 For vectors in two dimensions, which of the following are possible choices of a basis?

$$\{\hat{x}\}, \{\hat{x}, \hat{y}\}, \{-\hat{x}, \hat{x} + \hat{y}\}, \{\hat{x}, \hat{y}, \hat{x} + \hat{y}\}$$

▷ Solution, p. 1052 □

3 (a) Consider the set of vectors in two dimensions. This set $P$ is a vector space, and can be visualized as a plane, with each vector being like an arrow that extends from the origin to a particular point. Now consider the line $ℓ$ defined by the equation $y = x$ in Cartesian coordinates, and the ray $r$ defined by $y = x$ with $x \geq 0$. Sketch $ℓ$ and $r$. If we consider $ℓ$ and $r$ as subsets of the arrows in $P$, is $ℓ$ a vector space? Is $r$?

(b) Consider the set $C$ of angles $0 \leq θ < 2\pi$. Define addition on $C$ by adding the angles and then, if necessary, bringing the result back into the required range. For example, if $x = \pi$ and $y = 3\pi/2$, then $x + y = \pi/2$. Thus if we visualize $C$ as a circle, every point on the circle has a single number to represent it, not multiple representations such as $\pi/2$ and $5\pi/2$. Suppose we want to make $C$ into a vector space over the real numbers, so that elements of $C$ are the vectors, while a scalar $α$ can be any real number, not just a number from 0 to $2\pi$. Then for example if $α = 2$ is a scalar and $v = \pi$ is a vector, then $αv = 0$. Find an example to prove that $C$ is not a vector space, because it violates the distributive property $α(v + w) = αv + αw$.

▷ Solution, p. 1052 □

4 In the SI, we have three base units, the kilogram, the meter, and the second. From these, we form expressions such as $\text{m/s}$ to represent units of velocity, and $\text{kg}\cdot\text{m}/\text{s}^2$ for force. Show that these expressions form a vector space with the rational numbers as the scalars. What operation on the units should we take as the “addition” operation? What operation should scalar “multiplication” be?

▷ Solution, p. 1053 □
In problem 23 on p. 944, you showed that a wavefunction of the form

$$\Psi_0(x) = e^{-x^2/2}$$

was a solution of the Schrödinger equation for the quantum harmonic oscillator in one dimensions. (We ignore units, and the factor of 1/2 in the exponent is just a convention.) It represents the ground state. The wavefunction of the first excited state is

$$\Psi_1(x) = xe^{-x^2/2},$$

with the same value of $b$.

(a) Show that these states are orthogonal in the sense defined on p. 982.

(b) What is an observable that would distinguish them?

When an excited state in a nucleus undergoes gamma decay, the half-life depends on a variety of factors, but a fairly typical value would be about 1 ns. Find the uncertainty in energy imposed by the energy-time uncertainty relation, and compare with a typical excitation energy of 1 MeV.

(b) Some very neutron-rich nuclei are unstable with respect to emission of a neutron, and in these cases the half-life is typically on the order of $10^{-21}$ s. Carry out an estimate as in part a.

As you might have guessed from the equations given in problem 5, the $m$th excited state of the one-dimensional quantum harmonic oscillator has a wavefunction of the form

$$\Psi_m(x) = H_m(x)e^{-x^2/2}.$$  

Here $H_m$ is a polynomial of order $m$, and $H_m$ is an even function if $m$ is even, odd if $m$ is odd. Given these assumptions, it is possible to find $H_2$ simply from the requirement that it be orthogonal to $\Psi_0$ and $\Psi_1$, without having to solve the Schrödinger equation. Find $H_2$ by this method. (Don’t worry about normalization or phase.) Hint: Near the end of the calculation, you will encounter integrals of the form $\int_{-\infty}^{\infty} x^m e^{-x^2} \, dx$. This can be done using software, or you can use integration by parts to relate this integral to the corresponding integral for $m - 2$.

In example 14 on p. 989, we defined a very naughty energy operator

$$\hat{H}\Psi = i\Psi.$$  

Show that it is not hermitian, by directly using the definition on p. 982.
This problem refers to the analysis of the ammonia molecule in sec. 14.7.2, p. 992. (a) The bond lengths in this molecule are on the order of 0.1 nm. Use this fact to estimate the moment of inertia for rotation about the symmetry axis, and verify that states with \( L_z > 0 \) are likely to be populated at room temperature. 

(b) The original 1955 paper by Townes and Schawlow on the microwave spectroscopy of ammonia detected about 55 lines lying between 17 and 29 GHz. Each of these corresponds to a certain value of \( L \) and \( L_z \). Since there are many lines crowded together in this region of the spectrum, the issue arises of whether the resolution of the experiment will be sufficient to distinguish them. One of the factors limiting the resolution is that the molecules of ammonia gas have velocities that are random and randomly oriented, and this causes random Doppler shifts in the lines. Estimate the Doppler shifts at room temperature and determine whether or not they are likely to cause problems.

This problem refers to the analysis of the ammonia molecule in sec. 14.7.2, p. 992. (a) The text constructs the ground state \( |\text{g.s.}\rangle \), which has energy \(-|f| = f\). Use the same method to find the excited state, which has energy \(+|f| = -f\).

(b) Verify that these two states are orthogonal.

(c) Find normalized versions of the two states.

Consider the wavefunctions \( \Psi_1 = \bigwedge \) and \( \Psi_2 = \bigvee \) for a particle in a one-dimensional box. Suppose we have the superposition \( \Psi = A(2\Psi_1 + \Psi_2) \).

(a) If \( \Psi \) is to be properly normalized, what is \(|A|\)?

(b) Sketch the wavefunction.

(c) Suppose you can measure the position of the particle very accurately. What is the probability that the particle will be found in the left half of the box?

(d) Instead of measuring position, suppose you measure the energy of the state. What is the probability that you’ll measure the ground state energy?

(e) Suppose that the wavefunction had been \( \Phi = A(2\Psi_1 - \Psi_2) \). Which of your answers to parts a-d would remain the same, and which would change? (You need not redo the work for the ones that would change. Just give your reasoning as to whether they would or would not.) [Problem by B. Shotwell]
**12** This problem builds on the results of problems 13-21 (p. 944) and 13-38 (p. 947).

Suppose we have a three-dimensional box of dimensions $L \times L \times L/2$. Let the box be oriented so that the shorter dimension is along the $z$ direction. For convenience, define the quantity $\epsilon = h^2/8mL^2$, which has units of energy.

(a) What are the five lowest energies allowed in this box, expressed in terms of $\epsilon$? Give the quantum numbers for each energy, and find the degeneracy (p. 920) of each.

(b) Suppose we put five electrons in this box such that they have the lowest possible total energy. (Keep in mind that there is a limit to how many electrons can have the same spatial wavefunction.) What is the total energy of this state?

(c) What are the two lowest-energy photons that can excite one of the five electrons (from the situation described in part b) to an excited state?

[Problem by B. Shotwell.]

**13** In a helium nucleus, each particle feels a potential due to the attractive forces from the other three. This potential can be well approximated as

$$U = \frac{1}{2}kr^2,$$

where $k$ is a constant and $r$ is the distance from the center of mass in three dimensions. You will need the result of problem 23, p. 944. Refer also to sec. 14.2.4, p. 962.

(a) Show that the Schrödinger equation is separable in terms of $x$, $y$, and $z$ in this example.

(b) Find the ground-state wavefunction, expressed in terms of $r$ and the constant $b$ defined in problem 23, p. 944.

(c) Find the energy of the ground state, expressed in terms of the classical frequency $\omega$.

[Problem by B. Shotwell.]

**14** An entangled state of three particles is prepared, described by the wavefunction,

$$\Psi = k[-2|↑↑↑⟩ + 2i|↑↓↑⟩ + |↓↓↓⟩]$$

where the arrows are the $z$-components of the spins of particles A, B, and C, respectively.

(a) The wavefunction is not properly normalized. What value of $|k|$ will normalize the wavefunction?

(b) What is the probability that measuring the $z$-component of particle A’s spin will give spin up?

(c) Suppose that we first measure the spin of particle C and find that it is down, and then we measure the spin of particle A. What is the probability that we will find A’s spin to be up?

[Problem by B. Shotwell.]
15 Suppose that we replace the usual probability rule of quantum mechanics with one of the form \( P \propto |\langle \Psi | \Psi \rangle|^M \), with \( M > 2 \). Suppose \( M = 4 \). Show, by considering the example in discussion question B on p. 917, that this leads to nonconservation of probability.

16 In example 12 on p. 984, we defined unnormalized wavefunctions for the traveling-wave solutions to the Schrödinger equation in a “quantum moat,” and calculated the inner product \( \langle \text{ccw} | \text{cw} \rangle = 0 \) to verify that the counterclockwise and clockwise traveling waves were orthogonal, as must be the case for distinguishable states. Suppose we want to define a normalized version of the counterclockwise wave, \( |\text{ccw}\rangle = A e^{i\theta} \). Use an inner product to determine \( |A|^2 \), and show that normalization doesn’t depend on the phase of \( A \). (Do not assume that \( A \) is real.)

17 In example 12 on p. 984, we defined wavefunctions for the traveling-wave solutions to the Schrödinger equation in a “quantum moat,” and calculated the inner product \( \langle \text{ccw} | \text{cw} \rangle = 0 \) to verify that the counterclockwise and clockwise traveling waves were orthogonal, as must be the case for distinguishable states. Let’s now define standing-wave versions \( |c\rangle = \cos \theta \) and \( |s\rangle = \sin \theta \). Verify by direct calculation that \( \langle c | s \rangle = 0 \).

Remark: Note that, as discussed in the sidebar on p. 966, this does not contradict the principle that a quantum-mechanical phase is undetectable.

18 Consider the wavefunctions

\[
\begin{align*}
\Psi_1 &= e^{ikx}, \\
\Psi_2 &= e^{-ikx}, \\
\Phi_1 &= \cos kx, \quad \text{and} \\
\Phi_2 &= i \sin kx.
\end{align*}
\]

Show that \( \Psi_1 = \Phi_1 + \Phi_2 \). Similarly, express \( \Psi_2 \) in terms of the \( \Phi \)'s, and express each of the \( \Phi \)'s in terms of the \( \Psi \)'s. Relate this to the principle that there is no preferred basis in quantum mechanics (p. 987).

\[\checkmark\]
19 In section 14.5.1, p. 970, we found the solution to the Schrödinger equation for a particle arriving at a potential barrier, in the case where the far side of the barrier is classically allowed. We now consider the case in which the barrier is not just high enough to make the region beyond it classically forbidden — we let the height of the barrier be infinite. Let the potential be

\[ U(x) = \begin{cases} 
0, & x < 0 \\
+\infty, & x > 0, 
\end{cases} \]

and let the incident wave be

\[ \Psi_1 = e^{i(kx-\omega t)} \quad (x < 0). \]

Determine the form of the complete solution to the Schrödinger equation on the left side of the barrier, including both the incident wave and the reflected wave.

\[ \sqrt{20} \]

20 As discussed on p. 986, suppose that for a particle in a box, we have

\[ O_E \bigodot 1 = \bigodot, \]
\[ O_E \bigodot 4 = 4 \bigodot, \]
\[ \Psi = c \bigodot + c' \bigodot, \quad \text{and} \]
\[ |c| = |c'|. \]

Show that \( \langle \Psi | O_E | \Psi \rangle = 2.5. \) \( \sqrt{\triangleright} \) Solution, p. 1053

21 Consider the wavefunctions

\[ \Psi_1 = e^{x+it} \quad \text{and} \]
\[ \Psi_2 = e^{t+ix}. \]

To keep the writing simple, we use a system of units (not SI) such that these expressions make sense, and in which \( \hbar = 1. \)

(a) Show by direct substitution in the time-dependent Schrödinger equation (with \( U = \text{constant} \)) that one of these is a solution and the other is not.

(b) Make an independent argument, requiring no calculations, to the effect that the invalid one violates one of the fundamental principles 1-5 of quantum mechanics listed in section 14.6.5, p. 990.
Microscopic circuits are etched on the surface of a silicon chip. The equivalent of a wire in such an integrated circuit is called a “trace.” We consider the case where the trace is narrow enough to make quantum effects relevant, and we treat an electron inside the trace using the two-dimensional Schrödinger equation. We describe the trace as an infinite strip running parallel to the \( x \) axis, extending from \( y = 0 \) to \( y = b \). The potential is

\[
U = \begin{cases} 
0, & 0 < y < b \\
\infty, & y \leq 0 \text{ or } y \geq b 
\end{cases}
\]

For convenience of notation, let \( a = \pi/b \). Consider the following wavefunctions:

\[
\begin{align*}
\Psi_1 &= e^{i(-kx-\omega t)} \sin ay \\
\Psi_2 &= e^{-i\omega t} \sin kx e^{ay} \\
\Psi_3 &= e^{-i\omega t} e^{kx} \sin ay \\
\Psi_4 &= e^{-i\omega t} (\sin 2ax \sin ay + \sin ax \sin 2ay)
\end{align*}
\]

The symbols \( \omega \) and \( k \) stand for real constants. Identify the wavefunctions that have the following properties. Exactly one of the wavefunctions has each property. Explain all answers.

(a) cannot be a solution of the Schrödinger equation for this potential
(b) is a traveling wave solution
(c) is a solution that could represent the case where \( 0 < y < b \) is classically forbidden
(d) is not separable

Key to symbols:
- Easy
- Typical
- Challenging
- Difficult
- Very difficult
✓ An answer check is available at www.lightandmatter.com.
Three essential mathematical skills

More often than not when a search-and-rescue team finds a hiker dead in the wilderness, it turns out that the person won a Darwin Award by not carrying some item from a short list of essentials, such as water and a map. There are three mathematical essentials in this course.

1. Converting units

basic technique: subsection 0.1.9, p. 28; conversion of area, volume, etc.: subsection 0.2.1, p. 34

Examples:

\[ 0.7 \text{ kg} \times \frac{10^3 \text{ g}}{1 \text{ kg}} = 700 \text{ g}. \]

To check that we have the conversion factor the right way up \((10^3\) rather than \(1/10^3\)), we note that the smaller unit of grams has been compensated for by making the number larger.

For units like \(m^2\), \(kg/m^3\), etc., we have to raise the conversion factor to the appropriate power:

\[ 4 \text{ m}^3 \times \left( \frac{10^3 \text{ mm}}{1 \text{ m}} \right)^3 = 4 \times 10^9 \text{ m}^3 \times \frac{\text{mm}^3}{\text{m}^3} = 4 \times 10^9 \text{ mm}^3 \]

Examples with solutions — p. 47, #6; p. 51, #31

Problems you can check at lightandmatter.com/area1checker.html — p. 47, #5; p. 47, #4; p. 47, #7; p. 51, #22; p. 52, #40

2. Reasoning about ratios and proportionailities

The technique is introduced in subsection 0.2.2, p. 35, in the context of area and volume, but it applies more generally to any relationship in which one variable depends on another raised to some power.

Example: When a car or truck travels over a road, there is wear and tear on the road surface, which incurs a cost. Studies show that the cost per kilometer of travel \(C\) is given by

\[ C = kw^4, \]

where \(w\) is the weight per axle and \(k\) is a constant. The weight per axle is about 13 times higher for a semi-trailer than for my Honda Fit. How many times greater is the cost imposed on the federal government when the semi travels a given distance on an interstate freeway?
First we convert the equation into a proportionality by throwing out \( k \), which is the same for both vehicles:

\[ C \propto w^4 \]

Next we convert this proportionality to a statement about ratios:

\[ \frac{C_1}{C_2} = \left( \frac{w_1}{w_2} \right)^4 \approx 29,000 \]

Since the gas taxes paid by the trucker are nowhere near 29,000 times more than those I pay to drive my Fit the same distance, the federal government is effectively awarding a massive subsidy to the trucking company. Plus my Fit is cuter.

Examples with solutions — p. 51, #32; p. 51, #33; p. 52, #38; p. 49, #17

Problems you can check at lightandmatter.com/area1checker.html — p. 52, #37; p. 52, #39; p. 123, #24; p. 124, #27; p. 121, #9; p. 294, #3

3. Vector addition

subsection 3.4.3, p. 203

Example: The \( \Delta r \) vector from San Diego to Los Angeles has magnitude 190 km and direction 129° counterclockwise from east. The one from LA to Las Vegas is 370 km at 38° counterclockwise from east. Find the distance and direction from San Diego to Las Vegas.

Graphical addition is discussed on p. 203. Here we concentrate on analytic addition, which involves adding the \( x \) components to find the total \( x \) component, and similarly for \( y \). The trig needed in order to find the components of the second leg (LA to Vegas) is laid out in figure 1 on p. 201 and explained in detail in example 60 on p. 201:

\[ \Delta x_2 = (370 \text{ km}) \cos 38^\circ = 292 \text{ km} \]
\[ \Delta y_2 = (370 \text{ km}) \sin 38^\circ = 228 \text{ km} \]

(Since these are intermediate results, we keep an extra sig fig to avoid accumulating too much rounding error.) Once we understand the trig for one example, we don’t need to reinvent the wheel every time. The pattern is completely universal, provided that we first make sure to get the angle expressed according to the usual trig convention, counterclockwise from the \( x \) axis. Applying the pattern to the first leg, we have:

\[ \Delta x_1 = (190 \text{ km}) \cos 129^\circ = -120 \text{ km} \]
\[ \Delta y_1 = (190 \text{ km}) \sin 129^\circ = 148 \text{ km} \]

For the vector directly from San Diego to Las Vegas, we have

\[ \Delta x = \Delta x_1 + \Delta x_2 = 172 \text{ km} \]
\[ \Delta y = \Delta y_1 + \Delta y_2 = 376 \text{ km} . \]
The distance from San Diego to Las Vegas is found using the Pythagorean theorem,
\[ \sqrt{(172 \text{ km})^2 + (376 \text{ km})^2} = 410 \text{ km} \]
(rounded to two sig figs because it’s one of our final results). The direction is one of the two possible values of the inverse tangent
\[ \tan^{-1}(\Delta y/\Delta x) = \{65^\circ, 245^\circ\} \].
Consulting a sketch shows that the first of these values is the correct one.

Examples with solutions — p. 233, #62; p. 233, #63; p. 230, #45

Problems you can check at lightandmatter.com/area1checker.html — p. 232, #53; p. 232, #57; p. 233, #58; p. 238, #79; p. 230, #46
Mathematical Review

Algebra

Quadratic equation:

The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Logarithms and exponentials:

\[ \ln(ab) = \ln a + \ln b \]
\[ e^{a+b} = e^a e^b \]
\[ \ln e^x = x \]
\[ \ln(a^b) = b \ln a \]

Geometry, area, and volume

- area of a triangle of base $b$ and height $h = \frac{1}{2}bh$
- circumference of a circle of radius $r = 2\pi r$
- area of a circle of radius $r = \pi r^2$
- surface area of a sphere of radius $r = 4\pi r^2$
- volume of a sphere of radius $r = \frac{4}{3}\pi r^3$

Trigonometry with a right triangle

\[ \sin \theta = \frac{o}{h} \quad \cos \theta = \frac{a}{h} \quad \tan \theta = \frac{o}{a} \]

Pythagorean theorem: $h^2 = a^2 + o^2$

Trigonometry with any triangle

Law of Sines:

\[ \frac{\sin \alpha}{A} = \frac{\sin \beta}{B} = \frac{\sin \gamma}{C} \]

Law of Cosines:

\[ C^2 = A^2 + B^2 - 2AB \cos \gamma \]

Properties of the derivative and integral

Let $f$ and $g$ be functions of $x$, and let $c$ be a constant.

Linearity of the derivative:

\[ \frac{d}{dx}(cf) = c \frac{df}{dx} \]
\[ \frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx} \]

The chain rule:

\[ \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \]

Derivatives of products and quotients:

\[ \frac{d}{dx} (fg) = \frac{df}{dx} g + \frac{dg}{dx} f \]
\[ \frac{d}{dx} \left( \frac{f}{g} \right) = \frac{f'g - fg'}{g^2} \]

Some derivatives:

\[ \frac{d}{dx} x^m = mx^{m-1}, \quad \text{except for } m = 0 \]
\[ \frac{d}{dx} \sin x = \cos x \quad \frac{d}{dx} \cos x = -\sin x \]
\[ \frac{d}{dx} e^x = e^x \quad \frac{d}{dx} \ln x = \frac{1}{x} \]

Linearity of the integral:

\[ \int c f(x) \, dx = c \int f(x) \, dx \]
\[ \int [f(x) + g(x)] = \int f(x) \, dx + \int g(x) \, dx \]

The fundamental theorem of calculus:

The derivative and the integral undo each other, in the following sense:

\[ \int_a^b f'(x) \, dx = f(b) - f(a) \]
Approximations to Exponents and Logarithms

It is often useful to have certain approximations involving exponents and logarithms. As a simple numerical example, suppose that your bank balance grows by 1% for two years in a row. Then the result of compound interest is growth by a factor of $1.01^2 = 1.0201$, but the compounding effect is quite small, and the result is essentially 2% growth. That is, $1.01^2 \approx 1.02$. This is a special case of the more general approximation

$$(1 + \epsilon)^p \approx 1 + p\epsilon,$$

which holds for small values of $\epsilon$ and is used in example 4 on p. 408 relating to relativity.

Proof: Any real exponent $p$ can be approximated to the desired precision as $p = a/b$, where $a$ and $b$ are integers. Let $(1+\epsilon)^p = 1+x$. Then $(1+\epsilon)^a = (1+x)^b$. Multiplying out both sides gives $1 + a\epsilon + \ldots = 1 + bx + \ldots$, where $\ldots$ indicates higher powers. Neglecting these higher powers gives $x \approx (a/b)\epsilon \approx p\epsilon$.

We have considered an approximation that can be found by restricting the base of an exponential to be close to 1. It is often of interest as well to consider the case where the exponent is restricted to be small. Consider the base-$e$ case. One way of defining $e$ is that when we use it as a base, the rate of growth of the function $e^x$, for small $x$, equals 1. That is,

$$e^x \approx 1 + x$$

for small $x$. This can easily be generalized to other bases, since $a^x = e^{\ln(a^x)} = e^{x\ln a}$, giving

$$a^x \approx 1 + x\ln a.$$

Finally, since $e^x \approx 1 + x$, we also have

$$\ln(1 + x) \approx x.$$
Programming with Python

The purpose of this tutorial is to help you get familiar with a computer programming language called Python, which I’ve chosen because (a) it’s free, and (b) it’s easy to use interactively. I won’t assume you have any previous experience with computer programming; you won’t need to learn very much Python, and what little you do need to learn I’ll explain explicitly. If you really want to learn Python more thoroughly, there are a couple of excellent books that you can download for free on the Web:

- **How to Think Like a Computer Scientist (Python Version)**, Allen B. Downey, Jeffrey Elkner, Moshe Zadka, http://www.ibiblio.org/obp/
- **Dive Into Python**, Mark Pilgrim, http://diveintopython.net/

The first book is meant for people who have never programmed before, while the second is a more complete introduction aimed at veteran programmers who know a different language already.

### Using Python as a calculator

The easiest way to get Python going is to go to the web site ideone.com. Under “choose a language,” select Python. Inside the window where it says “paste your source code or insert template or sample,” type `print(2+2)`. Click on the “submit” button. The result, 4, is shown under “output.” In other words, you can use Python just like a calculator.

For compactness, I’ll show examples in the following style:

```python
>>> print(2+2)
4
```

Here the `>>>` is not something you would type yourself; it’s just a marker to distinguish your input from the program’s output. (In some versions of Python, the computer will actually print out `>>>` as a prompt to tell you it’s ready to type something.)

There are only a couple of things to watch out for. First, Python distinguishes between integers and real numbers, so the following gives an unexpected result:

```python
>>> print(2/3)
0
```

To get it to treat these values as real numbers, you have to use decimal points:

```python
>>> print(2./3.)
0.6666666666666666666663
```

Multiplication is represented by “*”:

```python
>>> print(2.*3.)
6.0
```

Also, Python doesn’t know about its own library of math functions unless you tell it explicitly to load them in:

```python
>>> import math
>>> print(math.sqrt(2.))
Traceback (most recent call last):
  File ‘<stdin>’, line 1, in ?
NameError: There is no variable named ‘sqrt’
```

Here are the steps you have to go through to calculate the square root of 2 successfully:

```python
>>> import math
>>> print(math.sqrt(2.))
1.4142135623730951
```

The first line is just something you can make a habit of doing every time you start up Python. In the second line, the name of the square root function had to be prefixed with “math.” to tell Python where you wanted to get this “sqrt” function from. (All of this may seem like a nuisance if you’re just using Python as a
calculator, but it’s a good way to design a pro-
gramming language so that names of functions
never conflict.)

Try it. Experiment and figure out whether
Python’s trig functions assume radians or
degrees.

Variables
Python lets you define variables and assign
values to them using an equals sign:

```python
>>> dwarfs=7
>>> print(dwarfs)
>>> print(dwarfs+3)
7
10
```

Note that a variable in computer program-
ming isn’t quite like a variable in algebra. In
algebra, if \( a=7 \) then \( a=7 \) always, throughout a
particular calculation. But in a programming
language, the variable name really represents
a place in memory where a number can be
stored, so you can change its value:

```python
>>> dwarfs=7
>>> dwarfs=37
>>> print(dwarfs)
37
```

You can even do stuff like this,

```python
>>> dwarfs=37
>>> dwarfs=dwarfs+1
>>> print(dwarfs)
38
```

In algebra it would be nonsense to have a vari-
able equal to itself plus one, but in a com-
puter program, it’s not an assertion that the
two things are equal, it’s a command to calcu-
late the value of the expression on the right
side of the equals, and then put that number
into the memory location referred to by the
variable name on the left.

Try it. What happens if you do `dwarfs+1 =
dwarfs`? Do you understand why?

Functions

Somebody had to teach Python how to do
functions like `sqrt`, and it’s handy to be able
to define your own functions in the same way.
Here’s how to do it:

```python
>>> def double(x):
...      return 2.*x
>>> print(double(5.))
10.0
```

Note that the indentation is mandatory. The
first and second lines define a function called
double. The final line evaluates that function
with an input of 5.

Loops

Suppose we want to add up all the numbers
from 0 to 99.

Automating this kind of thing is exactly
what computers are best at, and Python pro-
vides a mechanism for this called a loop:

```python
>>> sum=0
>>> for j in range(100):
...      sum=sum+j
>>> print(sum)
4950
```

The stuff that gets repeated — the inside of
the loop — has to be indented, just like in
a function definition. Python always counts
loops starting from 0, so for `j in range(100)`
actually causes `j` to range from 0 to 99, not
from 1 to 100.

Problems

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Appendix 2: Miscellany

Unphysical “hovering” solutions to conservation of energy

On page 83, I gave the following derivation for the acceleration of an object under the influence of gravity:

\[
\frac{dv}{dt} = \frac{dv}{dK} \left( \frac{dK}{dU} \right) \left( \frac{dU}{dy} \right) \left( \frac{dy}{dt} \right) = \left( \frac{1}{mv} \right) (-1)(mg)(v) = -g
\]

There is a loophole in this argument, however. When I say \( dv/dK = 1/(mv) \), that only works when the object is moving. If it’s at rest, \( v \) is nondifferentiable as a function of \( K \) (or we could say that the derivative is infinite). Energy can in fact be conserved by an object that simply hovers above the ground: its kinetic energy is constant, and its gravitational energy is also constant. Why, then, do we never observe such behavior, except in Coyote and Roadrunner cartoons when the Coyote runs off the edge of a cliff without noticing it at first?

Suppose we toss a baseball straight up, and pick a coordinate system in which upward velocities are positive. The ball’s velocity is a continuous function of time, and it changes from being positive to being negative, so there must be some instant at which it equals zero. Conservation of energy would be satisfied if the velocity were to remain at zero for a minute or an hour before the ball finally made the decision to fall. One thing that seems odd about all this is that there’s no obvious way for the ball to “decide” when it was time to go ahead and fall back down again. It violates the principle that the laws of physics are supposed to be deterministic.

One reason that we could never hope to observe such behavior in reality is that the ball would have to spend some time being exactly at rest, and yet no object can ever stay exactly at rest for any finite amount of time. Objects in the real world are buffeted by air currents, for example. At the atomic level, the interaction of these air currents with the ball consists of discrete collisions with whizzing air molecules, and a quick back-of-the-envelope estimate shows that for an object this size, the typical time between collisions is on the order of \( 10^{-27} \) s, which would limit the duration of the hovering to a time far too short to allow it to be observed. Nevertheless this is not a completely satisfying explanation. It makes us wonder whether we ought to apply to the government for a research grant to do an experiment in which a baseball would be shot upward in a chamber that had been pumped out to an ultra-high vacuum!

A somewhat better approach is to consider that motion is relative, so the ball’s velocity can only be zero in one particular frame of reference. It wouldn’t make sense for the ball to exhibit qualitatively different behavior when it was at rest, because different observers don’t even agree when the ball is at rest. But this argument also fails to resolve the issue completely, because this is a ball interacting with the planet Earth via gravitational forces, so it could make a difference whether the ball was at rest relative to the earth. Suppose we go into a frame of reference defined by an observer watching the ball as she descends in a glass elevator. At the moment when the
ball is at rest relative to the earth, she sees both the ball and the earth as moving upward at the same speed. It would be perfectly consistent with conservation of energy if she were to see them maintain this distance from one another for several minutes. In her frame, their kinetic energies would be nonzero, but constant, and the gravitational energy only depends on the separation between the ball and the earth, so it would be constant as well.

Now that we’re thinking of the ball and the earth as two objects interacting with one another, it becomes natural to think of them on the same footing. What about the motion of the Earth? The earth feels a gravitational attraction from the ball, just as the ball feels one from the Earth. To make this symmetry more evident, let’s imagine two planets of equal mass, Foo and Bar, initially at rest with respect to one another. The Fooites and Barians realize that the gravitational interaction between their planets will cause them to drop together and collide. It seems that they should get ready for the end of the world. And yet before they riot, get drunk, or tell their spouses that yes, they really do look fat in that dress, maybe they should consider the possibility that the two planets will simply hover in place for some amount of time, because that would satisfy conservation of energy. Now the physical implausibility of the hovering solution becomes even more apparent. Not only does one planet have to “decide” at precisely what microsecond to go ahead and fall, but the other planet has to make the same decision at the same instant, or else conservation of energy will be violated. There is no physical process or interaction between the two planets that could perfectly synchronize their “decisions” like this. (The mechanism can’t be gravity, because nothing about the gravitational interaction provides any kind of a count-down that would pick out one particular time as the one at which the planets should start moving.)

The key to making sense of all this is to realize that each planet can only “feel” the gravitational field in its own region of space. Its acceleration can only depend on the field, and not on the detailed arrangement of masses elsewhere in the universe that caused that field. Granting this kind of “real” status to fields can be considered as a logically necessary supplement to conservation of energy.

Automated search for the brachistochrone
See page 95.

```python
1  d=.01
2  c1=.61905
3  c2=-.94427
4  a = 1.
5  b = 1.
6  for i in range(100):
7      bestt = 99.
8  for j in range(3):
9      for k in range(3):
10         try_c1 = c1+(j-1)*d
11         try_c2 = c2+(k-1)*d
12         t = timeb(a,b,try_c1,try_c2,100000)
13         if t<bestt :
14             bestc1 = try_c1
15             bestc2 = try_c2
16             bestj = j
17             bestk = k
```
bestt = t

c1 = bestc1

c2 = bestc2

c3 = (b-c1*a-c2*a**2)/(a**3)

print(c1, c2, c3, bestt)

if (bestj == 1) and (bestk == 1):
    d = d*.5

Derivation of the steady state for damped, driven oscillations

Using the trig identities for the sine of a sum and cosine of a sum, we can change equation [2] on page 181 into the form

\[\left((-m\omega^2 + k) \cos \delta - b\omega \sin \delta - \frac{F_m}{A}\right) \sin \omega t
\]
\[+ \left((-m\omega^2 + k) \sin \delta + b\omega \cos \delta\right) \cos \omega t = 0.\]

Both the quantities in square brackets must equal zero, which gives us two equations we can use to determine the unknowns \(A\) and \(\delta\). The results are

\[\delta = \tan^{-1} \frac{b\omega}{m\omega^2 - k} = \tan^{-1} \frac{\omega \omega_0}{Q(\omega_0^2 - \omega^2)}\]

and

\[A = \frac{F_m}{\sqrt{(m\omega^2 - k)^2 + b^2\omega^2}} = \frac{F_m}{m \sqrt{(\omega^2 - \omega_0^2)^2 + \omega_0^2\omega^2Q^{-2}}}.\]

Proofs relating to angular momentum

Uniqueness of the cross product

The vector cross product as we have defined it has the following properties:

1. It does not violate rotational invariance.
2. It has the property \(A \times (B + C) = A \times B + A \times C\).
3. It has the property \(A \times (kB) = k(A \times B)\), where \(k\) is a scalar.

**Theorem:** The definition we have given is the only possible method of multiplying two vectors to make a third vector which has these properties, with the exception of trivial redefinitions which just involve multiplying all the results by the same constant or swapping the names of the axes. (Specifically, using a left-hand rule rather than a right-hand rule corresponds to multiplying all the results by \(-1\).)

**Proof:** We prove only the uniqueness of the definition, without explicitly proving that it has properties (1) through (3).

Using properties (2) and (3), we can break down any vector multiplication \((A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})\) into terms involving cross products of unit vectors.

A “self-term” like \(\hat{x} \times \hat{x}\) must either be zero or lie along the \(x\) axis, since any other direction would violate property (1). If it was not zero, then \((-\hat{x}) \times (-\hat{x})\) would have to lie in the opposite
direction to avoid breaking rotational invariance, but property (3) says that $(-\hat{x}) \times (-\hat{x})$ is the same as $\hat{x} \times \hat{x}$, which is a contradiction. Therefore the self-terms must be zero.

An “other-term” like $\hat{x} \times \hat{y}$ could conceivably have components in the $x$-$y$ plane and along the $z$ axis. If there was a nonzero component in the $x$-$y$ plane, symmetry would require that it lie along the diagonal between the $x$ and $y$ axes, and similarly the in-the-plane component of $(-\hat{x}) \times \hat{y}$ would have to be along the other diagonal in the $x$-$y$ plane. Property (3), however, requires that $(-\hat{x}) \times \hat{y}$ equal $-(-\hat{x} \times \hat{y})$, which would be along the original diagonal. The only way it can lie along both diagonals is if it is zero.

We now know that $\hat{x} \times \hat{y}$ must lie along the $z$ axis. Since we are not interested in trivial differences in definitions, we can fix $\hat{x} \times \hat{y} = \hat{z}$, ignoring peurile possibilities such as $\hat{x} \times \hat{y} = 7\hat{z}$ or the left-handed definition $\hat{x} \times \hat{y} = -\hat{z}$. Given $\hat{x} \times \hat{y} = \hat{z}$, the symmetry of space requires that similar relations hold for $\hat{y} \times \hat{z}$ and $\hat{z} \times \hat{x}$, with at most a difference in sign. A difference in sign could always be eliminated by swapping the names of some of the axes, so ignoring possible trivial differences in definitions we can assume that the cyclically related set of relations $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, and $\hat{z} \times \hat{x} = \hat{y}$ holds. Since the arbitrary cross-product with which we started can be broken down into these simpler ones, the cross product is uniquely defined.

**The choice of axis theorem**

**Theorem:** Suppose a closed system of material particles conserves angular momentum in one frame of reference, with the axis taken to be at the origin. Then conservation of angular momentum is unaffected if the origin is relocated or if we change to a frame of reference that is in constant-velocity motion with respect to the first one. The theorem also holds in the case where the system is not closed, but the total external force is zero.

**Proof:** In the original frame of reference, angular momentum is conserved, so we have $\frac{dL}{dt} = 0$. From example 28 on page 290, this derivative can be rewritten as

$$\frac{dL}{dt} = \sum_i r_i \times F_i,$$

where $F_i$ is the total force acting on particle $i$. In other words, we’re adding up all the torques on all the particles.

By changing to the new frame of reference, we have changed the position vector of each particle according to $r_i \rightarrow r_i + k - ut$, where $k$ is a constant vector that indicates the relative position of the new origin at $t = 0$, and $u$ is the velocity of the new frame with respect to the old one. The forces are all the same in the new frame of reference, however. In the new frame, the rate of change of the angular momentum is

$$\frac{dL}{dt} = \sum_i (r_i + k - ut) \times F_i$$

$$= \sum_i r_i \times F_i + (k - ut) \times \sum_i F_i.$$

The first term is the expression for the rate of change of the angular momentum in the original frame of reference, which is zero by assumption. The second term vanishes by Newton’s third law; since the system is closed, every force $F_i$ cancels with some force $F_j$. (If external forces act, but they add up to zero, then the sum can be broken up into a sum of internal forces and a sum of external forces, each of which is zero.) The rate of change of the angular momentum is therefore zero in the new frame of reference.
The spin theorem

Theorem: An object’s angular momentum with respect to some outside axis A can be found by adding up two parts:

1) The first part is the object’s angular momentum found by using its own center of mass as the axis, i.e. the angular momentum the object has because it is spinning.
2) The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

Proof: Let \( \mathbf{r}_{\text{cm}} \) be the position of the center of mass. The total angular momentum is

\[
\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i,
\]

which can be rewritten as

\[
\mathbf{L} = \sum_i \left( \mathbf{r}_{\text{cm}} + \mathbf{r}_i - \mathbf{r}_{\text{cm}} \right) \times \mathbf{p}_i,
\]

where \( \mathbf{r}_i - \mathbf{r}_{\text{cm}} \) is particle \( i \)'s position relative to the center of mass. We then have

\[
\mathbf{L} = \mathbf{r}_{\text{cm}} \times \sum_i \mathbf{p}_i + \sum_i \left( \mathbf{r}_i - \mathbf{r}_{\text{cm}} \right) \times \mathbf{p}_i
\]

\[
= \mathbf{r}_{\text{cm}} \times \mathbf{p}_{\text{total}} + \sum_i \left( \mathbf{r}_i - \mathbf{r}_{\text{cm}} \right) \times \mathbf{p}_i
\]

\[
= \mathbf{r}_{\text{cm}} \times m_{\text{total}} \mathbf{v}_{\text{cm}} + \sum_i \left( \mathbf{r}_i - \mathbf{r}_{\text{cm}} \right) \times \mathbf{p}_i.
\]

The first and second terms in this expression correspond to the quantities (2) and (1), respectively.

Different Forms of Maxwell’s Equations

First we reproduce Maxwell’s equations as stated on page 722, in integral form, using the SI (meter-kilogram-second) system of units, with the coupling constants written in terms of \( k \) and \( c \):

\[
\Phi_E = 4\pi k q_{in}
\]
\[
\Phi_B = 0
\]
\[
\Gamma_E = -\frac{\partial \Phi_B}{\partial t}
\]
\[
c^2 \Gamma_B = \frac{\partial \Phi_E}{\partial t} + 4\pi k I_{\text{through}}
\]

Homework problem 39 on page 753 deals with rewriting these in terms of \( \epsilon_o = 1/4\pi k \) and \( \mu_o = 4\pi k/c^2 \) rather than \( k \) and \( c \).

For the reader who has been studying the optional sections giving Maxwell’s equations in differential form, here is a summary:
\[
\begin{align*}
\text{div } \mathbf{E} &= 4\pi k \rho \\
\text{div } \mathbf{B} &= 0 \\
\text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\frac{c^2}{\text{curl } \mathbf{B}} &= \frac{\partial \mathbf{E}}{\partial t} + 4\pi k \mathbf{j}
\end{align*}
\]

Although all engineering and most scientific work these days is done in the SI (mks) system, one may still encounter the older cgs (centimeter-gram-second) system, especially in astronomy and particle physics. The mechanical units in this system include the dyne \( (g \cdot cm/s^2) \) for force, and the erg \( (g \cdot cm^2/s^2) \) for energy. The system is extended to electrical units by taking \( k = 1 \) as a matter of definition, so the Coulomb force law is \( F = q_1 q_2 / r^2 \). This equation indirectly defines a unit of charge called the elestrostatic unit, with \( 1 \text{ C} = 2.998 \times 10^9 \text{ esu} \), the factor of 2.998 arising from the speed of light. The unit of voltage is the statvolt, \( 1 \text{ statvolt} = 299.8 \text{ V} \). In this system, the electric and magnetic fields have the same units, dynes/esu, but to avoid confusion, magnetic fields are normally written using the equivalent unit of gauss, \( 1 \text{ gauss} = 1 \text{ dyne/esu} = 10^{-4} \text{ T} \). The force on a charged particle is \( \mathbf{F} = q \mathbf{E} + q \frac{\mathbf{v}}{c} \times \mathbf{B} \), which differs from the mks version by the \( 1/c \) factor in the magnetic term. Maxwell’s equations are:

\[
\begin{align*}
\Phi_E &= 4\pi q_{in} \\
\Phi_B &= 0 \\
\Gamma_E &= -\frac{1}{c} \frac{\partial \Phi_B}{\partial t} \\
\Gamma_B &= \frac{1}{c} \frac{\partial \Phi_E}{\partial t} + \frac{4\pi}{c} I_{\text{through}}
\end{align*}
\]
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Hints

Hints for chapter 2
Page 122, problem 16:
You can use either the chain-rule technique from page 83 or the technique prescribed in problem 15 on p. 122. The positions and velocities of the two masses are related to each other, and you’ll need to use this relationship to eliminate one mass’s position and velocity and get everything in terms of the other mass’s position and velocity. The relationship between the two positions will involve some extraneous variables like the length of the string, which won’t have any effect on your final result.

Page 122, problem 17:
This is similar to problem 16, but you’re trying to find the combination of masses that will result in zero acceleration. In this problem, the distance dropped by one weight is different from, but still related to, the distance by which the other weight rises. Try relating the heights of the two weights to each other, so you can get the total gravitational energy in terms of only one of these heights.

Page 122, problem 18:
This is similar to problem 17, in that you’re looking for a setup that will give zero acceleration, and the distance the middle weight rises or falls is not the same as the distance the other two weights fall or rise. The simplest approach is to get the three heights in terms of $\theta$, so that you can write the gravitational energy in terms of $\theta$.

Page 122, problem 19:
This is very similar to problems 16 and 17.

Page 122, problem 20:
The first two parts can be done more easily by setting $a = 1$, since the value of $a$ only changes the distance scale. One way to do part b is by graphing.

Page 123, problem 22:
The condition for a circular orbit contains three unknowns, $v$, $g$, and $r$, so you can’t just solve it for $r$. You’ll need more equations to make three equations in three unknowns. You’ll need a relationship between $g$ and $r$, and also a relationship between $v$ and $r$ that uses the given fact that it’s supposed to take 24 hours for an orbit.

Page 123, problem 25:
What does the total energy have to be if the projectile’s velocity is exactly escape velocity? Write down conservation of energy, change $v$ to $dr/dt$, separate the variables, and integrate.

Page 123, problem 26:
The analytic approach is a little cumbersome, although it can be done by using approximations like $1/\sqrt{1+\epsilon} \approx 1 - (1/2)\epsilon$. A more straightforward, brute-force method is simply to write a computer program that calculates $U/m$ for a given point in spherical coordinates. By trial and error, you can fairly rapidly find the $r$ that gives a desired value of $U/m$.

Page 125, problem 33:
Use calculus to find the minimum of $U$.

Page 125, problem 35:
The spring constant of this spring, $k$, is not the quantity you need in the equation for the period. What you need in that equation is the second derivative of the spring’s energy with respect to
the position of the thing that’s oscillating. You need to start by finding the energy stored in
the spring as a function of the vertical position, \( y \), of the mass. This is similar to example 23
on page 118.

**Page 126, problem 37:**
The variables \( x_1 \) and \( x_2 \) will adjust themselves to reach an equilibrium. Write down the total
energy in terms of \( x_1 \) and \( x_2 \), then eliminate one variable, and find the equilibrium value of the
other. Finally, eliminate both \( x_1 \) and \( x_2 \) from the total energy, getting it just in terms of \( b \).

**Hints for chapter 3**
**Page 225, problem 20:**
Write down two equations, one for Newton’s second law applied to each object. Solve these for
the two unknowns \( T \) and \( a \).

**Page 229, problem 41:**
The whole expression for the amplitude has maxima where the stuff inside the square root is at
a minimum, and vice versa, so you can save yourself a lot of work by just working on the stuff
inside the square root. For normal, large values of \( Q \), the there are two extrema, one at \( \omega = 0 \)
and one at resonance; one of these is a maximum and one is a minimum. You want to find out
at what value of \( Q \) the zero-frequency extremum switches over from being a maximum to being
a minimum.

**Page 234, problem 69:**
You can use the geometric interpretation of the dot product.

**Page 235, problem 70:**
The easiest way to do this problem is to use two different coordinate systems: one that’s tilted
to coincide with the upper slope, and one that’s tilted to coincide with the lower one.

**Hints for chapter 4**
**Page 294, problem 8:**
The choice of axis theorem only applies to a closed system, or to a system acted on by a total
force of zero. Even if the box is not going to rotate, its center of mass is going to accelerate,
and this can still cause a change in its angular momentum, unless the right axis is chosen. For
example, if the axis is chosen at the bottom right corner, then the box will start accumulating
clockwise angular momentum, even if it is just accelerating to the right without rotating. Only
by choosing the axis at the center of mass (or at some other point on the same horizontal line)
do we get a constant, zero angular momentum.

**Page 295, problem 11:**
There are four forces on the wheel at first, but only three when it lifts off. Normal forces are
always perpendicular to the surface of contact. Note that the corner of the step cannot be
perfectly sharp, so the surface of contact for this force really coincides with the surface of the
wheel.

**Page 301, problem 35:**
You’ll need the result of problem 19 in order to relate the energy and angular momentum of a
rigidly rotating body. Since this relationship involves a variable raised to a power, you can’t
just graph the data and get the moment of inertia directly. One way to get around this is to
manipulate one of the variables to make the graph linear. Here is an example of this technique
from another context. Suppose you were given a table of the masses, \( m \), of cubical pieces of
wood, whose sides had various lengths, \( b \). You want to find a best-fit value for the density of
the wood. The relationship is \( m = \rho b^3 \). The graph of \( m \) versus \( b \) would be a curve, and you would not have any easy way to get the density from such a graph. But by graphing \( m \) versus \( b^3 \), you can produce a graph that is linear, and whose slope equals the density.

**Hints for chapter 6**

**Page 392, problem 4:**
How could you change the values of \( x \) and \( t \) so that the value of \( y \) would remain the same? What would this represent physically?

**Page 393, problem 8:**
(a) The most straightforward approach is to apply the equation \( \partial^2 y / \partial t^2 = (T/\mu) \partial^2 y / \partial x^2 \). Although this equation was developed in the main text in the context of a straight string with a curvy wave on it, it works just as well for a circular loop; the left-hand side is simply the inward acceleration of any point on the rope. Note, however, that we’ve been assuming the string was (at least approximately) parallel to the \( x \) axis, which will only be true if you choose a specific value of \( x \). You need to get an equation for \( y \) in terms of \( x \) in order to evaluate the right-hand side.

**Page 394, problem 12:**
The answers to the two parts are not the same.

**Hints for chapter 7**

**Page 463, problem 28:**
Apply the equivalence principle.

**Hints for chapter 8**

**Page 526, problem 15:**
The force on the lithium ion is the vector sum of all the forces of all the quadrillions of sodium and chlorine atoms, which would obviously be too laborious to calculate. Nearly all of these forces, however, are canceled by a force from an ion on the opposite side of the lithium.

**Hints for chapter 9**

**Page 567, problem 20:**
The approach is similar to the one used for the other problem, but you want to work with voltage and electrical energy rather than force.

**Hints for chapter 10**

**Page 658, problem 15:**
Use the approximation \( (1 + \epsilon)^p \approx 1 + p \epsilon \), which is valid for small \( \epsilon \).

**Page 661, problem 25:**
First find the energy stored in a spherical shell extending from \( r \) to \( r + dr \), then integrate to find the total energy.

**Page 661, problem 26:**
Since we have \( t \ll r \), the volume of the membrane is essentially the same as if it was unrolled and flattened out, and the field’s magnitude is nearly constant.

**Page 662, problem 31:**
The math is messy if you put the origin of your polar coordinates at the center of the disk. It comes out much simpler if you put the origin at the edge, right on top of the point at which we’re trying to compute the voltage.
**Page 663, problem 37:**
There are various ways of doing this, but one easy and natural approach is to change the base of the exponent to $e$ using the same method that we would use for real numbers.

**Hints for chapter 11**

**Page 750, problem 24:**
A stable system has low energy; energy would have to be added to change its configuration.

**Page 754, problem 41:**
We’re ignoring the fact that the light consists of little wavepackets, and imagining it as a simple sine wave. But wait, there’s more good news! The energy density depends on the squares of the fields, which means the squares of some sine waves. Well, when you square a sine wave that varies from $-1$ to $+1$, you get a sine wave that goes from 0 to $+1$, and the average value of that sine wave is $1/2$. That means you don’t have to do an integral like $U = \int \left( \frac{dU}{dV} \right) dV$. All you have to do is throw in the appropriate factor of $1/2$, and you can pretend that the fields have their constant values $\tilde{E}$ and $\tilde{B}$ everywhere.

**Page 754, problem 42:**
Use Faraday’s law, and choose an Ampèrian surface that is a disk of radius $R$ sandwiched between the plates.

**Page 756, problem 51:**
(a) Magnetic fields are created by currents, so once you’ve decided how currents behave under time-reversal, you can figure out how magnetic fields behave.

**Hints for chapter 12**

**Page 842, problem 60:**
Expand $\sin \theta$ in a Taylor series around $\theta = 90^\circ$.

**Solutions to selected problems**

**Solutions for chapter 0**

**Page 47, problem 6:**

\[
134 \text{ mg} \times \frac{10^{-3} \text{ g}}{1 \text{ mg}} \times \frac{10^{-3} \text{ kg}}{1 \text{ g}} = 1.34 \times 10^{-4} \text{ kg}
\]

**Page 47, problem 8:**
(a) Let’s do 10.0 g and 1000 g. The arithmetic mean is 505 grams. It comes out to be 0.505 kg, which is consistent. (b) The geometric mean comes out to be 100 g or 0.1 kg, which is consistent. (c) If we multiply meters by meters, we get square meters. Multiplying grams by grams should give square grams! This sounds strange, but it makes sense. Taking the square root of square grams ($g^2$) gives grams again. (d) No. The superduper mean of two quantities with units of grams wouldn’t even be something with units of grams! Related to this shortcoming is the fact that the superduper mean would fail the kind of consistency test carried out in the first two parts of the problem.

**Page 48, problem 12:**
(a) They’re all defined in terms of the ratio of side of a triangle to another. For instance, the tangent is the length of the opposite side over the length of the adjacent side. Dividing meters
by meters gives a unitless result, so the tangent, as well as the other trig functions, is unitless.  
(b) The tangent function gives a unitless result, so the units on the right-hand side had better cancel out. They do, because the top of the fraction has units of meters squared, and so does the bottom.

Page 49, problem 17:
\[ \Delta x = \frac{1}{2}at^2, \] 
so for a fixed value of \( \Delta x \), we have \( t \propto 1/\sqrt{a} \). Translating this into the language of ratios gives \( t_M/t_E = \sqrt{a_E/a_M} = \sqrt{3} = 1.7 \).

Page 50, problem 19:
(a) Solving for \( \Delta x = \frac{1}{2}at^2 \) for \( a \), we find \( a = 2\Delta x/t^2 = 5.51 \text{ m/s}^2 \).  
(b) \( v = \sqrt{2a\Delta x} = 66.6 \text{ m/s} \).  
(c) The actual car’s final velocity is less than that of the idealized constant-acceleration car. If the real car and the idealized car covered the quarter mile in the same time but the real car was moving more slowly at the end than the idealized one, the real car must have been going faster than the idealized car at the beginning of the race. The real car apparently has a greater acceleration at the beginning, and less acceleration at the end. This make sense, because every car has some maximum speed, which is the speed beyond which it cannot accelerate.

Page 51, problem 31:
\[ 1 \text{ mm}^2 \times \left( \frac{1 \text{ cm}}{10 \text{ mm}} \right)^2 = 10^{-2} \text{ cm}^2 \]

Page 51, problem 32:
The bigger scope has a diameter that’s ten times greater. Area scales as the square of the linear dimensions, so \( A \propto d^2 \), or in the language of ratios \( A_1/A_2 = (d_1/d_2)^2 = 100 \). Its light-gathering power is a hundred times greater.

Page 51, problem 33:
Since they differ by two steps on the Richter scale, the energy of the bigger quake is \( 10^4 \) times greater. The wave forms a hemisphere, and the surface area of the hemisphere over which the energy is spread is proportional to the square of its radius, \( A \propto r^2 \), or \( r \propto \sqrt{A} \), which means \( r_1/r_2 = \sqrt{A_1/A_2} \). If the amount of vibration was the same, then the surface areas must be in the ratio \( A_1/A_2 = 10^4 \), which means that the ratio of the radii is 10^2.

Page 52, problem 38:
The cone of mixed gin and vermouth is the same shape as the cone of vermouth, but its linear dimensions are doubled. Translating the proportionality \( V \propto L^3 \) into an equation about ratios, we have \( V_1/V_2 = (L_1/L_2)^3 = 8 \). Since the ratio of the whole thing to the vermouth is 8, the ratio of gin to vermouth is 7.

Page 52, problem 40:
The proportionality \( V \propto L^3 \) can be restated in terms of ratios as \( V_1/V_2 = (L_1/L_2)^3 = (1/10)^3 = 1/1000 \), so scaling down the linear dimensions by a factor of 1/10 reduces the volume by 1/1000, to a milliliter.

Page 53, problem 41:
Let’s estimate the Great Wall’s volume, and then figure out how many bricks that would represent. The wall is famous because it covers pretty much all of China’s northern border, so let’s say it’s 1000 km long. From pictures, it looks like it’s about 10 m high and 10 m wide, so the total volume would be \( 10^6 \text{ m} \times 10 \text{ m} \times 10 \text{ m} = 10^8 \text{ m}^3 \). If a single brick has a volume of 1 liter, or \( 10^{-3} \text{ m}^3 \), then this represents about \( 10^{11} \) bricks. If one person can lay 10 bricks in an hour
(taking into account all the preparation, etc.), then this would be $10^{10}$ man-hours.

**Page 53, problem 44:**
Directly guessing the number of jelly beans would be like guessing volume directly. That would be a mistake. Instead, we start by estimating the linear dimensions, in units of beans. The contents of the jar look like they’re about 10 beans deep. Although the jar is a cylinder, its exact geometrical shape doesn’t really matter for the purposes of our order-of-magnitude estimate. Let’s pretend it’s a rectangular jar. The horizontal dimensions are also something like 10 beans, so it looks like the jar has about $10 \times 10 \times 10$ or $\sim 10^3$ beans inside.

**Solutions for chapter 1**

**Page 71, problem 12:**
To the person riding the moving bike, bug A is simply going in circles. The only difference between the motions of the two wheels is that one is traveling through space, but motion is relative, so this doesn’t have any effect on the bugs. It’s equally hard for each of them.

**Solutions for chapter 2**

**Page 120, problem 1:**
(a) The energy stored in the gasoline is being changed into heat via frictional heating, and also probably into sound and into energy of water waves. Note that the kinetic energy of the propeller and the boat are not changing, so they are not involved in the energy transformation. (b) The cruising speed would be greater by a factor of the cube root of 2, or about a 26% increase.

**Page 120, problem 2:**
We don’t have actual masses and velocities to plug in to the equation, but that’s OK. We just have to reason in terms of ratios and proportionalities. Kinetic energy is proportional to mass and to the square of velocity, so B’s kinetic energy equals $(13.4 \text{ J})(3.77)/(2.34)^2 = 9.23 \text{ J}$.

**Page 120, problem 3:**
Room temperature is about 20$^\circ$C. The fraction of the energy that actually goes into heating the water is

$$\frac{(250 \text{ g})/(0.24 \text{ g}$·$^\circ$C/J) \times (100^\circ \text{C} - 20^\circ \text{C})}{(1.25 \times 10^3 \text{ J/s})(126 \text{ s})} = 0.53$$

So roughly half of the energy is wasted. The wasted energy might be in several forms: heating of the cup, heating of the oven itself, or leakage of microwaves from the oven.

**Page 120, problem 5:**

$$E_{\text{total, } i} = E_{\text{total, } f}$$

$$PE_i + \text{heat}_i = PE_f + KE_f + \text{heat}_f$$

$$\frac{1}{2}mv^2 = PE_i - PE_f + \text{heat}_i - \text{heat}_f$$

$$= -\Delta PE - \Delta \text{heat}$$

$$v = \sqrt{2 \left( \frac{-\Delta PE - \Delta \text{heat}}{m} \right)}$$

$$= 6.4 \text{ m/s}$$
Solutions for chapter 3

Page 222, problem 4:
A conservation law is about addition: it says that when you add up a certain thing, the total always stays the same. Funkosity would violate the additive nature of conservation laws, because a two-kilogram mass would have twice as much funkosity as a pair of one-kilogram masses moving at the same speed.

Page 223, problem 12:
Momentum is a vector. The total momentum of the molecules is always zero, since the momenta in different directions cancel out on the average. Cooling changes individual molecular momenta, but not the total.

Page 224, problem 15:
\[ a = \frac{\Delta v}{\Delta t}, \] and also \[ a = \frac{F}{m}, \] so
\[ \Delta t = \frac{\Delta v}{a} = \frac{m\Delta v}{F} = \frac{(1000 \text{ kg})(50 \text{ m/s} - 20 \text{ m/s})}{3000 \text{ N}} = 10 \text{ s} \]

Page 225, problem 23:
(a) This is a measure of the box’s resistance to a change in its state of motion, so it measures the box’s mass. The experiment would come out the same in lunar gravity.
(b) This is a measure of how much gravitational force it feels, so it’s a measure of weight. In lunar gravity, the box would make a softer sound when it hit.
(c) As in part a, this is a measure of its resistance to a change in its state of motion: its mass. Gravity isn’t involved at all.

Page 228, problem 34:
(a) The swimmer’s acceleration is caused by the water’s force on the swimmer, and the swimmer makes a backward force on the water, which accelerates the water backward. (b) The club’s normal force on the ball accelerates the ball, and the ball makes a backward normal force on the club, which decelerates the club. (c) The bowstring’s normal force accelerates the arrow, and the arrow also makes a backward normal force on the string. This force on the string causes the string to accelerate less rapidly than it would if the bow’s force was the only one acting on it.
(d) The tracks’ backward frictional force slows the locomotive down. The locomotive’s forward frictional force causes the whole planet earth to accelerate by a tiny amount, which is too small to measure because the earth’s mass is so great.

Page 228, problem 37:
(a) Spring constants in parallel add, so the spring constant has to be proportional to the cross-sectional area. Two springs in series give half the spring constant, three springs in series give 1/3, and so on, so the spring constant has to be inversely proportional to the length. Summarizing, we have \( k \propto \frac{A}{L} \).
(b) With the Young’s modulus, we have \( k = (A/L)E \). The spring constant has units of N/m, so the units of E would have to be N/m².

Page 230, problem 44:
By conservation of momentum, the total momenta of the pieces after the explosion is the same
as the momentum of the firework before the explosion. However, there is no law of conservation of kinetic energy, only a law of conservation of energy. The chemical energy in the gunpowder is converted into heat and kinetic energy when it explodes. All we can say about the kinetic energy of the pieces is that their total is greater than the kinetic energy before the explosion.

Page 230, problem 45:
Let $m$ be the mass of the little puck and $M = 2.3m$ be the mass of the big one. All we need to do is find the direction of the total momentum vector before the collision, because the total momentum vector is the same after the collision. Given the two components of the momentum vector $p_x = Mv$ and $p_y = mv$, the direction of the vector is $\tan^{-1}(p_y/p_x) = 23^\circ$ counterclockwise from the big puck’s original direction of motion.

Page 233, problem 62:
We want to find out about the velocity vector $v_{BG}$ of the bullet relative to the ground, so we need to add Annie’s velocity relative to the ground $v_{AG}$ to the bullet’s velocity vector $v_{BA}$ relative to her. Letting the positive $x$ axis be east and $y$ north, we have

$$v_{BA,x} = (140 \text{ mi/hr}) \cos 45^\circ$$
$$= 100 \text{ mi/hr}$$
$$v_{BA,y} = (140 \text{ mi/hr}) \sin 45^\circ$$
$$= 100 \text{ mi/hr}$$

and

$$v_{AG,x} = 0$$
$$v_{AG,y} = 30 \text{ mi/hr}.$$ 

The bullet's velocity relative to the ground therefore has components

$$v_{BG,x} = 100 \text{ mi/hr}$$

and

$$v_{BG,y} = 130 \text{ mi/hr}.$$ 

Its speed on impact with the animal is the magnitude of this vector

$$|v_{BG}| = \sqrt{(100 \text{ mi/hr})^2 + (130 \text{ mi/hr})^2}$$
$$= 160 \text{ mi/hr}$$

(rounded off to two significant figures).

Page 233, problem 63:
Since its velocity vector is constant, it has zero acceleration, and the sum of the force vectors acting on it must be zero. There are three forces acting on the plane: thrust, lift, and gravity. We are given the first two, and if we can find the third we can infer the plane’s mass. The sum of the $y$ components of the forces is zero, so

$$0 = F_{\text{thrust},y} + F_{\text{lift},y} + F_{g,y}$$
$$= |F_{\text{thrust}}| \sin \theta + |F_{\text{lift}}| \cos \theta - mg.$$
The mass is

\[ m = \frac{(|F_{thrust}| \sin \theta + |F_{lift}| \cos \theta)}{g} = 7.0 \times 10^4 \text{ kg.} \]

Page 234, problem 64:
(a) Since the wagon has no acceleration, the total forces in both the \(x\) and \(y\) directions must be zero. There are three forces acting on the wagon: \(T\), \(F_g\), and the normal force from the ground, \(F_n\). If we pick a coordinate system with \(x\) being horizontal and \(y\) vertical, then the angles of these forces measured counterclockwise from the \(x\) axis are \(90^\circ - \phi\), \(270^\circ\), and \(90^\circ + \theta\), respectively. We have

\[ F_{x,\text{total}} = T \cos(90^\circ - \phi) + F_g \cos(270^\circ) + F_n \cos(90^\circ + \theta) \]
\[ F_{y,\text{total}} = T \sin(90^\circ - \phi) + F_g \sin(270^\circ) + F_n \sin(90^\circ + \theta), \]

which simplifies to

\[ 0 = T \sin \phi - F_n \sin \theta \]
\[ 0 = T \cos \phi - F_g + F_n \cos \theta. \]

The normal force is a quantity that we are not given and do not wish to find, so we should choose it to eliminate. Solving the first equation for \(F_n = \frac{(\sin \phi)}{(\sin \theta)} T\), we eliminate \(F_n\) from the second equation,

\[ 0 = T \cos \phi - F_g + T \sin \phi \cos \theta / \sin \theta \]

and solve for \(T\), finding

\[ T = \frac{F_g}{\cos \phi + \sin \phi \cos \theta / \sin \theta} \]

Multiplying both the top and the bottom of the fraction by \(\sin \theta\), and using the trig identity for \(\sin(\theta + \phi)\) gives the desired result,

\[ T = \frac{\sin \theta}{\sin(\theta + \phi)} F_g \]

(b) The case of \(\phi = 0\), i.e. pulling straight up on the wagon, results in \(T = F_g\): we simply support the wagon and it glides up the slope like a chair-lift on a ski slope. In the case of \(\phi = 180^\circ - \theta\), \(T\) becomes infinite. Physically this is because we are pulling directly into the ground, so no amount of force will suffice.

Page 234, problem 65:
(a) If there was no friction, the angle of repose would be zero, so the coefficient of static friction, \(\mu_s\), will definitely matter. We also make up symbols \(\theta\), \(m\) and \(g\) for the angle of the slope, the mass of the object, and the acceleration of gravity. The forces form a triangle just like the one in example 68 on page 207, but instead of a force applied by an external object, we have static friction, which is less than \(\mu_s F_n\). As in that example, \(F_s = mg \sin \theta\), and \(F_s < \mu_s F_n\), so

\[ mg \sin \theta < \mu_s F_n. \]

From the same triangle, we have \(F_n = mg \cos \theta\), so

\[ mg \sin \theta < \mu_s mg \cos \theta. \]
Rearranging,

\[ \theta < \tan^{-1} \mu_s. \]

(b) Both \( m \) and \( g \) canceled out, so the angle of repose would be the same on an asteroid.

**Page 242, problem 88:**

(a) Based on units, we must have \( g = kG\lambda/y \), where \( k \) is a unitless universal constant.

(b) For the actual calculation, we have

\[
g = \int \frac{dg_y}{y} = G \int \frac{dm}{r^2} \cos \theta,
\]

where \( \theta \) is the angle between the perpendicular and the \( r \) vector. Then \( dm = \lambda dx \), \( \cos \theta = \frac{y}{r} \), and \( r = \sqrt{x^2 + y^2} \), so

\[
g = G \int_{-\infty}^{\infty} \frac{\lambda dx}{x^2 + y^2} \cdot \frac{b}{\sqrt{x^2 + y^2}} = G\lambda y \int_{-\infty}^{\infty} (x^2 + y^2)^{-3/2} dx.
\]

Even though this has limits of integration, this is an indefinite integral because it contains the variable \( y \). It’s nicer to clean this up by doing a change of variable to the unitless quantity \( u = x/y \), giving

\[
g = \frac{G\lambda}{y} \int_{-\infty}^{\infty} (u^2 + 1)^{-3/2} du.
\]

The definite integral is the sort of thing that sane people these days will do using computer software. It equals 2. The result for the field is

\[
g = \frac{2G\lambda}{y}.
\]

**Solutions for chapter 4**

**Page 294, problem 1:**

The pliers are not moving, so their angular momentum remains constant at zero, and the total torque on them must be zero. Not only that, but each half of the pliers must have zero total torque on it. This tells us that the magnitude of the torque at one end must be the same as that at the other end. The distance from the axis to the nut is about 2.5 cm, and the distance from the axis to the centers of the palm and fingers are about 8 cm. The angles are close enough to 90° that we can pretend they’re 90 degrees, considering the rough nature of the other assumptions and measurements. The result is \( (300 \text{ N})(2.5 \text{ cm}) = (F)(8 \text{ cm}) \), or \( F = 90 \text{ N} \).

**Page 301, problem 37:**

The foot of the rod is moving in a circle relative to the center of the rod, with speed \( v = \pi b/T \), and acceleration \( v^2/(b/2) = (\pi^2/8)g \). This acceleration is initially upward, and is greater in magnitude than \( g \), so the foot of the rod will lift off without dragging. We could also worry about whether the foot of the rod would make contact with the floor again before the rod finishes up flat on its back. This is a question that can be settled by graphing, or simply by inspection of figure i on page 282. The key here is that the two parts of the acceleration are
both independent of $m$ and $b$, so the result is universal, and it does suffice to check a graph in a single example. In practical terms, this tells us something about how difficult the trick is to do. Because $\pi^2/8 = 1.23$ isn’t much greater than unity, a hit that is just a little too weak (by a factor of $1.23^{1/2} = 1.11$) will cause a fairly obvious qualitative change in the results. This is easily observed if you try it a few times with a pencil.

**Page 303, problem 45:**

The moment of inertia is $I = \int r^2 \, dm$. Let the ring have total mass $M$ and radius $b$. The proportionality

\[
\frac{M}{2\pi} = \frac{dm}{d\theta}
\]

gives a change of variable that results in

\[
I = \frac{M}{2\pi} \int_0^{2\pi} r^2 \, d\theta.
\]

If we measure $\theta$ from the axis of rotation, then $r = b \sin \theta$, so this becomes

\[
I = \frac{Mb^2}{2\pi} \int_0^{2\pi} \sin^2 \theta \, d\theta.
\]

The integrand averages to 1/2 over the $2\pi$ range of integration, so the integral equals $\pi$. We therefore have $I = \frac{1}{2} Mb^2$. This is, as claimed, half the value for rotation about the symmetry axis.

**Solutions for chapter 5**

**Page 349, problem 11:**

(a) We have

\[
dP = \rho g \, dy
\]

\[
\Delta P = \int \rho g \, dy,
\]

and since we’re taking water to be incompressible, and $g$ doesn’t change very much over 11 km of height, we can treat $\rho$ and $g$ as constants and take them outside the integral.

\[
\Delta P = \rho g \Delta y
\]

\[
= (1.0 \text{ g/cm}^3)(9.8 \text{ m/s}^2)(11.0 \text{ km})
\]

\[
= (1.0 \times 10^3 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(1.10 \times 10^4 \text{ m})
\]

\[
= 1.0 \times 10^8 \text{ Pa}
\]

\[
= 1.0 \times 10^3 \text{ atm}.
\]

The precision of the result is limited to a few percent, due to the compressibility of the water, so we have at most two significant figures. If the change in pressure were exactly a thousand atmospheres, then the pressure at the bottom would be 1001 atmospheres; however, this distinction is not relevant at the level of approximation we’re attempting here.

(b) Since the air in the bubble is in thermal contact with the water, it’s reasonable to assume that it keeps the same temperature the whole time. The ideal gas law is $PV = nkT$, and rewriting this as a proportionality gives

\[
V \propto P^{-1},
\]
or

\[ \frac{V_f}{V_i} = \left( \frac{P_f}{P_i} \right)^{-1} \approx 10^3. \]

Since the volume is proportional to the cube of the linear dimensions, the growth in radius is about a factor of 10.

**Page 349, problem 12:**
\( (a) \) Roughly speaking, the thermal energy is \( \sim k_B T \) (where \( k_B \) is the Boltzmann constant), and we need this to be on the same order of magnitude as \( ke^2/r \) (where \( k \) is the Coulomb constant). For this type of rough estimate it’s not especially crucial to get all the factors of two right, but let’s do so anyway. Each proton’s average kinetic energy due to motion along a particular axis is \( (1/2)k_B T \). If two protons are colliding along a certain line in the center-of-mass frame, then their average combined kinetic energy due to motion along that axis is \( 2(1/2)k_B T = k_B T \). So in fact the factors of 2 cancel. We have \( T = ke^2/k_B r \).
\( (b) \) The units are \( K = (J \cdot m/C^2)/(C^2)/(J/K)\cdot m \), which does work out.
\( (c) \) The numerical result is \( \sim 10^{10} \) K, which as suggested is much higher than the temperature at the core of the sun.

**Page 350, problem 13:**
If the full-sized brick A undergoes some process, such as heating it with a blowtorch, then we want to be able to apply the equation \( \Delta S = Q/T \) to either the whole brick or half of it, which would be identical to B. When we redefine the boundary of the system to contain only half of the brick, the quantities \( \Delta S \) and \( Q \) are each half as big, because entropy and energy are additive quantities. \( T \), meanwhile, stays the same, because temperature isn’t additive — two cups of coffee aren’t twice as hot as one. These changes to the variables leave the equation consistent, since each side has been divided by 2.

**Page 350, problem 14:**
\( (a) \) If the expression \( 1 + by \) is to make sense, then \( by \) has to be unitless, so \( b \) has units of \( m^{-1} \). The input to the exponential function also has to be unitless, so \( k \) also has of \( m^{-1} \). The only factor with units on the right-hand side is \( P_o \), so \( P_o \) must have units of pressure, or Pa.
\( (b) \)

\[ dP = \rho g \, dy \]

\[ \rho = \frac{1}{g} \, \frac{dP}{dy} \]

\[ \rho = \frac{P_o}{g} e^{-ky}(-k - kby + b) \]

\( (c) \) The three terms inside the parentheses on the right all have units of \( m^{-1} \), so it makes sense to add them, and the factor in parentheses has those units. The units of the result from b then look like

\[ \frac{kg}{m^3} = \frac{Pa}{m/s^2} m^{-1} = \frac{N/m^2}{m^2/s^2} = \frac{kg \cdot m^{-1} \cdot s^{-2}}{m^2/s^2}, \]

which checks out.
Solutions for chapter 7
Page 461, problem 17:
(a) Plugging in, we find
\[ \sqrt{\frac{1-w}{1+w}} = \sqrt{\frac{1-u}{1+u}} \sqrt{\frac{1-v}{1+v}}. \]

(b) First let’s simplify by squaring both sides.
\[ \frac{1-w}{1+w} = \frac{1-u}{1+u} \cdot \frac{1-v}{1+v}. \]

For convenience, let’s write \( A \) for the right-hand side of this equation. We then have
\[ \frac{1-w}{1+w} = A \]
\[ 1-w = A + Aw. \]

Solving for \( w \),
\[ w = \frac{1-A}{1+A} \]
\[ = \frac{(1+u)(1+v) - (1-u)(1-v)}{(1+u)(1+v) + (1-u)(1-v)} \]
\[ = \frac{2(u+v)}{2(1+uv)} \]
\[ = \frac{u+v}{1+uv} \]

(c) This is all in units where \( c = 1 \). The correspondence principle says that we should get \( w \approx u + v \) when both \( u \) and \( v \) are small compared to 1. Under those circumstances, \( uv \) is the product of two very small numbers, which makes it very, very small. Neglecting this term in the denominator, we recover the nonrelativistic result.

Page 461, problem 18:
Among the spacelike vectors, \( a \) and \( e \) are clearly congruent, because they’re the same except for a rotation in space; this is the same as the definition of congruence in ordinary Euclidean geometry, where rotation doesn’t matter. Vector \( b \) is also congruent to these, since it represents an interval \( 3^2 - 5^2 = -4^2 \), just like the other two.

The lightlike vectors \( c \) and \( d \) both represent intervals of zero, so they’re congruent, even though \( c \) is a double-scale version of \( d \).

The timelike vectors \( f \) and \( g \) are not congruent to each other or to any of the others; \( f \) represents an interval of \( 2^2 \), while \( g \)’s interval is \( 4^2 \).

Page 462, problem 22:
At the center of each of the large triangle’s sides, the angles add up to 180° because they form a straight line. Therefore \( 4s = S + 3 \times 180° \) because they form a straight line. Therefore \( 4s = S + 3 \times 180° \), so \( S - 180° = 4(s - 180°) \).

Page 463, problem 28:
By the equivalence principle, we can adopt a frame tied to the tossed clock, B, and in this frame there is no gravitational field. We see a desk and clock A go by. The desk applies a force to clock A, decelerating it and then reaccelerating it so that it comes back. We’ve already established that the effect of motion is to slow down time, so clock A reads a smaller time interval.
Page 464, problem 32:
To make the units make sense, we need to make sure that both sides of the $\approx$ sign have the same units, and also that both terms on the right-hand side have the same units. Everything is unitless except for the second term on the right, so we add a factor of $c^{-2}$ to fix it:

$$\gamma \approx 1 + \frac{v^2}{2c^2}.$$  

Solutions for chapter 9

Page 564, problem 1:
$\Delta t = \Delta q/I = e/I = 0.16 \mu s$

Page 565, problem 12:
In series, they give 11 kΩ. In parallel, they give $(1/1 \, \text{kΩ} + 1/10 \, \text{kΩ})^{-1} = 0.9 \, \text{kΩ}$.

Page 568, problem 25:
The actual shape is irrelevant; all we care about is what’s connected to what. Therefore, we can draw the circuit flattened into a plane. Every vertex of the tetrahedron is adjacent to every other vertex, so any two vertices to which we connect will give the same resistance. Picking two arbitrarily, we have this:

This is unfortunately a circuit that cannot be converted into parallel and series parts, and that’s what makes this a hard problem! However, we can recognize that by symmetry, there is zero current in the resistor marked with an asterisk. Eliminating this one, we recognize the whole arrangement as a triple parallel circuit consisting of resistances $R$, $2R$, and $2R$. The resulting resistance is $R/2$.

Page 569, problem 29:
(a) Conservation of energy gives

$$U_A = U_B + K_B$$
$$K_B = U_A - U_B$$
$$\frac{1}{2}mv^2 = e\Delta V$$
$$v = \sqrt{\frac{2e\Delta V}{m}}$$

(b) Plugging in numbers, we get $5.9 \times 10^7$ m/s. This is about 20% of the speed of light, so the nonrelativistic assumption was good to at least a rough approximation.

Page 570, problem 32:
It’s much more practical to measure voltage differences. To measure a current, you have to break the circuit somewhere and insert the meter there, but it’s not possible to disconnect the circuits sealed inside the board.

Solutions for chapter 10
Page 658, problem 16:
By symmetry, the field is always directly toward or away from the center. We can therefore calculate it along the x axis, where \( r = x \), and the result will be valid for any location at that distance from the center.

\[
E = -\frac{dV}{dx} = -\frac{d}{dx} \left( x^{-1}e^{-x} \right) = x^{-2}e^{-x} + x^{-1}e^{-x}
\]

At small \( x \), near the proton, the first term dominates, and the exponential is essentially 1, so we have \( E \propto x^{-2} \), as we expect from the Coulomb force law. At large \( x \), the second term dominates, and the field approaches zero faster than an exponential.

Page 666, problem 56:
\[
\sin(a + b) = \left( e^{i(a+b)} - e^{-i(a+b)} \right) / 2i
\]
\[
= \left( e^{ia}e^{ib} - e^{-ia}e^{-ib} \right) / 2i
\]
\[
= \left[ (\cos a + i \sin a)(\cos b + i \sin b) - (\cos a - i \sin a)(\cos b - i \sin b) \right] / 2i
\]
\[
= \cos a \sin b + \sin a \cos b
\]

By a similar computation, we find \( \cos(a + b) = \cos a \cos b - \sin a \sin b \).

Page 666, problem 57:
If \( z^3 = 1 \), then we know that \( |z| = 1 \), since cubing \( z \) cubes its magnitude. Cubing \( z \) triples its argument, so the argument of \( z \) must be a number that, when tripled, is equivalent to an angle of zero. There are three possibilities: \( 0 \times 3 = 0 \), \( (2\pi/3) \times 3 = 2\pi \), and \( (4\pi/3) \times 3 = 4\pi \). (Other possibilities, such as \( (32\pi/3) \), are equivalent to one of these.) The solutions are:
\[
z = 1, \ e^{2\pi i/3}, \ e^{4\pi i/3}
\]

Page 666, problem 59:
We have \( D = q\ell \) and \( F_x = q\beta\ell = D\partial E_x / \partial x \), which, as claimed, is consistent with the result of example 7 on p. 589 and depends on \( q \) and \( \ell \) only via \( D \).

**Solutions for chapter 11**

Page 756, problem 51:
(a) For a material object, \( p = mv \). The velocity vector reverses itself, but mass is still positive, so the momentum vector is reversed.

(b) In the forward-time universe, conservation of momentum is \( p_{1,i} + p_{2,i} = p_{1,f} + p_{2,f} \). In the backward-time universe, all the momenta are reversed, but that just negates both sides of the equation, so momentum is still conserved.

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backward-time universe, all the momenta are reversed, but that just negates both sides of the equation, so momentum is still conserved.

**Page 758, problem 54:**
Note that in the Biot-Savart law, the variable $r$ is defined as a vector that points from the current to the point at which the field is being calculated, whereas in the polar coordinates used to express the equation of the spiral, the vector more naturally points the opposite way. This requires some fiddling with signs, which I’ll suppress, and simply identify $d\ell$ with $dr$.

$$B = \frac{kI}{c^2} \int \frac{d\ell \times r}{r^3}$$

The vector $dr$ has components $dx = w\cos(\theta - \theta \sin \theta)$ and $dy = w(\sin \theta + \theta \cos \theta)$. Evaluating the vector cross product, and substituting $\theta/w$ for $r$, we find

$$B = \frac{kI}{c^2w} \int \frac{\theta(\cos \theta \sin \theta - \theta \sin^2 \theta - \cos \theta \sin \theta - \theta \cos^2 \theta) d\theta}{\theta^3}$$

$$= \frac{kI}{c^2w} \int \frac{d\theta}{\theta}$$

$$= \frac{kI}{c^2w} \ln \frac{\theta_2}{\theta_1}$$

$$= \frac{kI}{c^2w} \ln \frac{b}{a}$$

**Solutions for chapter 12**

**Page 827, problem 4:**
Because the surfaces are flat, you get specular reflection. In specular reflection, all the reflected rays go in one direction. Unless the plane is directly overhead, that direction won’t be the right direction to make the rays come back to the radar station.

![Diagram of specular reflection](image)

This is different from a normal plane, which has complicated, bumpy surfaces. These surfaces give diffuse reflection, which spreads the reflected rays randomly in more or less every possible direction.

**Page 827, problem 5:**
It spells “bonk.”

![Diagram of a polygon](image)
Page 828, problem 6:
(a) The rays all cross at pretty much the same place, given the accuracy with which we can draw them.
(b) It could be used to cook food, for instance. All the sunlight is concentrated in a small area.
(c) Put the lightbulb at the point where the rays cross. The outgoing rays will then form a parallel beam going out to the right.

Page 829, problem 11:
The magnification is the ratio of the image’s size to the object’s size. It has nothing to do with the person’s location. The angular magnification, however, does depend on the person’s location, because things farther away subtend smaller angles. The distance to the actual object is not changed significantly, since it’s zillions of miles away in outer space, but the distance to the image does change if the observer’s point of view changes. If you can get closer to the image, the angular magnification is greater.

Page 830, problem 15:
For a flat mirror, $d_i$ and $d_o$ are equal, so the magnification is 1, i.e., the image is the same size as the object.

Page 830, problem 16:
See the ray diagram below. Decreasing $\theta_o$ decreases $\theta_i$, so the equation \( \theta_f = \pm \theta_i \pm \theta_o \) must have opposite signs on the right. Since $\theta_o$ is bigger than $\theta_i$, the only way to get a positive $\theta_f$ is if the signs are $\theta_f = -\theta_i + \theta_o$. This gives $1/f = -1/d_i + 1/d_o$.

Page 830, problem 19:
(a) The object distance is less than the focal length, so the image is virtual: because the object is so close, the cone of rays is diverging too strongly for the mirror to bring it back to a focus.
(b) Now the object distance is greater than the focal length, so the image is real. (c),(d) A diverging mirror can only make virtual images.

**Page 830, problem 20:**
(a) In problem #2 we found that the equation relating the object and image distances was of the form \(1/f = -1/d_i + 1/d_o\). Let’s make \(f = 1.00\) m. To get a virtual image we need \(d_o < f\), so let \(d_o = 0.50\) m. Solving for \(d_i\), we find \(d_i = 1/(1/d_o - 1/f) = 1.00\) m. The magnification is \(M = d_i/d_o = 2.00\). If we change \(d_o\) to 0.55 m, the magnification becomes 2.22. The magnification changes somewhat with distance, so the store’s ad must be assuming you’ll use the mirror at a certain distance. It can’t have a magnification of 5 at all distances.

(b) Theoretically yes, but in practical terms no. If you go through a calculation similar to the one in part a, you’ll find that the images of both planets are formed at almost exactly the same \(d_i\), \(d_i = f\), since \(1/d_o\) is pretty close to zero for any astronomical object. The more distant planet has an image half as big \((M = d_i/d_o\), and \(d_o\) is doubled), but we’re talking about *angular* magnification here, so what we care about is the angular size of the image compared to the angular size of the object. The more distant planet has half the angular size, but its image has half the angular size as well, so the angular magnification is the same. If you think about it, it wouldn’t make much sense for the angular magnification to depend on the planet’s distance — if it did, then determining astronomical distances would be much easier than it actually is!

**Page 830, problem 21:**
(a) This occurs when the \(d_i\) is infinite. Let’s say it’s a converging mirror creating a virtual image, as in problems 2 and 3. Then we’d get an infinite \(d_i\) if we put \(d_o = f\), i.e., the object is at the focal point of the mirror. The image is infinitely large, but it’s also infinitely far away, so its angular size isn’t infinite; an angular size can never be more than about 180° since you can’t see in back of your head!

(b) It’s not possible to make the magnification infinite by having \(d_o = 0\). The image location and object location are related by \(1/f = 1/d_o - 1/d_i\), so \(1/d_i = 1/d_o - 1/f\). If \(d_o\) is zero, then \(1/d_o\) is infinite, \(1/d_i\) is infinite, and \(d_i\) is zero as well. In other words, as \(d_o\) approaches zero, so does \(d_i\), and \(d_i/d_o\) doesn’t blow up. Physically, the mirror’s curvature becomes irrelevant from the point of view of a tiny flea sitting on its surface: the mirror seems flat to the flea. So physically the magnification would be 1, not infinity, for very small values of \(d_o\).

**Page 832, problem 27:**
The refracted ray that was bent closer to the normal in the plastic when the plastic was in air will be bent farther from the normal in the plastic when the plastic is in water. It will become a diverging lens.

**Page 832, problem 29:**
Refraction occurs only at the boundary between two substances, which in this case means the surface of the lens. Light doesn’t get bent at all inside the lens, so the thickness of the lens isn’t really what’s important. What matters is the angles of the lens’ surfaces at various points.

Ray 1 makes an angle of zero with respect to the normal as it enters the lens, so it doesn’t get bent at all, and likewise at the back.

At the edge of the lens, 2, the front and back are not parallel, so a ray that traverses the lens at the edge ends up being bent quite a bit.

Although I drew both ray 1 and ray 2 coming in along the axis of the lens, it really doesn’t matter. For instance, ray 3 bends on the way in, but bends an equal amount on the way out, so it still emerges from the lens moving in the same direction as the direction it originally had.
Summarizing and systematizing these observations, we can say that for a ray that enters the lens at the center, where the surfaces are parallel, the sum of the two deflection angles is zero. Since the total deflection is zero at the center, it must be larger away from the center.

Page 833, problem 31:
Normally, in air, your eyes do most of their focusing at the air-eye boundary. When you swim without goggles, there is almost no difference in speed at the water-eye interface, so light is not strongly refracted there (see figure), and the image is far behind the retina.

Goggles fix this problem for the following reason. The light rays cross a water-air boundary as they enter the goggles, but they’re coming in along the normal, so they don’t get bent. At the air-eye boundary, they get bent the same amount they normally would when you weren’t swimming.
Page 833, problem 33:
(a) See the figure below. The first refraction clearly bends it inward. However, the back surface of the lens is more slanted, so the ray makes a bigger angle with respect to the normal at the back surface. The bending at the back surface is therefore greater than the bending at the front surface, and the ray ends up being bent outward more than inward.

(b) Lens 2 must act the same as lens 1. It’s diverging. One way of knowing this is time-reversal symmetry: if we flip the original figure over and then reverse the direction of the ray, it’s still a valid diagram.

Lens 3 is diverging like lens 1 on top, and diverging like lens 2 on the bottom. It’s a diverging lens.

As for lens 4, any close-up diagram we draw of a particular ray passing through it will look exactly like the corresponding close-up diagram for some part of lens 1. Lens 4 behaves the same as lens 1.

Page 834, problem 39:
Since $d_o$ is much greater than $d_i$, the lens-film distance $d_i$ is essentially the same as $f$. (a) Splitting the triangle inside the camera into two right triangles, straightforward trigonometry
gives
\[ \theta = 2 \tan^{-1} \frac{w}{2f} \]
for the field of view. This comes out to be 39° and 64° for the two lenses. (b) For small angles, the tangent is approximately the same as the angle itself, provided we measure everything in radians. The equation above then simplifies to
\[ \theta = \frac{w}{f} \]
The results for the two lenses are .70 rad = 40°, and 1.25 rad = 72°. This is a decent approximation.

(c) With the 28-mm lens, which is closer to the film, the entire field of view we had with the 50-mm lens is now confined to a small part of the film. Using our small-angle approximation \( \theta = w/f \), the amount of light contained within the same angular width \( \theta \) is now striking a piece of the film whose linear dimensions are smaller by the ratio 28/50. Area depends on the square of the linear dimensions, so all other things being equal, the film would now be overexposed by a factor of \((50/28)^2 = 3.2\). To compensate, we need to shorten the exposure by a factor of 3.2.

Page 838, problem 48:
You don’t want the wave properties of light to cause all kinds of funny-looking diffraction effects. You want to see the thing you’re looking at in the same way you’d see a big object. Diffraction effects are most pronounced when the wavelength of the light is relatively large compared to the size of the object the light is interacting with, so red would be the worst. Blue light is near the short-wavelength end of the visible spectrum, which would be the best.

Page 838, problem 49:
(a) You can tell it’s a single slit because of the double-width central fringe.
(b) Four fringes on the top pattern are about 23.5 mm, while five fringes on the bottom one are about 14.5 mm. The spacings are 5.88 and 2.90 mm, with a ratio of 2.03. A smaller \( d \) leads to larger diffraction angles, so the width of the slit used to make the bottom pattern was almost exactly twice as wide as the one used to make the top one.

Page 839, problem 51:
For the size of the diffraction blob, we have:
\[ \frac{\lambda}{d} \sim \sin \theta \]
\[ \approx \theta \]
\[ \theta \sim \frac{700 \text{ nm}}{10 \text{ m}} \]
\[ \approx 10^{-7} \text{ radians} \]
For the actual angular size of the star, the small-angle approximation gives
\[ \theta \sim \frac{10^9 \text{ m}}{10^{17} \text{ m}} \]
\[ = 10^{-8} \text{ radians} \]
The diffraction blob is ten times bigger than the actual disk of the star, so we can never make an image of the star itself in this way.