18. In the figure, the battery is 9 V.
(a) What are the voltage differences across each light bulb? ✓
(b) What current flows through each of the three components of the circuit? ✓
(c) If a new wire is added to connect points A and B, how will the appearances of the bulbs change? What will be the new voltages and currents?
(d) Suppose no wire is connected from A to B, but the two bulbs are switched. How will the results compare with the results from the original setup as drawn?

19. A student in a biology lab is given the following instructions: “Connect the cerebral eraser (C.E.) and the neural depolarizer (N.D.) in parallel with the power supply (P.S.). (Under no circumstances should you ever allow the cerebral eraser to come within 20 cm of your head.) Connect a voltmeter to measure the voltage across the cerebral eraser, and also insert an ammeter in the circuit so that you can make sure you don’t put more than 100 mA through the neural depolarizer.” The diagrams show two lab groups’ attempts to follow the instructions.
(a) Translate diagram 1 into a standard-style schematic. What is correct and incorrect about this group’s setup?
(b) Do the same for diagram 2.

20. Referring back to problem 15 on page 509 about the sodium chloride crystal, suppose the lithium ion is going to jump from the gap it is occupying to one of the four closest neighboring gaps. Which one will it jump to, and if it starts from rest, how fast will it be going by the time it gets there? (It will keep on moving and accelerating after that, but that does not concern us.) [Hint: The approach is similar to the one used for the other problem, but you want to work with voltage and electrical energy rather than force.]

21. A 1.0 Ω toaster and a 2.0 Ω lamp are connected in parallel with the 110-V supply of your house. (Ignore the fact that the voltage is AC rather than DC.)
(a) Draw a schematic of the circuit.
(b) For each of the three components in the circuit, find the current passing through it and the voltage drop across it.
(c) Suppose they were instead hooked up in series. Draw a schematic and calculate the same things.

22. The heating element of an electric stove is connected in series with a switch that opens and closes many times per second. When you turn the knob up for more power, the fraction of the time that the switch is closed increases. Suppose someone suggests a simpler alternative for controlling the power by putting the heating element in series with a variable resistor controlled by the knob. (With the
Problem 27. knob turned all the way clockwise, the variable resistor’s resistance is nearly zero, and when it’s all the way counterclockwise, its resistance is essentially infinite.) (a) Draw schematics. (b) Why would the simpler design be undesirable?

23 You have a circuit consisting of two unknown resistors in series, and a second circuit consisting of two unknown resistors in parallel.
(a) What, if anything, would you learn about the resistors in the series circuit by finding that the currents through them were equal?
(b) What if you found out the voltage differences across the resistors in the series circuit were equal?
(c) What would you learn about the resistors in the parallel circuit from knowing that the currents were equal?
(d) What if the voltages in the parallel circuit were equal?

24 How many different resistance values can be created by combining three unequal resistors? (Don’t count possibilities in which not all the resistors are used, i.e., ones in which there is zero current in one or more of them.)

25 Suppose six identical resistors, each with resistance $R$, are connected so that they form the edges of a tetrahedron (a pyramid with three sides in addition to the base, i.e., one less side than an Egyptian pyramid). What resistance value or values can be obtained by making connections onto any two points on this arrangement?

Solution, p. 941

26 A person in a rural area who has no electricity runs an extremely long extension cord to a friend’s house down the road so she can run an electric light. The cord is so long that its resistance, $x$, is not negligible. Show that the lamp’s brightness is greatest if its resistance, $y$, is equal to $x$. Explain physically why the lamp is dim for values of $y$ that are too small or too large.

27 All three resistors have the same resistance, $R$. Find the three unknown currents in terms of $V_1$, $V_2$, and $R$. √
28 You are given a battery, a flashlight bulb, and a single piece of wire. Draw at least two configurations of these items that would result in lighting up the bulb, and at least two that would not light it. (Don’t draw schematics.) If you’re not sure what’s going on, borrow the materials from your instructor and try it. Note that the bulb has two electrical contacts: one is the threaded metal jacket, and the other is the tip (at the bottom in the figure). [Problem by Arnold Arons.]

29 The figure shows a simplified diagram of an electron gun such as the one that creates the electron beam in a TV tube. Electrons that spontaneously emerge from the negative electrode (cathode) are then accelerated to the positive electrode, which has a hole in it. (Once they emerge through the hole, they will slow down. However, if the two electrodes are fairly close together, this slowing down is a small effect, because the attractive and repulsive forces experienced by the electron tend to cancel.)
(a) If the voltage difference between the electrodes is $\Delta V$, what is the velocity of an electron as it emerges at B? Assume that its initial velocity, at A, is negligible, and that the velocity is nonrelativistic. (If you haven’t read ch. 7 yet, don’t worry about the remark about relativity.)
(b) Evaluate your expression numerically for the case where $\Delta V = 10$ kV, and compare to the speed of light. If you’ve read ch. 7 already, comment on whether the assumption of nonrelativistic motion was justified.

30 (a) Many battery-operated devices take more than one battery. If you look closely in the battery compartment, you will see that the batteries are wired in series. Consider a flashlight circuit. What does the loop rule tell you about the effect of putting several batteries in series in this way?
(b) The cells of an electric eel’s nervous system are not that different from ours — each cell can develop a voltage difference across it of somewhere on the order of one volt. How, then, do you think an electric eel can create voltages of thousands of volts between different parts of its body?

31 The figure shows two possible ways of wiring a flashlight with a switch. Both will serve to turn the bulb on and off, although the switch functions in the opposite sense. Why is method (1) preferable?
32. You have to do different things with a circuit to measure current than to measure a voltage difference. Which would be more practical for a printed circuit board, in which the wires are actually strips of metal embedded inside the board?  
Solution, p. 942

33. The bulbs are all identical. Which one doesn’t light up?

34. Each bulb has a resistance of one ohm. How much power is drawn from the one-volt battery?

35. The bulbs all have unequal resistances. Given the three currents shown in the figure, find the currents through bulbs A, B, C, and D.

36. A silk thread is uniformly charged by rubbing it with llama fur. The thread is then dangled vertically above a metal plate and released. As each part of the thread makes contact with the conducting plate, its charge is deposited onto the plate. Since the thread is accelerating due to gravity, the rate of charge deposition increases with time, and by time \( t \), the cumulative amount of charge is \( q = ct^2 \), where \( c \) is a constant. (a) Find the current flowing onto the plate. \( \checkmark \) (b) Suppose that the charge is immediately carried away through a resistance \( R \). Find the power dissipated as heat. \( \checkmark \)

37. In example 9 on p. 528, suppose that the larger sphere has radius \( a \), the smaller one \( b \). (a) Use the result of problem 9 show that the ratio of the charges on the two spheres is \( q_a/q_b = a/b \). (b) Show that the density of charge (charge per unit area) is the other way around: the charge density on the smaller sphere is greater than that on the larger sphere in the ratio \( a/b \).

38. (a) Recall that the gravitational energy of two gravitationally interacting spheres is given by \( PE = -Gm_1m_2/r \), where \( r \) is the center-to-center distance. Sketch a graph of \( PE \) as a function of \( r \), making sure that your graph behaves properly at small values of \( r \), where you’re dividing by a small number, and at large ones, where you’re dividing by a large one. Check that your graph behaves properly when a rock is dropped from a larger \( r \) to a smaller one; the rock should lose potential energy as it gains kinetic energy. (b) Electrical forces are closely analogous to gravitational ones, since both depend on \( 1/r^2 \). Since the forces are analogous, the potential energies should also behave analogously. Using this analogy, write down the expression for the electrical potential energy of two interacting charged particles. The main uncertainty here is the sign out in front. Like masses attract, but like charges repel. To figure out whether you have the right sign in your equation, sketch graphs in the case where both charges are positive, and also in the case where one is positive and one negative; make sure that in both cases, when the charges are released near one another, their motion causes them
to lose PE while gaining KE.

39  Find the current drawn from the battery.
**Exercises**

**Exercise 9A: Voltage and Current**

1. How many different currents could you measure in this circuit? Make a prediction, and then try it.

![Circuit Diagram 1](image1)

What do you notice? How does this make sense in terms of the roller coaster metaphor introduced in discussion question 9.1.3A on page 523?

What is being *used up* in the resistor?

2. By connecting probes to these points, how many ways could you measure a voltage? How many of them would be different numbers? Make a prediction, and then do it.

![Circuit Diagram 2](image2)

What do you notice? Interpret this using the roller coaster metaphor, and color in parts of the circuit that represent constant voltages.

3. The resistors are unequal. How many *different* voltages and currents can you measure? Make a prediction, and then try it.

![Circuit Diagram 3](image3)

What do you notice? Interpret this using the roller coaster metaphor, and color in parts of the circuit that represent constant voltages.
Exercise 9B: The Loop and Junction Rules

Apparatus:
- DC power supply
- multimeter
- resistors

1. The junction rule

Construct a circuit like this one, using the power supply as your voltage source. To make things more interesting, don’t use equal resistors. Use nice big resistors (say 100 kΩ to 1 MΩ) — this will ensure that you don’t burn up the resistors, and that the multimeter’s small internal resistance when used as an ammeter is negligible in comparison.

![Circuit Diagram]

Insert your multimeter in the circuit to measure all three currents that you need in order to test the junction rule.

2. The loop rule

Now come up with a circuit to test the loop rule. Since the loop rule is always supposed to be true, it’s hard to go wrong here! Make sure you have at least three resistors in a loop, and make sure you hook in the power supply in a way that creates non-zero voltage differences across all the resistors. Measure the voltage differences you need to measure to test the loop rule. Here it is best to use fairly small resistances, so that the multimeter’s large internal resistance when used in parallel as a voltmeter will not significantly reduce the resistance of the circuit. Do not use resistances of less than about 100 Ω, however, or you may blow a fuse or burn up a resistor.
Exercise 9C: Reasoning About Circuits

The questions in this exercise can all be solved using some combination of the following approaches:

a) There is constant voltage throughout any conductor.

b) Ohm’s law can be applied to any part of a circuit.

c) Apply the loop rule.

d) Apply the junction rule.

In each case, discuss the question, decide what you think is the right answer, and then try the experiment.

1. A wire is added in parallel with one bulb.

Which reasoning is correct?

- Each bulb still has 1.2 V across it, so both bulbs are still lit up.

- All parts of a wire are at the same voltage, and there is now a wire connection from one side of the right-hand bulb to the other. The right-hand bulb has no voltage difference across it, so it goes out.

2. The series circuit is changed as shown.

Which reasoning is correct?

- Each bulb now has its sides connected to the two terminals of the battery, so each now has 2.4 V across it instead of 1.2 V. They get brighter.

- Just as in the original circuit, the current goes through one bulb, then the other. It’s just that now the current goes in a figure-8 pattern. The bulbs glow the same as before.
3. A wire is added as shown to the original circuit.

What is wrong with the following reasoning?

*The top right bulb will go out, because its two sides are now connected with wire, so there will be no voltage difference across it. The other three bulbs will not be affected.*

4. A wire is added as shown to the original circuit.

What is wrong with the following reasoning?

*The current flows out of the right side of the battery. When it hits the first junction, some of it will go left and some will keep going up. The part that goes up lights the top right bulb. The part that turns left then follows the path of least resistance, going through the new wire instead of the bottom bulb. The top bulb stays lit, the bottom one goes out, and others stay the same.*

5. What happens when one bulb is unscrewed, leaving an air gap?
Chapter 10
Fields

“Okay. Your duties are as follows: Get Breen. I don’t care how you get
him, but get him soon. That faker! He posed for twenty years as a scientist
without ever being apprehended. Well, I’m going to do some apprehend-
ing that’ll make all previous apprehending look like no apprehension at all.
You with me?”

“Yes,” said Battle, very much confused. “What’s that thing you have?”

“Piggy-back heat-ray. You transpose the air in its path into an unstable
isotope which tends to carry all energy as heat. Then you shoot your juice
light, or whatever along the isotopic path and you burn whatever’s on the
receiving end. You want a few?”

“No,” said Battle. “I have my gats. What else have you got for offense
and defense?” Underbottam opened a cabinet and proudly waved an
arm. “Everything,” he said.

“Disintegraters, heat-rays, bombs of every type. And impenetrable
shields of energy, massive and portable. What more do I need?”

From THE REVERSIBLE REVOLUTIONS by Cecil Corwin, Cosmic
Stories, March 1941. Art by Morey, Bok, Kyle, Hunt, Forte. Copyright
expired.

10.1 Fields of Force

Cutting-edge science readily infiltrates popular culture, though som-
times in garbled form. The Newtonian imagination populated the
universe mostly with that nice solid stuff called matter, which was
made of little hard balls called atoms. In the early twentieth cen-
tury, consumers of pulp fiction and popularized science began to
hear of a new image of the universe, full of x-rays, N-rays, and
Hertzian waves. What they were beginning to soak up through
their skins was a drastic revision of Newton’s concept of a universe
made of chunks of matter which happened to interact via forces. In
the newly emerging picture, the universe was made of force, or, to
be more technically accurate, of ripples in universal fields of force.
Unlike the average reader of Cosmic Stories in 1941, you now pos-
sess enough technical background to understand what a “force field”
really is.

10.1.1 Why fields?
A bar magnet's atoms are (partially) aligned.

A bar magnet interacts with our magnetic planet.

Magnets aligned north-south.

The second magnet is reversed.

Both magnets are reversed.

Time delays in forces exerted at a distance

What convinced physicists that they needed this new concept of a field of force? Although we have been dealing mostly with electrical forces, let’s start with a magnetic example. (In fact the main reason I’ve delayed a detailed discussion of magnetism for so long is that mathematical calculations of magnetic effects are handled much more easily with the concept of a field of force.) First a little background leading up to our example. A bar magnet, a, has an axis about which many of the electrons’ orbits are oriented. The earth itself is also a magnet, although not a bar-shaped one. The interaction between the earth-magnet and the bar magnet, b, makes them want to line up their axes in opposing directions (in other words such that their electrons rotate in parallel planes, but with one set rotating clockwise and the other counterclockwise as seen looking along the axes). On a smaller scale, any two bar magnets placed near each other will try to align themselves head-to-tail, c.

Now we get to the relevant example. It is clear that two people separated by a paper-thin wall could use a pair of bar magnets to signal to each other. Each person would feel her own magnet trying to twist around in response to any rotation performed by the other person’s magnet. The practical range of communication would be very short for this setup, but a sensitive electrical apparatus could pick up magnetic signals from much farther away. In fact, this is not so different from what a radio does: the electrons racing up and down the transmitting antenna create forces on the electrons in the distant receiving antenna. (Both magnetic and electric forces are involved in real radio signals, but we don’t need to worry about that yet.)

A question now naturally arises as to whether there is any time delay in this kind of communication via magnetic (and electric) forces. Newton would have thought not, since he conceived of physics in terms of instantaneous action at a distance. We now know, however, that there is such a time delay. If you make a long-distance phone call that is routed through a communications satellite, you should easily be able to detect a delay of about half a second over the signal’s round trip of 50,000 miles. Modern measurements have shown that electric, magnetic, and gravitational forces all travel at the speed of light, $3 \times 10^8$ m/s. (In fact, we will soon discuss how light itself is made of electricity and magnetism.)

If it takes some time for forces to be transmitted through space, then apparently there is some thing that travels through space. The fact that the phenomenon travels outward at the same speed in all directions strongly evokes wave metaphors such as ripples on a pond.
More evidence that fields of force are real: they carry energy.

The smoking-gun argument for this strange notion of traveling force ripples comes from the fact that they carry energy.

First suppose that the person holding the bar magnet on the right decides to reverse hers, resulting in configuration d. She had to do mechanical work to twist it, and if she releases the magnet, energy will be released as it flips back to c. She has apparently stored energy by going from c to d. So far everything is easily explained without the concept of a field of force.

But now imagine that the two people start in position c and then simultaneously flip their magnets extremely quickly to position e, keeping them lined up with each other the whole time. Imagine, for the sake of argument, that they can do this so quickly that each magnet is reversed while the force signal from the other is still in transit. (For a more realistic example, we’d have to have two radio antennas, not two magnets, but the magnets are easier to visualize.) During the flipping, each magnet is still feeling the forces arising from the way the other magnet used to be oriented. Even though the two magnets stay aligned during the flip, the time delay causes each person to feel resistance as she twists her magnet around. How can this be? Both of them are apparently doing mechanical work, so they must be storing magnetic energy somehow. But in the traditional Newtonian conception of matter interacting via instantaneous forces at a distance, interaction energy arises from the relative positions of objects that are interacting via forces. If the magnets never changed their orientations relative to each other, how can any magnetic energy have been stored?

The only possible answer is that the energy must have gone into the magnetic force ripples crisscrossing the space between the magnets. Fields of force apparently carry energy across space, which is strong evidence that they are real things.

This is perhaps not as radical an idea to us as it was to our ancestors. We are used to the idea that a radio transmitting antenna consumes a great deal of power, and somehow spews it out into the universe. A person working around such an antenna needs to be careful not to get too close to it, since all that energy can easily cook flesh (a painful phenomenon known as an “RF burn”).

10.1.2 The gravitational field

Given that fields of force are real, how do we define, measure, and calculate them? A fruitful metaphor will be the wind patterns experienced by a sailing ship. Wherever the ship goes, it will feel a certain amount of force from the wind, and that force will be in a certain direction. The weather is ever-changing, of course, but for now let’s just imagine steady wind patterns. Definitions in physics are operational, i.e., they describe how to measure the thing being measured.
defined. The ship’s captain can measure the wind’s “field of force” by going to the location of interest and determining both the direction of the wind and the strength with which it is blowing. Charting all these measurements on a map leads to a depiction of the field of wind force like the one shown in the figure. This is known as the “sea of arrows” method of visualizing a field.

Now let’s see how these concepts are applied to the fundamental force fields of the universe. We’ll start with the gravitational field, which is the easiest to understand. As with the wind patterns, we’ll start by imagining gravity as a static field, even though the existence of the tides proves that there are continual changes in the gravity field in our region of space. When the gravitational field was introduced in chapter 2, I avoided discussing its direction explicitly, but defining it is easy enough: we simply go to the location of interest and measure the direction of the gravitational force on an object, such as a weight tied to the end of a string.

In chapter 2, I defined the gravitational field in terms of the energy required to raise a unit mass through a unit distance. However, I’m going to give a different definition now, using an approach that will be more easily adapted to electric and magnetic fields. This approach is based on force rather than energy. We couldn’t carry out the energy-based definition without dividing by the mass of the object involved, and the same is true for the force-based definition. For example, gravitational forces are weaker on the moon than on the earth, but we cannot specify the strength of gravity simply by giving a certain number of newtons. The number of newtons of gravitational force depends not just on the strength of the local gravitational field but also on the mass of the object on which we’re testing gravity, our “test mass.” A boulder on the moon feels a stronger gravitational force than a pebble on the earth. We can get around this problem by defining the strength of the gravitational field as the force acting on an object, divided by the object’s mass:

\[
g = \frac{F}{m_t}
\]

The gravitational field vector, \( \mathbf{g} \), at any location in space is found by placing a test mass \( m_t \) at that point. The field vector is then given by \( \mathbf{g} = \frac{\mathbf{F}}{m_t} \), where \( \mathbf{F} \) is the gravitational force on the test mass.

We now have three ways of representing a gravitational field. The magnitude of the gravitational field near the surface of the earth, for instance, could be written as 9.8 N/kg, 9.8 J/kg · m, or 9.8 m/s\(^2\). If we already had two names for it, why invent a third? The main reason is that it prepares us with the right approach for defining other fields.

The most subtle point about all this is that the gravitational field tells us about what forces would be exerted on a test mass by the earth, sun, moon, and the rest of the universe, if we inserted a test mass at the point in question. The field still exists at all the
places where we didn’t measure it.

| Gravitational field of the earth example 1 |
▷ What is the magnitude of the earth’s gravitational field, in terms of its mass, \( M \), and the distance \( r \) from its center?
▷ Substituting \( |\mathbf{F}| = \frac{GMm}{r^2} \) into the definition of the gravitational field, we find \( |\mathbf{g}| = \frac{GM}{r^2} \). This expression could be used for the field of any spherically symmetric mass distribution, since the equation we assumed for the gravitational force would apply in any such case.

Sources and sinks

If we make a sea-of-arrows picture of the gravitational fields surrounding the earth, \( \mathbf{g} \), the result is evocative of water going down a drain. For this reason, anything that creates an inward-pointing field around itself is called a sink. The earth is a gravitational sink. The term “source” can refer specifically to things that make outward fields, or it can be used as a more general term for both “outies” and “innies.” However confusing the terminology, we know that gravitational fields are only attractive, so we will never find a region of space with an outward-pointing field pattern.

Knowledge of the field is interchangeable with knowledge of its sources (at least in the case of a static, unchanging field). If aliens saw the earth’s gravitational field pattern they could immediately infer the existence of the planet, and conversely if they knew the mass of the earth they could predict its influence on the surrounding gravitational field.

Superposition of fields

A very important fact about all fields of force is that when there is more than one source (or sink), the fields add according to the rules of vector addition. The gravitational field certainly will have this property, since it is defined in terms of the force on a test mass, and forces add like vectors. Superposition is an important characteristics of waves, so the superposition property of fields is consistent with the idea that disturbances can propagate outward as waves in a field.
The average gravitational field on Jupiter’s moon Io is 1.81 N/kg. By how much is this reduced when Jupiter is directly overhead? Io’s orbit has a radius of $4.22 \times 10^8$ m, and Jupiter’s mass is $1.899 \times 10^{27}$ kg.

By the shell theorem, we can treat the Jupiter as if its mass was all concentrated at its center, and likewise for Io. If we visit Io and land at the point where Jupiter is overhead, we are on the same line as these two centers, so the whole problem can be treated one-dimensionally, and vector addition is just like scalar addition. Let’s use positive numbers for downward fields (toward the center of Io) and negative for upward ones. Plugging the appropriate data into the expression derived in example 1, we find that the Jupiter’s contribution to the field is $-0.71$ N/kg. Superposition says that we can find the actual gravitational field by adding up the fields created by Io and Jupiter: $1.81 - 0.71$ N/kg = 1.1 N/kg. You might think that this reduction would create some spectacular effects, and make Io an exciting tourist destination. Actually you would not detect any difference if you flew from one side of Io to the other. This is because your body and Io both experience Jupiter’s gravity, so you follow the same orbital curve through the space around Jupiter.

Gravitational waves

A source that sits still will create a static field pattern, like a steel ball sitting peacefully on a sheet of rubber. A moving source will create a spreading wave pattern in the field, like a bug thrashing on the
surface of a pond. Although we have started with the gravitational field as the simplest example of a static field, stars and planets do more stately gliding than thrashing, so gravitational waves are not easy to detect. Newton’s theory of gravity does not describe gravitational waves, but they are predicted by Einstein’s general theory of relativity. J.H. Taylor and R.A. Hulse were awarded the Nobel Prize in 1993 for giving indirect evidence that Einstein’s waves actually exist. They discovered a pair of exotic, ultra-dense stars called neutron stars orbiting one another very closely, and showed that they were losing orbital energy at the rate predicted by Einstein’s theory.

A Caltech-MIT collaboration has built a pair of gravitational wave detectors called LIGO to search for more direct evidence of gravitational waves. Since they are essentially the most sensitive vibration detectors ever made, they are located in quiet rural areas, and signals will be compared between them to make sure that they were not due to passing trucks. The project began operating at full sensitivity in 2005, and is now able to detect a vibration that causes a change of $10^{-18}$ m in the distance between the mirrors at the ends of the 4-km vacuum tunnels. This is a thousand times less than the size of an atomic nucleus! There is only enough funding to keep the detectors operating for a few more years, so the physicists can only hope that during that time, somewhere in the universe, a sufficiently violent cataclysm will occur to make a detectable gravitational wave. (More accurately, they want the wave to arrive in our solar system during that time, although it will have been produced millions of years before.)

10.1.3 The electric field

Definition

The definition of the electric field is directly analogous to, and has the same motivation as, the definition of the gravitational field:

The electric field vector, $\mathbf{E}$, at any location in space is found by placing a test charge $q_t$ at that point. The electric field vector is then given by $\mathbf{E} = \mathbf{F}/q_t$, where $\mathbf{F}$ is the electric force on the test charge.

Charges are what create electric fields. Unlike gravity, which is always attractive, electricity displays both attraction and repulsion. A positive charge is a source of electric fields, and a negative one is a sink.

The most difficult point about the definition of the electric field is that the force on a negative charge is in the opposite direction compared to the field. This follows from the definition, since dividing a vector by a negative number reverses its direction. It’s as though we had some objects that fell upward instead of down.
**Self-check A**

Find an equation for the magnitude of the field of a single point charge \(Q\).

\[ \text{Answer, p. 929} \]

**Superposition of electric fields**

Charges \(q\) and \(-q\) are at a distance \(b\) from each other, as shown in the figure. What is the electric field at the point \(P\), which lies at a third corner of the square?

The field at \(P\) is the vector sum of the fields that would have been created by the two charges independently. Let positive \(x\) be to the right and let positive \(y\) be up.

Negative charges have fields that point at them, so the charge \(-q\) makes a field that points to the right, i.e., has a positive \(x\) component. Using the answer to the self-check, we have

\[
E_{-q,x} = \frac{kq}{b^2}
\]

\[
E_{-q,y} = 0
\]

Note that if we had blindly ignored the absolute value signs and plugged in \(-q\) to the equation, we would have incorrectly concluded that the field went to the left.

By the Pythagorean theorem, the positive charge is at a distance \(\sqrt{2}b\) from \(P\), so the magnitude of its contribution to the field is \(E = kq/2b^2\). Positive charges have fields that point away from them, so the field vector is at an angle of \(135^\circ\) counterclockwise from the \(x\) axis.

\[
E_{q,x} = \frac{kq}{2b^2} \cos 135^\circ = -\frac{kq}{2^{3/2}b^2}
\]

\[
E_{q,y} = \frac{kq}{2b^2} \sin 135^\circ = \frac{kq}{2^{3/2}b^2}
\]

The total field is

\[
E_x = (1 - 2^{-3/2}) \frac{kq}{b^2}
\]

\[
E_y = \frac{kq}{2^{3/2}b^2}
\]

**Dipoles**

The simplest set of sources that can occur with electricity but not with gravity is the *dipole*, consisting of a positive charge and a
negative charge with equal magnitudes. More generally, an electric
dipole can be any object with an imbalance of positive charge on
one side and negative on the other. A water molecule, \( l \), is a dipole
because the electrons tend to shift away from the hydrogen atoms
and onto the oxygen atom.

Your microwave oven acts on water molecules with electric fields.
Let us imagine what happens if we start with a uniform electric field,
\( m/1 \), made by some external charges, and then insert a dipole, \( m/2 \),
consisting of two charges connected by a rigid rod. The dipole dis-
turbs the field pattern, but more important for our present purposes
is that it experiences a torque. In this example, the positive charge
feels an upward force, but the negative charge is pulled down. The
result is that the dipole wants to align itself with the field, \( m/3 \). The
microwave oven heats food with electrical (and magnetic) waves.
The alternation of the torque causes the molecules to wiggle and in-
crease the amount of random motion. The slightly vague definition
of a dipole given above can be improved by saying that a dipole is
any object that experiences a torque in an electric field.

What determines the torque on a dipole placed in an externally
created field? Torque depends on the force, the distance from the
axis at which the force is applied, and the angle between the force
and the line from the axis to the point of application. Let a dipole
consisting of charges \( +q \) and \( -q \) separated by a distance \( \ell \) be placed
in an external field of magnitude \( |E| \), at an angle \( \theta \) with respect to
the field. The total torque on the dipole is

\[
\tau = \frac{\ell q |E| \sin \theta}{2} + \frac{\ell q |E| \sin \theta}{2} = \ell q |E| \sin \theta.
\]

(Note that even though the two forces are in opposite directions,
the torques do not cancel, because they are both trying to twist
the dipole in the same direction.) The quantity is called the dipole
moment, notated \( D \). (More complex dipoles can also be assigned

m/1. A uniform electric field created by some charges “off-stage.”
2. A dipole is placed in the field. 3. The dipole aligns with the field.

\[
\tau = \frac{\ell q |E| \sin \theta}{2} + \frac{\ell q |E| \sin \theta}{2} = \ell q |E| \sin \theta.
\]
a dipole moment — they are defined as having the same dipole moment as the two-charge dipole that would experience the same torque.)

Employing a little more mathematical elegance, we can define a dipole moment vector,

\[ \mathbf{D} = \sum q_i \mathbf{r}_i, \]

where \( \mathbf{r}_i \) is the position vector of the charge labeled by the index \( i \). We can then write the torque in terms of a vector cross product (page 281),

\[ \mathbf{\tau} = \mathbf{D} \times \mathbf{E}. \]

No matter how we notate it, the definition of the dipole moment requires that we choose point from which we measure all the position vectors of the charges. However, in the commonly encountered special case where the total charge of the object is zero, the dipole moment is the same regardless of this choice.

<table>
<thead>
<tr>
<th>Dipole moment of a molecule of NaCl gas example 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>In a molecule of NaCl gas, the center-to-center distance between the two atoms is about 0.6 nm. Assuming that the chlorine completely steals one of the sodium’s electrons, compute the magnitude of this molecule’s dipole moment.</td>
</tr>
<tr>
<td>The total charge is zero, so it doesn’t matter where we choose the origin of our coordinate system. For convenience, let’s choose it to be at one of the atoms, so that the charge on that atom doesn’t contribute to the dipole moment. The magnitude of the dipole moment is then</td>
</tr>
</tbody>
</table>
| \[ D = (6 \times 10^{-10} \text{ m})(e) \]
| \[ = (6 \times 10^{-10} \text{ m})(1.6 \times 10^{-19} \text{ C}) \]
| \[ = 1 \times 10^{-28} \text{ C} \cdot \text{m} \] |

<table>
<thead>
<tr>
<th>Dipole moments as vectors example 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>The horizontal and vertical spacing between the charges in the figure is ( b ). Find the dipole moment.</td>
</tr>
<tr>
<td>Let the origin of the coordinate system be at the leftmost charge.</td>
</tr>
</tbody>
</table>
| \[ \mathbf{D} = \sum q_i \mathbf{r}_i \]
| \[ = (q)(0) + (-q)(b\hat{x}) + (q)(b\hat{x} + b\hat{y}) + (-q)(2b\hat{x}) \]
| \[ = -2bq\hat{x} + bq\hat{y} \] |
Alternative definition of the electric field

The behavior of a dipole in an externally created field leads us to an alternative definition of the electric field:

The electric field vector, \( \mathbf{E} \), at any location in space is defined by observing the torque exerted on a test dipole \( \mathbf{D}_t \) placed there. The direction of the field is the direction in which the field tends to align a dipole (from \( - \) to \( + \)), and the field’s magnitude is \( |\mathbf{E}| = \tau / \mathbf{D}_t \sin \theta \). In other words, the field vector is the vector that satisfies the equation \( \tau = \mathbf{D}_t \times \mathbf{E} \) for any test dipole \( \mathbf{D}_t \) placed at that point in space.

The main reason for introducing a second definition for the same concept is that the magnetic field is most easily defined using a similar approach.

Discussion Questions

A  In the definition of the electric field, does the test charge need to be 1 coulomb? Does it need to be positive?

B  Does a charged particle such as an electron or proton feel a force from its own electric field?

C  Is there an electric field surrounding a wall socket that has nothing plugged into it, or a battery that is just sitting on a table?

D  In a flashlight powered by a battery, which way do the electric fields point? What would the fields be like inside the wires? Inside the filament of the bulb?

E  Criticize the following statement: “An electric field can be represented by a sea of arrows showing how current is flowing.”

F  The field of a point charge, \( |\mathbf{E}| = kQ / r^2 \), was derived in a self-check. How would the field pattern of a uniformly charged sphere compare with the field of a point charge?

G  The interior of a perfect electrical conductor in equilibrium must have zero electric field, since otherwise the free charges within it would be drifting in response to the field, and it would not be in equilibrium. What about the field right at the surface of a perfect conductor? Consider the possibility of a field perpendicular to the surface or parallel to it.

H  Compare the dipole moments of the molecules and molecular ions shown in the figure.

I  Small pieces of paper that have not been electrically prepared in any way can be picked up with a charged object such as a charged piece of tape. In our new terminology, we could describe the tape’s charge as inducing a dipole moment in the paper. Can a similar technique be used to induce not just a dipole moment but a charge?

10.2 Voltage Related To Field

Section 10.2  Voltage Related To Field
10.2.1 One dimension

Voltage is electrical energy per unit charge, and electric field is force per unit charge. For a particle moving in one dimension, along the $x$ axis, we can therefore relate voltage and field if we start from the relationship between interaction energy and force,

$$dU = -F_x \, dx,$$

and divide by charge,

$$\frac{dU}{q} = -\frac{F_x}{q} \, dx,$$

giving

$$dV = -E_x \, dx,$$

or

$$\frac{dV}{dx} = -E_x.$$

The interpretation is that a strong electric field occurs in a region of space where the voltage is rapidly changing. By analogy, a steep hillside is a place on the map where the altitude is rapidly changing.

Field generated by an electric eel example 6

* Suppose an electric eel is 1 m long, and generates a voltage difference of 1000 volts between its head and tail. What is the electric field in the water around it?

* We are only calculating the amount of field, not its direction, so we ignore positive and negative signs. Subject to the possibly inaccurate assumption of a constant field parallel to the eel’s body, we have

$$|E| = \frac{dV}{dx} \approx \frac{\Delta V}{\Delta x} \quad \text{[assumption of constant field]}$$

$$= 1000 \text{ V/m}.$$

Relating the units of electric field and voltage example 7

From our original definition of the electric field, we expect it to have units of newtons per coulomb, N/C. The example above, however, came out in volts per meter, V/m. Are these inconsistent? Let’s reassure ourselves that this all works. In this kind of situation, the best strategy is usually to simplify the more complex units so that they involve only mks units and coulombs. Since
voltage is defined as electrical energy per unit charge, it has units of \( \text{J/C} \):

\[
\frac{V}{m} = \frac{\text{J}}{\text{C} \cdot \text{m}}.
\]

To connect joules to newtons, we recall that work equals force times distance, so \( J = N \cdot m \), so

\[
\frac{V}{m} = \frac{N \cdot m}{C \cdot m} = \frac{N}{C}.
\]

As with other such difficulties with electrical units, one quickly begins to recognize frequently occurring combinations.

\( ^1 \text{Voltage associated with a point charge} \)  

\( \triangleright \) What is the voltage associated with a point charge?

\( \triangleright \) As derived previously in self-check A on page 567, the field is

\[
|E| = \frac{kQ}{r^2}
\]

The difference in voltage between two points on the same radius line is

\[
\Delta V = \int dV = -\int E_x \, dx
\]

In the general discussion above, \( x \) was just a generic name for distance traveled along the line from one point to the other, so in this case \( x \) really means \( r \).

\[
\Delta V = \int_{r_1}^{r_2} E_r \, dr = -\int_{r_1}^{r_2} \frac{kQ}{r^2} \, dr = \frac{kQ}{r} \bigg|_{r_1}^{r_2} = \frac{kQ}{r_2} - \frac{kQ}{r_1}.
\]

The standard convention is to use \( r_1 = \infty \) as a reference point, so that the voltage at any distance \( r \) from the charge is

\[
V = \frac{kQ}{r}.
\]
The interpretation is that if you bring a positive test charge closer to a positive charge, its electrical energy is increased; if it was released, it would spring away, releasing this as kinetic energy.

**self-check B**

Show that you can recover the expression for the field of a point charge by evaluating the derivative $E_x = -dV/dx$. ▷ Answer, p. 929
10.2.2 Two or three dimensions

The topographical map in figure a suggests a good way to visualize the relationship between field and voltage in two dimensions. Each contour on the map is a line of constant height; some of these are labeled with their elevations in units of feet. Height is related to gravitational energy, so in a gravitational analogy, we can think of height as representing voltage. Where the contour lines are far apart, as in the town, the slope is gentle. Lines close together indicate a steep slope.

If we walk along a straight line, say straight east from the town, then height (voltage) is a function of the east-west coordinate $x$. Using the usual mathematical definition of the slope, and writing $V$ for the height in order to remind us of the electrical analogy, the slope along such a line is $dV/dx$ (the rise over the run).

What if everything isn’t confined to a straight line? Water flows downhill. Notice how the streams on the map cut perpendicularly through the lines of constant height.

It is possible to map voltages in the same way, as shown in figure b. The electric field is strongest where the constant-voltage curves are closest together, and the electric field vectors always point perpendicular to the constant-voltage curves.

The one-dimensional relationship $E = -dV/dx$ generalizes to three dimensions as follows:

$$
E_x = -\frac{dV}{dx} \\
E_y = -\frac{dV}{dy} \\
E_z = -\frac{dV}{dz}
$$

This can be notated as a gradient (page 215),

$$
\mathbf{E} = \nabla V ,
$$

and if we know the field and want to find the voltage, we can use a line integral,

$$
\Delta V = \int_C \mathbf{E} \cdot d\mathbf{r} ,
$$

where the quantity inside the integral is a vector dot product.

**self-check C**

Imagine that figure a represents voltage rather than height. (a) Consider the stream that starts near the center of the map. Determine the positive and negative signs of $dV/dx$ and $dV/dy$, and relate these to the direction of the force that is pushing the current forward against the resistance of friction. (b) If you wanted to find a lot of electric charge on this map, where would you look? 

Answer, p. 929
Figure c shows some examples of ways to visualize field and voltage patterns.

- **Two-dimensional field and voltage patterns.** Top: A uniformly charged rod. Bottom: A dipole. In each case, the diagram on the left shows the field vectors and constant-voltage curves, while the one on the right shows the voltage (up-down coordinate) as a function of x and y. Interpreting the field diagrams: Each arrow represents the field at the point where its tail has been positioned. For clarity, some of the arrows in regions of very strong field strength are not shown — they would be too long to show. Interpreting the constant-voltage curves: In regions of very strong fields, the curves are not shown because they would merge together to make solid black regions. Interpreting the perspective plots: Keep in mind that even though we're visualizing things in three dimensions, these are really two-dimensional voltage patterns being represented. The third (up-down) dimension represents voltage, not position.
10.3 Fields by Superposition

10.3.1 Electric field of a continuous charge distribution

Charge really comes in discrete chunks, but often it is mathematically convenient to treat a set of charges as if they were like a continuous fluid spread throughout a region of space. For example, a charged metal ball will have charge spread nearly uniformly all over its surface, and for most purposes it will make sense to ignore the fact that this uniformity is broken at the atomic level. The electric field made by such a continuous charge distribution is the sum of the fields created by every part of it. If we let the “parts” become infinitesimally small, we have a sum of an infinitely many infinitesimal numbers: an integral. If it was a discrete sum, as in example 3 on page 568, we would have a total electric field in the $x$ direction that was the sum of all the $x$ components of the individual fields, and similarly we’d have sums for the $y$ and $z$ components. In the continuous case, we have three integrals. Let’s keep it simple by starting with a one-dimensional example.

Field of a uniformly charged rod example 9

A rod of length $L$ has charge $Q$ spread uniformly along it. Find the electric field at a point a distance $d$ from the center of the rod, along the rod’s axis.

This is a one-dimensional situation, so we really only need to do a single integral representing the total field along the axis. We imagine breaking the rod down into short pieces of length $dz$, each with charge $dq$. Since charge is uniformly spread along the rod, we have $dq = \lambda \, dz$, where $\lambda = Q/L$ (Greek lambda) is the charge per unit length, in units of coulombs per meter. Since the pieces are infinitesimally short, we can treat them as point charges and use the expression $k \, dq/r^2$ for their contributions to the field, where $r = d - z$ is the distance from the charge at $z$ to the point in which we are interested.

$$E_z = \int \frac{k \, dq}{r^2}$$

$$= \int_{-L/2}^{+L/2} \frac{k \lambda \, dz}{r^2}$$

$$= k \lambda \int_{-L/2}^{+L/2} \frac{dz}{(d - z)^2}$$

The integral can be looked up in a table, or reduced to an elementary form by substituting a new variable for $d - z$. The result is

$$E_z = k \lambda \left( \frac{1}{d - z} \right)_{-L/2}^{+L/2}$$

$$= kQ \frac{1}{L} \left( \frac{1}{d - L/2} - \frac{1}{d + L/2} \right)$$
For large values of $d$, this expression gets smaller for two reasons: (1) the denominators of the fractions become large, and (2) the two fractions become nearly the same, and tend to cancel out. This makes sense, since the field should get weaker as we get farther away from the charge. In fact, the field at large distances must approach $kQ/d^2$ (homework problem 2).

It’s also interesting to note that the field becomes infinite at the ends of the rod, but is not infinite on the interior of the rod. Can you explain physically why this happens?

Example 9 was one-dimensional. In the general three-dimensional case, we might have to integrate all three components of the field. However, there is a trick that lets us avoid this much complication. The voltage is a scalar, so we can find the voltage by doing just a single integral, then use the voltage to find the field.

Voltage, then field example 10

> A rod of length $L$ is uniformly charged with charge $Q$. Find the field at a point lying in the midplane of the rod at a distance $R$.

> By symmetry, the field has only a radial component, $E_R$, pointing directly away from the rod (or toward it for $Q < 0$). The brute-force approach, then, would be to evaluate the integral $E = \int |dE|\cos \theta$, where $dE$ is the contribution to the field from a charge $dq$ at some point along the rod, and $\theta$ is the angle $dE$ makes with the radial line.

It’s easier, however, to find the voltage first, and then find the field from the voltage. Since the voltage is a scalar, we simply integrate the contribution $dV$ from each charge $dq$, without even worrying about angles and directions. Let $z$ be the coordinate that measures distance up and down along the rod, with $z = 0$ at the center of the rod. Then the distance between a point $z$ on the rod and the point of interest is $r = \sqrt{z^2 + R^2}$, and we have

$$V = \int \frac{k\,dq}{r} = k\lambda \int_{-L/2}^{+L/2} \frac{dz}{r} = k\lambda \int_{-L/2}^{+L/2} \frac{dz}{\sqrt{z^2 + R^2}}$$

The integral can be looked up in a table, or evaluated using com-
puter software:

\[ V = k\lambda \ln \left( z + \sqrt{z^2 + R^2} \right) \bigg|_{-L/2}^{+L/2} \]
\[ = k\lambda \ln \left( \frac{L/2 + \sqrt{L^2/4 + R^2}}{-L/2 + \sqrt{L^2/4 + R^2}} \right) \]

The expression inside the parentheses can be simplified a little. Leaving out some tedious algebra, the result is

\[ V = 2k\lambda \ln \left( \frac{L}{2R} + \sqrt{1 + \frac{L^2}{4R^2}} \right) \]

This can readily be differentiated to find the field:

\[ E_R = -\frac{dV}{dR} \]
\[ = (-2k\lambda) \frac{-L/2R^2 + (1/2)(1 + L^2/4R^2)^{-1/2}(-L^2/2R^3)}{L/2R + (1 + L^2/4R^2)^{1/2}} \]
\[ \text{or, after some simplification,} \]
\[ E_R = \frac{k\lambda L}{R^2 \sqrt{1 + L^2/4R^2}} \]

For large values of \( R \), the square root approaches one, and we have simply \( E_R \approx \frac{k\lambda L}{R^2} = \frac{kQ}{R^2} \). In other words, the field very far away is the same regardless of whether the charge is a point charge or some other shape like a rod. This is intuitively appealing, and doing this kind of check also helps to reassure one that the final result is correct.

The preceding example, although it involved some messy algebra, required only straightforward calculus, and no vector operations at all, because we only had to integrate a scalar function to find the voltage. The next example is one in which we can integrate either the field or the voltage without too much complication.

\begin{itemize}
  \item \textbf{On-axis field of a ring of charge example 11} \textit{\textbf{example 11}}
  \item Find the voltage and field along the axis of a uniformly charged ring.
  \item Integrating the voltage is straightforward.
  \[ V = \int \frac{k}{r} dq \]
  \[ = k \int \frac{dq}{\sqrt{b^2 + z^2}} \]
  \[ = \frac{k}{\sqrt{b^2 + z^2}} \int dq \]
  \[ = \frac{kQ}{\sqrt{b^2 + z^2}} \]
\end{itemize}
where $Q$ is the total charge of the ring. This result could have been derived without calculus, since the distance $r$ is the same for every point around the ring, i.e., the integrand is a constant. It would also be straightforward to find the field by differentiating this expression with respect to $z$ (homework problem 10).

Instead, let’s see how to find the field by direct integration. By symmetry, the field at the point of interest can have only a component along the axis of symmetry, the $z$ axis:

$$E_x = 0$$
$$E_y = 0$$

To find the field in the $z$ direction, we integrate the $z$ components contributed to the field by each infinitesimal part of the ring.

$$E_z = \int dE_z$$
$$= \int |dE| \cos \theta,$$

where $\theta$ is the angle shown in the figure.

$$E_z = \int \frac{k \ dq}{r^2} \cos \theta$$
$$= k \int \frac{dq}{b^2 + z^2} \cos \theta$$

Everything inside the integral is a constant, so we have

$$E_z = \frac{k}{b^2 + z^2} \cos \theta \int dq$$
$$= \frac{kQ}{b^2 + z^2} \cos \theta$$
$$= \frac{kQ}{b^2 + z^2} \frac{z}{r}$$
$$= \frac{kQz}{(b^2 + z^2)^{3/2}}$$

In all the examples presented so far, the charge has been confined to a one-dimensional line or curve. Although it is possible, for example, to put charge on a piece of wire, it is more common to encounter practical devices in which the charge is distributed over a two-dimensional surface, as in the flat metal plates used in Thomson’s experiments. Mathematically, we can approach this type of calculation with the divide-and-conquer technique: slice the surface into lines or curves whose fields we know how to calculate, and then add up the contributions to the field from all these slices. In the limit where the slices are imagined to be infinitesimally thin, we have an integral.
A capacitor consisting of two disks with opposite charges.

We're given that every part of the disk has the same charge per unit area, so rather than working with \( Q \), the total charge, it will be easier to use the charge per unit area, conventionally notated \( \sigma \) (Greek sigma), \( \sigma = Q/\pi b^2 \).

Since we already know the field due to a ring of charge, we can solve the problem by slicing the disk into rings, with each ring extending from \( r \) to \( r + dr \). The area of such a ring equals its circumference multiplied by its width, i.e., \( 2\pi r dr \), so its charge is \( dq = 2\pi \sigma r dr \), and from the result of example 11, its contribution to the field is

\[
dE_z = \frac{kz dq}{(r^2 + z^2)^{3/2}} = 2\pi \sigma k z r dr \left( \frac{r}{r^2 + z^2} \right)^{3/2}
\]

The total field is

\[
E_z = \int dE_z = 2\pi \sigma k z \int_0^b r dr \left( \frac{r}{r^2 + z^2} \right)^{3/2} = 2\pi \sigma k \left( 1 - \frac{z}{\sqrt{r^2 + z^2}} \right)
\]

The result of example 12 has some interesting properties. First, we note that it was derived on the unspoken assumption of \( z > 0 \). By symmetry, the field on the other side of the disk must be equally strong, but in the opposite direction, as shown in figures e and g. Thus there is a discontinuity in the field at \( z = 0 \). In reality, the disk will have some finite thickness, and the switching over of the field will be rapid, but not discontinuous.

At large values of \( z \), i.e., \( z \gg b \), the field rapidly approaches the \( 1/r^2 \) variation that we expect when we are so far from the disk that the disk's size and shape cannot matter (homework problem 2).
Example 12: variation of the field ($\sigma > 0$).

A practical application is the case of a capacitor, having two parallel circular plates very close together. In normal operation, the charges on the plates are opposite, so one plate has fields pointing into it and the other one has fields pointing out. In a real capacitor, the plates are a metal conductor, not an insulator, so the charge will tend to arrange itself more densely near the edges, rather than spreading itself uniformly on each plate. Furthermore, we have only calculated the on-axis field in example 12; in the off-axis region, each disk’s contribution to the field will be weaker, and it will also point away from the axis a little. But if we are willing to ignore these complications for the sake of a rough analysis, then the fields superimpose as shown in figure f: the fields cancel the outside of the capacitor, but between the plates its value is double that contributed by a single plate. This cancellation on the outside is a very useful property for a practical capacitor. For instance, if you look at the printed circuit board in a typical piece of consumer electronics, there are many capacitors, often placed fairly close together. If their exterior fields didn’t cancel out nicely, then each capacitor would interact with its neighbors in a complicated way, and the behavior of the circuit would depend on the exact physical layout, since the interaction would be stronger or weaker depending on distance. In reality, a capacitor does create weak external electric fields, but their effects are often negligible, and we can then use the lumped-circuit approximation, which states that each component’s behavior depends only on the currents that flow in and out of it, not on the interaction of its fields with the other components.
10.3.2 The field near a charged surface

From a theoretical point of view, there is something even more intriguing about example 12: the magnitude of the field for small values of $z$ ($z \ll b$) is $E = 2\pi k\sigma$, which doesn’t depend on $b$ at all for a fixed value of $\sigma$. If we made a disk with twice the radius, and covered it with the same number of coulombs per square meter (resulting in a total charge four times as great), the field close to the disk would be unchanged! That is, a flea living near the center of the disk, $h$, would have no way of determining the size of her flat “planet” by measuring the local field and charge density. (Only by leaping off the surface into outer space would she be able to measure fields that were dependent on $b$. If she traveled very far, to $z \gg b$, she would be in the region where the field is well approximated by $|E| \approx kQ/z^2 = k\pi b^2\sigma/z^2$, which she could solve for $b$.)

What is the reason for this surprisingly simple behavior of the field? Is it a piece of mathematical trivia, true only in this particular case? What if the shape was a square rather than a circle? In other words, the flea gets no information about the size of the disk from measuring $E$, since $E = 2\pi k\sigma$, independent of $b$, but what if she didn’t know the shape, either? If the result for a square had some other geometrical factor in front instead of $2\pi$, then she could tell which shape it was by measuring $E$. The surprising mathematical fact, however, is that the result for a square, indeed for any shape whatsoever, is $E = 2\pi\sigma k$. It doesn’t even matter whether the surface is flat or warped, or whether the density of charge is different at parts of the surface which are far away compared to the flea’s distance above the surface.

This universal $E_\perp = 2\pi k\sigma$ field perpendicular to a charged surface can be proved mathematically based on Gauss’s law\(^1\) (section 10.6), but we can understand what’s happening on qualitative grounds. Suppose on night, while the flea is asleep, someone adds more surface area, also positively charged, around the outside edge of her disk-shaped world, doubling its radius. The added charge, however, has very little effect on the field in her environment, as long as she stays at low altitudes above the surface. As shown in figure i, the new charge to her west contributes a field, $T$, that is almost purely “horizontal” (i.e., parallel to the surface) and to the east. It has a negligible upward component, since the angle is so shallow. This new eastward contribution to the field is exactly canceled out by the westward field, $S$, created by the new charge to her east. There is likewise almost perfect cancellation between any other pair of opposite compass directions.

A similar argument can be made as to the shape-independence of the result, as long as the shape is symmetric. For example, suppose that the next night, the tricky real estate developers decide to add

---

\(^1\)rhymes with “mouse”

h / Close to the surface, the relationship between $E$ and $\sigma$ is a fixed one, regardless of the geometry. The flea can’t determine the size or shape of her world by comparing $E$ and $\sigma$. 

i / Fields contributed by nearby parts of the surface, $P$, $Q$, and $R$, contribute to $E_\perp$. Fields due to distant charges, $S$, and $T$, have very small contributions to $E_\perp$ because of their shallow angles.
corners to the disk and transform it into a square. Each corner’s contribution to the field measured at the center is canceled by the field due to the corner diagonally across from it.

What if the flea goes on a trip away from the center of the disk? The perfect cancellation of the “horizontal” fields contributed by distant charges will no longer occur, but the “vertical” field (i.e., the field perpendicular to the surface) will still be $E_\perp = 2\pi k\sigma$, where $\sigma$ is the local charge density, since the distant charges can’t contribute to the vertical field. The same result applies if the shape of the surface is asymmetric, and doesn’t even have any well-defined geometric center: the component perpendicular to the surface is $E_\perp = 2\pi k\sigma$, but we may have $E_\parallel \neq 0$. All of the above arguments can be made more rigorous by discussing mathematical limits rather than using words like “very small.” There is not much point in giving a rigorous proof here, however, since we will be able to demonstrate this fact as a corollary of Gauss’ Law in section 10.6. The result is as follows:

At a point lying a distance $z$ from a charged surface, the component of the electric field perpendicular to the surface obeys

$$\lim_{z \to 0} E_\perp = 2\pi k\sigma,$$

where $\sigma$ is the charge per unit area. This is true regardless of the shape or size of the surface.

The field near a point, line, or surface charge example 13

Compare the variation of the electric field with distance, $d$, for small values of $d$ in the case of a point charge, an infinite line of charge, and an infinite charged surface.

For a point charge, we have already found $E \propto d^{-2}$ for the magnitude of the field, where we are now using $d$ for the quantity we would ordinarily notate as $r$. This is true for all values of $d$, not just for small $d$ — it has to be that way, because the point charge has no size, so if $E$ behaved differently for small and large $d$, there would be no way to decide what $d$ had to be small or large relative to.

For a line of charge, the result of example 10 is

$$E = \frac{k\lambda L}{d^2 \sqrt{1 + L^2/4d^2}}.$$

In the limit of $d \ll L$, the quantity inside the square root is dominated by the second term, and we have $E \propto d^{-1}$.

Finally, in the case of a charged surface, the result is simply $E = 2\pi \sigma k$, or $E \propto d^0$.

Notice the lovely simplicity of the pattern, as shown in figure j. A point is zero-dimensional: it has no length, width, or breadth. A line is one-dimensional, and a surface is two-dimensional. As the
dimensionality of the charged object changes from 0 to 1, and then to 2, the exponent in the near-field expression goes from 2 to 1 to 0.


10.4 Energy In Fields

10.4.1 Electric field energy

Fields possess energy, as argued on page 563, but how much energy? The answer can be found using the following elegant approach. We assume that the electric energy contained in an infinitesimal volume of space $dV$ is given by $dU_e = f(E) dV$, where $f$ is some function, which we wish to determine, of the field $E$. It might seem that we would have no easy way to determine the function $f$, but many of the functions we could cook up would violate the symmetry of space. For instance, we could imagine $f(E) = aE_y$, where $a$ is some constant with the appropriate units. However, this would violate the symmetry of space, because it would give the $y$ axis a different status from $x$ and $z$. As discussed on page 212, if we wish to calculate a scalar based on some vectors, the dot product is the only way to do it that has the correct symmetry properties. If all we have is one vector, $E$, then the only scalar we can form is $E \cdot E$, which is the square of the magnitude of the electric field vector.

In principle, the energy function we are seeking could be proportional to $E \cdot E$, or to any function computed from it, such as $\sqrt{E \cdot E}$ or $(E \cdot E)^7$. On physical grounds, however, the only possibility that works is $E \cdot E$. Suppose, for instance, that we pull apart two oppositely charged capacitor plates, as shown in figure a. We are doing work by pulling them apart against the force of their electrical attraction, and this quantity of mechanical work equals the increase in electrical energy, $U_e$. Using our previous approach to energy, we would have thought of $U_e$ as a quantity which depended on the distance of the positive and negative charges from each other, but now we’re going to imagine $U_e$ as being stored within the electric field that exists in the space between and around the charges. When the plates are touching, their fields cancel everywhere, and there is zero electrical energy. When they are separated, there is still approximately zero field on the outside, but the field between the plates is nonzero, and holds some energy. Now suppose we carry out the whole process, but with the plates carrying double their previous charges. Since Coulomb’s law involves the product $q_1 q_2$ of two charges, we have quadrupled the force between any given pair of charged particles, and the total attractive force is therefore also four times greater than before. This means that the work done in separating the plates is four times greater, and so is the energy $U_e$ stored in the field. The field, however, has merely been doubled at any given location: the electric field $E_+$ due to the positively charged plate is doubled, and similarly for the contribution $E_-$ from the negative one, so the total electric field $E_+ + E_-$ is also doubled. Thus doubling the field results in an electrical energy which is four times greater, i.e., the energy density must be proportional to the square of the field, $dU_e \propto (E \cdot E) dV$. For ease of notation, we write this
as \( \text{d}U_e \propto E^2 \text{d}v \), or \( \text{d}U_e = aE^2 \text{d}v \), where \( a \) is a constant of proportionality. Note that we never really made use of any of the details of the geometry of figure a, so the reasoning is of general validity. In other words, not only is \( \text{d}U_e = aE^2 \text{d}v \) the function that works in this particular case, but there is every reason to believe that it would work in other cases as well.

It now remains only to find \( a \). Since the constant must be the same in all situations, we only need to find one example in which we can compute the field and the energy, and then we can determine \( a \). The situation shown in figure a is just about the easiest example to analyze. We let the square capacitor plates be uniformly covered with charge densities \(+\sigma\) and \(-\sigma\), and we write \( b \) for the lengths of their sides. Let \( h \) be the gap between the plates after they have been separated. We choose \( h \ll b \), so that the field experienced by the negative plate due to the positive plate is \( E_+ = 2\pi k\sigma \). The charge of the negative plate is \(-\sigma b^2\), so the magnitude of the force attracting it back toward the positive plate is \((\text{force}) = (\text{charge})(\text{field}) = 2\pi k\sigma^2 b^2\). The amount of work done in separating the plates is \((\text{work}) = (\text{force})(\text{distance}) = 2\pi k\sigma^2 b^2 h\). This is the amount of energy that has been stored in the field between the two plates, \( U_e = 2\pi k\sigma^2 b^2 h = 2\pi k\sigma^2 v \), where \( v \) is the volume of the region between the plates.

We want to equate this to \( U_e = aE^2 v \). (We can write \( U_e \) and \( v \) rather than \( \text{d}U_e \) and \( \text{d}v \), since the field is constant in the region between the plates.) The field between the plates has contributions from both plates, \( E = E_+ + E_- = 4\pi k\sigma \). (We only used half this value in the computation of the work done on the moving plate, since the moving plate can’t make a force on itself. Mathematically, each plate is in a region where its own field is reversing directions, so we can think of its own contribution to the field as being zero within itself.) We then have \( aE^2 v = a \cdot 16\pi^2 k^2 \sigma^2 \cdot v \), and setting this equal to \( U_e = 2\pi k\sigma^2 v \) from the result of the work computation, we find \( a = 1/8\pi k \). Our final result is as follows:

The electric energy possessed by an electric field \( \mathbf{E} \) occupying an infinitesimal volume of space \( \text{d}v \) is given by

\[
\text{d}U_e = \frac{1}{8\pi k} E^2 \text{d}v
\]

where \( E^2 = \mathbf{E} \cdot \mathbf{E} \) is the square of the magnitude of the electric field.

This is reminiscent of how waves behave: the energy content of a wave is typically proportional to the square of its amplitude.
We can think of the quantity \( dU_e / dv \) as the energy density due to the electric field, i.e., the number of joules per cubic meter needed in order to create that field. (a) How does this quantity depend on the components of the field vector, \( E_x \), \( E_y \), and \( E_z \)? (b) Suppose we have a field with \( E_x \neq 0 \), \( E_y = 0 \), and \( E_z = 0 \). What would happen to the energy density if we reversed the sign of \( E_x \)?

A numerical example

A capacitor has plates whose areas are \( 10^{-4} \) m\(^2\), separated by a gap of \( 10^{-5} \) m. A 1.5-volt battery is connected across it. How much energy is sucked out of the battery and stored in the electric field between the plates? (A real capacitor typically has an insulating material between the plates whose molecules interact electrically with the charge in the plates. For this example, we’ll assume that there is just a vacuum in between the plates. The plates are also typically rolled up rather than flat.)

To connect this with our previous calculations, we need to find the charge density on the plates in terms of the voltage we were given. Our previous examples were based on the assumption that the gap between the plates was small compared to the size of the plates. Is this valid here? Well, if the plates were square, then the area of \( 10^{-4} \) m\(^2\) would imply that their sides were \( 10^{-2} \) m in length. This is indeed very large compared to the gap of \( 10^{-5} \) m, so this assumption appears to be valid (unless, perhaps, the plates have some very strange, long and skinny shape).

Based on this assumption, the field is relatively uniform in the whole volume between the plates, so we can use a single symbol, \( E \), to represent its magnitude, and the relation \( E = dV/\Delta x \) is equivalent to \( E = \Delta V/\Delta x = (1.5 \text{ V})/(\text{gap}) = 1.5 \times 10^5 \text{ V/m} \).

Since the field is uniform, we can dispense with the calculus, and replace \( dU_e = (1/8\pi k)E^2 \) \( dv \) with \( U_e = (1/8\pi k)E^2 \) \( v \). The volume equals the area multiplied by the gap, so we have

\[
U_e = (1/8\pi k)E^2 \text{(area) (gap)}
\]

\[
= \frac{1}{8\pi \times 9 \times 10^9 \text{ N m}^2/\text{C}^2} (1.5 \times 10^5 \text{ V/m})^2 (10^{-4} \text{ m}^2) (10^{-5} \text{ m})
\]

\[
= 1 \times 10^{-10} \text{ J}
\]

Why \( k \) is on the bottom

It may also seem strange that the constant \( k \) is in the denominator of the equation \( dU_e = (1/8\pi k)E^2 \) \( dv \). The Coulomb constant \( k \) tells us how strong electric forces are, so shouldn’t it be on top?
No. Consider, for instance, an alternative universe in which electric forces are twice as strong as in ours. The numerical value of \( k \) is doubled. Because \( k \) is doubled, all the electric field strengths are doubled as well, which quadruples the quantity \( E^2 \). In the expression \( E^2/8\pi k \), we’ve quadrupled something on top and doubled something on the bottom, which makes the energy twice as big. That makes perfect sense.

1Potential energy of a pair of opposite charges example 16
Imagine taking two opposite charges, \( b \), that were initially far apart and allowing them to come together under the influence of their electrical attraction.

According to our old approach, electrical energy is lost because the electric force did positive work as it brought the charges together. (This makes sense because as they come together and accelerate it is their electrical energy that is being lost and converted to kinetic energy.)

By the new method, we must ask how the energy stored in the electric field has changed. In the region indicated approximately by the shading in the figure, the superposing fields of the two charges undergo partial cancellation because they are in opposing directions. The energy in the shaded region is reduced by this effect. In the unshaded region, the fields reinforce, and the energy is increased.

It would be quite a project to do an actual numerical calculation of the energy gained and lost in the two regions (this is a case where the old method of finding energy gives greater ease of computation), but it is fairly easy to convince oneself that the energy is less when the charges are closer. This is because bringing the charges together shrinks the high-energy unshaded region and enlarges the low-energy shaded region.

A spherical capacitor example 17

▷ A spherical capacitor, \( c \), consists of two concentric spheres of radii \( a \) and \( b \). Find the energy required to charge up the capacitor so that the plates hold charges \( +q \) and \( -q \).

▷ On page 102, I proved that for gravitational forces, the interaction of a spherical shell of mass with other masses outside it is the same as if the shell’s mass was concentrated at its center. On the interior of such a shell, the forces cancel out exactly. Since gravity and the electric force both vary as \( 1/r^2 \), the same proof carries over immediately to electrical forces. The magnitude of the outward electric field contributed by the charge \( +q \) of the central sphere is therefore

\[
|E_+| = \begin{cases} 
0, & r < a \\
\frac{kq}{r^2}, & r > a
\end{cases}
\]

where \( r \) is the distance from the center. Similarly, the magnitude
of the \textit{inward} field contributed by the outside sphere is

\[ |E_-| = \begin{cases} 
0, & r < b \\
\frac{kq}{r^2}, & r > b 
\end{cases}. \]

In the region outside the whole capacitor, the two fields are equal in magnitude, but opposite in direction, so they cancel. We then have for the total field

\[ |E| = \begin{cases} 
0, & r < a \\
\frac{kq}{r^2}, & a < r < b \\
0, & r > b 
\end{cases}, \]

so to calculate the energy, we only need to worry about the region \( a < r < b \). The energy density in this region is

\[ \frac{dU_e}{dv} = \frac{1}{8\pi k} E^2 = \frac{kq^2}{8\pi} r^{-4}. \]

This expression only depends on \( r \), so the energy density is constant across any sphere of radius \( r \). We can slice the region \( a < r < b \) into concentric spherical layers, like an onion, and the energy within one such layer, extending from \( r \) to \( r + dr \) is

\[ dU_e = \frac{dU_e}{dv} dv = \frac{dU_e}{dv} (\text{area of shell})(\text{thickness of shell}) = \left( \frac{kq^2}{8\pi} r^{-4} \right) (4\pi r^2) (dr) = \frac{kq^2}{2} r^{-2} dr. \]

Integrating over all the layers to find the total energy, we have

\[ U_e = \int dU_e = \int_a^b \frac{kq^2}{2} r^{-2} dr = \frac{kq^2}{2} \left| r^{-1} \right|^b_a = \frac{kq^2}{2} \left( \frac{1}{a} - \frac{1}{b} \right). \]
Discussion Questions

A The figure shows a positive charge in the gap between two capacitor plates. Compare the energy of the electric fields in the two cases. Does this agree with what you would have expected based on your knowledge of electrical forces?

B The figure shows a spherical capacitor. In the text, the energy stored in its electric field is shown to be

\[ U_e = \frac{kq^2}{2} \left( \frac{1}{a} - \frac{1}{b} \right) \]

What happens if the difference between \( b \) and \( a \) is very small? Does this make sense in terms of the mechanical work needed in order to separate the charges? Does it make sense in terms of the energy stored in the electric field? Should these two energies be added together?

Similarly, discuss the cases of \( b \to \infty \) and \( a \to 0 \).

C Criticize the following statement: “A solenoid makes a charge in the space surrounding it, which dissipates when you release the energy.”

D In example 16 on page 589, I argued that for the charges shown in the figure, the fields contain less energy when the charges are closer together, because the region of cancellation expanded, while the region of reinforcing fields shrunk. Perhaps a simpler approach is to consider the two extreme possibilities: the case where the charges are infinitely far apart, and the one in which they are at zero distance from each other, i.e., right on top of each other. Carry out this reasoning for the case of (1) a positive charge and a negative charge of equal magnitude, (2) two positive charges of equal magnitude, (3) the gravitational energy of two equal masses.

10.4.2 Gravitational field energy

Example B depended on the close analogy between electric and gravitational forces. In fact, every argument, proof, and example discussed so far in this section is equally valid as a gravitational example, provided we take into account one fact: only positive mass exists, and the gravitational force between two masses is attractive. This is the opposite of what happens with electrical forces, which are repulsive in the case of two positive charges. As a consequence of this, we need to assign a negative energy density to the gravitational field! For a gravitational field, we have

\[ dU_g = -\frac{1}{8\pi G} g^2 dv \]

where \( g^2 = g \cdot g \) is the square of the magnitude of the gravitational field.

10.4.3 Magnetic field energy

So far we’ve only touched in passing on the topic of magnetic fields, which will deal with in detail in chapter 11. Magnetism is an interaction between moving charge and moving charge, i.e., between currents and currents. Since a current has a direction in
space, while charge doesn’t, we can anticipate that the mathematical rule connecting a magnetic field to its source-currents will have to be completely different from the one relating the electric field to its source-charges. However, if you look carefully at the argument leading to the relation \( \frac{dU_e}{dv} = \frac{E^2}{8\pi k} \), you’ll see that these mathematical details were only necessary to the part of the argument in which we fixed the constant of proportionality. To establish \( \frac{dU_e}{dv} \propto E^2 \), we only had to use three simple facts:

- The field is proportional to the source.
- Forces are proportional to fields.
- Field contributed by multiple sources add like vectors.

All three of these statements are true for the magnetic field as well, so without knowing anything more specific about magnetic fields — not even what units are used to measure them! — we can state with certainty that the energy density in the magnetic field is proportional to the square of the magnitude of the magnetic field. The constant of proportionality is given on p. 669.

\footnote{Current is a scalar, since the definition \( I = \frac{dq}{dt} \) is the derivative of a scalar. However, there is a closely related quantity called the current \emph{density}, \( \mathbf{J} \), which is a vector, and \( \mathbf{J} \) is in fact the more fundamentally important quantity.}
The long road leading from the light bulb to the computer started with one very important step: the introduction of feedback into electronic circuits. Although the principle of feedback has been understood and applied to mechanical systems for centuries, and to electrical ones since the early twentieth century, for most of us the word evokes an image of Jimi Hendrix (or some more recent guitar hero) intentionally creating earsplitting screeches, or of the school principal doing the same inadvertently in the auditorium. In the guitar example, the musician stands in front of the amp and turns it up so high that the sound waves coming from the speaker come back to the guitar string and make it shake harder. This is an example of *positive* feedback: the harder the string vibrates, the stronger the sound waves, and the stronger the sound waves, the harder the string vibrates. The only limit is the power-handling ability of the amplifier.

Negative feedback is equally important. Your thermostat, for example, provides negative feedback by kicking the heater off when the house gets warm enough, and by firing it up again when it gets too cold. This causes the house’s temperature to oscillate back and forth within a certain range. Just as out-of-control exponential freak-outs are a characteristic behavior of positive-feedback systems, oscillation is typical in cases of negative feedback. You have already studied negative feedback extensively in section 3.3 in the case of a mechanical system, although we didn’t call it that.

### 10.5.1 Capacitance and inductance

In a mechanical oscillation, energy is exchanged repetitively between potential and kinetic forms, and may also be siphoned off in the form of heat dissipated by friction. In an electrical circuit, resistors are the circuit elements that dissipate heat. What are the electrical analogs of storing and releasing the potential and kinetic energy of a vibrating object? When you think of energy storage in an electrical circuit, you are likely to imagine a battery, but even rechargeable batteries can only go through 10 or 100 cycles before they wear out. In addition, batteries are not able to exchange energy on a short enough time scale for most applications. The circuit in a musical synthesizer may be called upon to oscillate thousands of times a second, and your microwave oven operates at gigahertz frequencies. Instead of batteries, we generally use capacitors and inductors to store energy in oscillating circuits. Capacitors, which you’ve already encountered, store energy in electric fields. An inductor does the same with magnetic fields.

**Capacitors**

A capacitor’s energy exists in its surrounding electric fields. It is proportional to the square of the field strength, which is proportional
The symbol for a capacitor.

Some capacitors.

Two common geometries for inductors. The cylindrical shape on the left is called a solenoid.

The symbol for an inductor.

Some inductors.

to the charges on the plates. If we assume the plates carry charges that are the same in magnitude, \( +q \) and \( -q \), then the energy stored in the capacitor must be proportional to \( q^2 \). For historical reasons, we write the constant of proportionality as \( 1/2C \),

\[
U_C = \frac{1}{2C}q^2.
\]

The constant \( C \) is a geometrical property of the capacitor, called its capacitance.

Based on this definition, the units of capacitance must be coulombs squared per joule, and this combination is more conveniently abbreviated as the farad, \( 1 \text{ F} = 1 \text{ C}^2/\text{J} \). “Condenser” is a less formal term for a capacitor. Note that the labels printed on capacitors often use MF to mean \( \mu \text{F} \), even though MF should really be the symbol for megafarads, not microfarads. Confusion doesn’t result from this nonstandard notation, since picofarad and microfarad values are the most common, and it wasn’t until the 1990’s that even millifarad and farad values became available in practical physical sizes. Figure a shows the symbol used in schematics to represent a capacitor.

*A parallel-plate capacitor*  

Suppose a capacitor consists of two parallel metal plates with area \( A \), and the gap between them is \( h \). The gap is small compared to the dimensions of the plates. What is the capacitance?

Since the plates are metal, the charges on each plate are free to move, and will tend to cluster themselves more densely near the edges due to the mutual repulsion of the other charges in the same plate. However, it turns out that if the gap is small, this is a small effect, so we can get away with assuming uniform charge density on each plate. The result of example 14 then applies, and for the region between the plates, we have \( E = 4\pi k \sigma = 4\pi kq/A \) and \( U_e = (1/8\pi k)E^2Ah \). Substituting the first expression into the second, we find \( U_e = 2\pi kq^2h/A \). Comparing this to the definition of capacitance, we end up with \( C = A/4\pi kh \).

Inductors

Any current will create a magnetic field, so in fact every current-carrying wire in a circuit acts as an inductor! However, this type of “stray” inductance is typically negligible, just as we can usually ignore the stray resistance of our wires and only take into account the actual resistors. To store any appreciable amount of magnetic energy, one usually uses a coil of wire designed specifically to be an inductor. All the loops’ contribution to the magnetic field add together to make a stronger field. Unlike capacitors and resistors, practical inductors are easy to make by hand. One can for instance spool some wire around a short wooden dowel. An inductor like this, in the form cylindrical coil of wire, is called a solenoid, c, and
a stylized solenoid, \( d \), is the symbol used to represent an inductor in a circuit regardless of its actual geometry.

How much energy does an inductor store? The energy density is proportional to the square of the magnetic field strength, which is in turn proportional to the current flowing through the coiled wire, so the energy stored in the inductor must be proportional to \( I^2 \). We write \( L/2 \) for the constant of proportionality, giving

\[
U_L = \frac{L}{2} I^2.
\]

As in the definition of capacitance, we have a factor of \( 1/2 \), which is purely a matter of definition. The quantity \( L \) is called the inductance of the inductor, and we see that its units must be joules per ampere squared. This clumsy combination of units is more commonly abbreviated as the henry, \( 1 \) henry \( = 1 \) J/A\(^2\). Rather than memorizing this definition, it makes more sense to derive it when needed from the definition of inductance. Many people know inductors simply as “coils,” or “choke,” and will not understand you if you refer to an “inductor,” but they will still refer to \( L \) as the “inductance,” not the “coilance” or “chokeance!”

There is a lumped circuit approximation for inductors, just like the one for capacitors (p. 582). For a capacitor, this means assuming that the electric fields are completely internal, so that components only interact via currents that flow through wires, not due to the physical overlapping of their fields in space. Similarly for an inductor, the lumped circuit approximation is the assumption that the magnetic fields are completely internal.

\[\text{Identical inductances in series} \] example 19

If two inductors are placed in series, any current that passes through the combined double inductor must pass through both its parts. If we assume the lumped circuit approximation, the two inductors’ fields don’t interfere with each other, so the energy is doubled for a given current. Thus by the definition of inductance, the inductance is doubled as well. In general, inductances in series add, just like resistances. The same kind of reasoning also shows that the inductance of a solenoid is approximately proportional to its length, assuming the number of turns per unit length is kept constant. (This is only approximately true, because putting two solenoids end-to-end causes the fields just outside their mouths to overlap and add together in a complicated manner. In other words, the lumped-circuit approximation may not be very good.)

\[\text{Identical capacitances in parallel} \] example 20

When two identical capacitances are placed in parallel, any charge deposited at the terminals of the combined double capacitor will divide itself evenly between the two parts. The electric fields sur-
A variable capacitor example 21

Figure h/1 shows the construction of a variable capacitor out of two parallel semicircles of metal. One plate is fixed, while the other can be rotated about their common axis with a knob. The opposite charges on the two plates are attracted to one another, and therefore tend to gather in the overlapping area. This overlapping area, then, is the only area that effectively contributes to the capacitance, and turning the knob changes the capacitance. The simple design can only provide very small capacitance values, so in practice one usually uses a bank of capacitors, wired in parallel, with all the moving parts on the same shaft.

Discussion Questions

A Suppose that two parallel-plate capacitors are wired in parallel, and are placed very close together, side by side, so that the lumped circuit approximation is not very accurate. Will the resulting capacitance be too small, or too big? Could you twist the circuit into a different shape and make the effect be the other way around, or make the effect vanish? How about the case of two inductors in series?

B Most practical capacitors do not have an air gap or vacuum gap between the plates; instead, they have an insulating substance called a dielectric. We can think of the molecules in this substance as dipoles that are free to rotate (at least a little), but that are not free to move around, since it is a solid. The figure shows a highly stylized and unrealistic way of visualizing this. We imagine that all the dipoles are initially turned sideways, (1), and that as the capacitor is charged, they all respond by turning through a certain angle, (2). (In reality, the scene might be much more random, and the alignment effect much weaker.)

For simplicity, imagine inserting just one electric dipole into the vacuum...
gap. For a given amount of charge on the plates, how does this affect the amount of energy stored in the electric field? How does this affect the capacitance?

Now redo the analysis in terms of the mechanical work needed in order to charge up the plates.

10.5.2 Oscillations

Figure j shows the simplest possible oscillating circuit. For any useful application it would actually need to include more components. For example, if it was a radio tuner, it would need to be connected to an antenna and an amplifier. Nevertheless, all the essential physics is there.

We can analyze it without any sweat or tears whatsoever, simply by constructing an analogy with a mechanical system. In a mechanical oscillator, k, we have two forms of stored energy.

\[ U_{spring} = \frac{1}{2} kx^2 \]  
\[ K = \frac{1}{2} mv^2 \]

In the case of a mechanical oscillator, we have usually assumed a friction force of the form that turns out to give the nicest mathematical results, \( F = -bv \). In the circuit, the dissipation of energy into heat occurs via the resistor, with no mechanical force involved, so in order to make the analogy, we need to restate the role of the friction force in terms of energy. The power dissipated by friction equals the mechanical work it does in a time interval \( dt \), divided by \( dt \),

\[ P = \frac{W}{dt} = F \frac{dx}{dt} = Fv = -bv^2 \]

so

\[ \text{rate of heat dissipation} = -bv^2 \]

**self-check F**

Equation (1) has \( x \) squared, and equations (2) and (3) have \( v \) squared. Because they’re squared, the results don’t depend on whether these variables are positive or negative. Does this make physical sense?  

Answer, p. 930

In the circuit, the stored forms of energy are

\[ U_C = \frac{1}{2} Cq^2 \]
\[ U_L = \frac{1}{2} LI^2 \]

and the rate of heat dissipation in the resistor is

\[ \text{rate of heat dissipation} = -RI^2 \]

Comparing the two sets of equations, we first form analogies between quantities that represent the state of the system at some moment.
in time:

\[ x \leftrightarrow q \]
\[ v \leftrightarrow I \]

**self-check G**
How is \( v \) related mathematically to \( x \)? How is \( I \) connected to \( q \)? Are the two relationships analogous?  \( \triangleright \) Answer, p. 930

Next we relate the ones that describe the system’s permanent characteristics:

\[ k \leftrightarrow 1/C \]
\[ m \leftrightarrow L \]
\[ b \leftrightarrow R \]

Since the mechanical system naturally oscillates with a frequency\(^3\)
\[ \omega \approx \sqrt{k/m} \], we can immediately solve the electrical version by analogy, giving

\[ \omega \approx \frac{1}{\sqrt{L/C}} \].

Since the resistance \( R \) is analogous to \( b \) in the mechanical case, we find that the \( Q \) (quality factor, not charge) of the resonance is inversely proportional to \( R \), and the width of the resonance is directly proportional to \( R \).

---

**Tuning a radio receiver**  \( \text{example 22} \)
A radio receiver uses this kind of circuit to pick out the desired station. Since the receiver resonates at a particular frequency, stations whose frequencies are far off will not excite any response in the circuit. The value of \( R \) has to be small enough so that only one station at a time is picked up, but big enough so that the tuner isn’t too touchy. The resonant frequency can be tuned by adjusting either \( L \) or \( C \), but variable capacitors are easier to build than variable inductors.

**A numerical calculation**  \( \text{example 23} \)
The phone company sends more than one conversation at a time over the same wire, which is accomplished by shifting each voice signal into different range of frequencies during transmission. The number of signals per wire can be maximized by making each range of frequencies (known as a bandwidth) as small as possible. It turns out that only a relatively narrow range of frequencies is necessary in order to make a human voice intelligible, so the

\(^3\)As in chapter 2, we use the word “frequency” to mean either \( f \) or \( \omega = 2\pi f \) when the context makes it clear which is being referred to.
If the filter consists of an LRC circuit with a broad resonance centered around 1.0 kHz, and the capacitor is 1 µF (microfarad), what inductance value must be used?

Solving for \( L \), we have

\[
L = \frac{1}{C \omega^2} = \frac{1}{(10^{-6} \text{ F})(2\pi \times 10^3 \text{ s}^{-1})^2} = 2.5 \times 10^{-3} \text{ F}^{-1} \text{s}^2
\]

Checking that these really are the same units as henries is a little tedious, but it builds character:

\[
F^{-1} \text{s}^2 = \left(\frac{C^2}{J}\right)^{-1} \text{s}^2 = J \cdot C^{-2} \text{s}^2 = J/A^2 = H
\]

The result is 25 mH (millihenries).

This is actually quite a large inductance value, and would require a big, heavy, expensive coil. In fact, there is a trick for making this kind of circuit small and cheap. There is a kind of silicon chip called an op-amp, which, among other things, can be used to simulate the behavior of an inductor. The main limitation of the op-amp is that it is restricted to low-power applications.

### 10.5.3 Voltage and current

What is physically happening in one of these oscillating circuits? Let's first look at the mechanical case, and then draw the analogy to the circuit. For simplicity, let's ignore the existence of damping, so there is no friction in the mechanical oscillator, and no resistance in the electrical one.

Suppose we take the mechanical oscillator and pull the mass away from equilibrium, then release it. Since friction tends to resist the spring's force, we might naively expect that having zero friction would allow the mass to leap instantaneously to the equilibrium position. This can't happen, however, because the mass would have to have infinite velocity in order to make such an instantaneous leap. Infinite velocity would require infinite kinetic energy, but the only kind of energy that is available for conversion to kinetic is the energy stored in the spring, and that is finite, not infinite. At each step on its way back to equilibrium, the mass's velocity is controlled exactly by the amount of the spring's energy that has so far been converted into kinetic energy. After the mass reaches equilibrium, it overshoots...
due to its own momentum. It performs identical oscillations on both sides of equilibrium, and it never loses amplitude because friction is not available to convert mechanical energy into heat.

Now with the electrical oscillator, the analog of position is charge. Pulling the mass away from equilibrium is like depositing charges \(+q\) and \(-q\) on the plates of the capacitor. Since resistance tends to resist the flow of charge, we might imagine that with no friction present, the charge would instantly flow through the inductor (which is, after all, just a piece of wire), and the capacitor would discharge instantly. However, such an instant discharge is impossible, because it would require infinite current for one instant. Infinite current would create infinite magnetic fields surrounding the inductor, and these fields would have infinite energy. Instead, the rate of flow of current is controlled at each instant by the relationship between the amount of energy stored in the magnetic field and the amount of current that must exist in order to have that strong a field. After the capacitor reaches \(q = 0\), it overshoots. The circuit has its own kind of electrical “inertia,” because if charge was to stop flowing, there would have to be zero current through the inductor. But the current in the inductor must be related to the amount of energy stored in its magnetic fields. When the capacitor is at \(q = 0\), all the circuit’s energy is in the inductor, so it must therefore have strong magnetic fields surrounding it and quite a bit of current going through it.

The only thing that might seem spooky here is that we used to speak as if the current in the inductor caused the magnetic field, but now it sounds as if the field causes the current. Actually this is symptomatic of the elusive nature of cause and effect in physics. It’s equally valid to think of the cause and effect relationship in either way. This may seem unsatisfying, however, and for example does not really get at the question of what brings about a voltage difference across the resistor (in the case where the resistance is finite); there must be such a voltage difference, because without one, Ohm’s law would predict zero current through the resistor.

Voltage, then, is what is really missing from our story so far.

Let’s start by studying the voltage across a capacitor. Voltage is electrical potential energy per unit charge, so the voltage difference between the two plates of the capacitor is related to the amount by which its energy would increase if we increased the absolute values