Chapter 4
Conservation of Angular Momentum

4.1 Angular momentum in two dimensions

4.1.1 Angular momentum

“Sure, and maybe the sun won’t come up tomorrow.” Of course, the sun only appears to go up and down because the earth spins, so the cliche should really refer to the unlikelihood of the earth’s stopping its rotation abruptly during the night. Why can’t it stop? It wouldn’t violate conservation of momentum, because the earth’s rotation doesn’t add anything to its momentum. While California spins in one direction, some equally massive part of India goes the opposite way, canceling its momentum. A halt to Earth’s rotation would entail a drop in kinetic energy, but that energy could simply be converted into some other form, such as heat.

The jumper can’t move his legs counterclockwise without moving his arms clockwise. (Thomas Eakins.)

Other examples along these lines are not hard to find. An atom spins at the same rate for billions of years. A high-diver who is rotating when he comes off the board does not need to make any physical effort to continue rotating, and indeed would be unable to stop rotating before he hit the water.

These observations have the hallmarks of a conservation law:

A closed system is involved. Nothing is making an effort to twist the earth, the hydrogen atom, or the high-diver. They are isolated from rotation-changing influences, i.e., they are closed systems.

Something remains unchanged. There appears to be a numerical quantity for measuring rotational motion such that the total amount of that quantity remains constant in a closed system.
Something can be transferred back and forth without changing the total amount. In the photo of the old-fashioned high jump, a, the jumper wants to get his feet out in front of him so he can keep from doing a “face plant” when he lands. Bringing his feet forward would involve a certain quantity of counterclockwise rotation, but he didn’t start out with any rotation when he left the ground. Suppose we consider counterclockwise as positive and clockwise as negative. The only way his legs can acquire some positive rotation is if some other part of his body picks up an equal amount of negative rotation. This is why he swings his arms up behind him, clockwise.

What numerical measure of rotational motion is conserved? Car engines and old-fashioned LP records have speeds of rotation measured in rotations per minute (r.p.m.), but the number of rotations per minute (or per second) is not a conserved quantity. A twirling figure skater, for instance, can pull her arms in to increase her r.p.m.’s. The first section of this chapter deals with the numerical definition of the quantity of rotation that results in a valid conservation law.

When most people think of rotation, they think of a solid object like a wheel rotating in a circle around a fixed point. Examples of this type of rotation, called rigid rotation or rigid-body rotation, include a spinning top, a seated child’s swinging leg, and a helicopter’s spinning propeller. Rotation, however, is a much more general phenomenon, and includes noncircular examples such as a comet in an elliptical orbit around the sun, or a cyclone, in which the core completes a circle more quickly than the outer parts.

If there is a numerical measure of rotational motion that is a conserved quantity, then it must include nonrigid cases like these, since nonrigid rotation can be traded back and forth with rigid rotation. For instance, there is a trick for finding out if an egg is raw or hardboiled. If you spin a hardboiled egg and then stop it briefly with your finger, it stops dead. But if you do the same with a raw egg, it springs back into rotation because the soft interior was still swirling around within the momentarily motionless shell. The pattern of flow of the liquid part is presumably very complex and nonuniform due to the asymmetric shape of the egg and the different consistencies of the yolk and the white, but there is apparently some way to describe the liquid’s total amount of rotation with a single number, of which some percentage is given back to the shell when you release it.

The best strategy is to devise a way of defining the amount of rotation of a single small part of a system. The amount of rotation of a system such as a cyclone will then be defined as the total of all the contributions from its many small parts.

The quest for a conserved quantity of rotation even requires us to broaden the rotation concept to include cases where the motion...
doesn’t repeat or even curve around. If you throw a piece of putty at a door, b, the door will recoil and start rotating. The putty was traveling straight, not in a circle, but if there is to be a general conservation law that can cover this situation, it appears that we must describe the putty as having had some “rotation,” which it then gave up to the door. The best way of thinking about it is to attribute rotation to any moving object or part of an object that changes its angle in relation to the axis of rotation. In the putty-and-door example, the hinge of the door is the natural point to think of as an axis, and the putty changes its angle as seen by someone standing at the hinge, c. For this reason, the conserved quantity we are investigating is called angular momentum. The symbol for angular momentum can’t be “a” or “m,” since those are used for acceleration and mass, so the letter $L$ is arbitrarily chosen instead.

Imagine a 1 kg blob of putty, thrown at the door at a speed of 1 m/s, which hits the door at a distance of 1 m from the hinge. We define this blob to have 1 unit of angular momentum. When it hits the door, the door will recoil and start rotating. We can use the speed at which the door recoils as a measure of the angular momentum the blob brought in.\(^1\)

Experiments show, not surprisingly, that a 2 kg blob thrown in the same way makes the door rotate twice as fast, so the angular momentum of the putty blob must be proportional to mass,

$$L \propto m.$$  

Similarly, experiments show that doubling the velocity of the blob will have a doubling effect on the result, so its angular momentum must be proportional to its velocity as well,

$$L \propto mv.$$  

You have undoubtedly had the experience of approaching a closed door with one of those bar-shaped handles on it and pushing on the wrong side, the side close to the hinges. You feel like an idiot, because you have so little leverage that you can hardly budge the door. The same would be true with the putty blob. Experiments would show that the amount of rotation the blob can give to the door is proportional to the distance, $r$, from the axis of rotation, so angular momentum must be proportional to $r$ as well,

$$L \propto mvr.$$  

We are almost done, but there is one missing ingredient. We know on grounds of symmetry that a putty ball thrown directly

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\(^1\)We assume that the door is much more massive than the blob. Under this assumption, the speed at which the door recoils is much less than the original speed of the blob, so the blob has lost essentially all its angular momentum, and given it to the door.
Only the component of the velocity vector perpendicular to the line connecting the object to the axis should be counted into the definition of angular momentum.

More generally, \( v_\perp \) should be thought of as the component of the object’s velocity vector that is perpendicular to the line joining the object to the axis of rotation.

We find that this equation agrees with the definition of the original putty blob as having one unit of angular momentum, and we can now see that the units of angular momentum are \((\text{kg} \cdot \text{m/s}) \cdot \text{m}\), i.e., \(\text{kg} \cdot \text{m}^2/\text{s}\). Summarizing, we have

\[
L = mv_\perp r
\]

where \( m \) is the particle’s mass, \( v_\perp \) is the component of its velocity vector perpendicular to the line joining it to the axis of rotation, and \( r \) is its distance from the axis. (Note that \( r \) is not necessarily the radius of a circle.) Positive and negative signs of angular momentum are used to describe opposite directions of rotation. The angular momentum of a finite-sized object or a system of many objects is found by dividing it up into many small parts, applying the equation to each part, and adding to find the total amount of angular momentum. (As implied by the word “particle,” matter isn’t the only thing that can have angular momentum. Light can also have angular momentum, and the above equation would not apply to light.)

Conservation of angular momentum has been verified over and over again by experiment, and is now believed to be one of the most fundamental principles of physics, along with conservation of mass, energy, and momentum.
Earth’s slowing rotation and the receding moon example 2

The earth’s rotation is actually slowing down very gradually, with the kinetic energy being dissipated as heat by friction between the land and the tidal bulges raised in the seas by the earth’s gravity. Does this mean that angular momentum is not really perfectly conserved? No, it just means that the earth is not quite a closed system by itself. If we consider the earth and moon as a system, then the angular momentum lost by the earth must be gained by the moon somehow. In fact very precise measurements of the distance between the earth and the moon have been carried out by bouncing laser beams off of a mirror left there by astronauts, and these measurements show that the moon is receding from the earth at a rate of 4 centimeters per year! The moon’s greater value of $r$ means that it has a greater angular momentum, and the increase turns out to be exactly the amount lost by the earth. In the days of the dinosaurs, the days were significantly shorter, and the moon was closer and appeared bigger in the sky.

But what force is causing the moon to speed up, drawing it out into a larger orbit? It is the gravitational forces of the earth’s tidal bulges. In figure g, the earth’s rotation is counterclockwise (arrow). The moon’s gravity creates a bulge on the side near it, because its gravitational pull is stronger there, and an “anti-bulge” on the far side, since its gravity there is weaker. For simplicity, let’s focus on the tidal bulge closer to the moon. Its frictional force is trying to slow down the earth’s rotation, so its force on the earth’s solid crust is toward the bottom of the figure. By Newton’s third law, the crust must thus make a force on the bulge which is toward the top of the figure. This causes the bulge to be pulled forward at a slight angle, and the bulge’s gravity therefore pulls the moon forward, accelerating its orbital motion about the earth and flinging it outward.

The result would obviously be extremely difficult to calculate directly, and this is one of those situations where a conservation law allows us to make precise quantitative statements about the outcome of a process when the calculation of the process itself would be prohibitively complex.

Restriction to rotation in a plane

Is angular momentum a vector, or a scalar? It does have a direction in space, but it’s a direction of rotation, not a straight-line direction like the directions of vectors such as velocity or force. It turns out that there is a way of defining angular momentum as a vector, but in this section the examples will be confined to a single plane of rotation, i.e., effectively two-dimensional situations. In this special case, we can choose to visualize the plane of rotation from one side or the other, and to define clockwise and counterclockwise rotation as having opposite signs of angular momentum. “Effec-
tively” two-dimensional means that we can deal with objects that aren’t flat, as long as the velocity vectors of all their parts lie in a plane.

Discussion Questions

A Conservation of plain old momentum, $p$, can be thought of as the greatly expanded and modified descendant of Galileo’s original principle of inertia, that no force is required to keep an object in motion. The principle of inertia is counterintuitive, and there are many situations in which it appears superficially that a force is needed to maintain motion, as maintained by Aristotle. Think of a situation in which conservation of angular momentum, $L$, also seems to be violated, making it seem incorrectly that something external must act on a closed system to keep its angular momentum from “running down.”

4.1.2 Application to planetary motion

We now discuss the application of conservation of angular momentum to planetary motion, both because of its intrinsic importance and because it is a good way to develop a visual intuition for angular momentum.

Kepler’s law of equal areas states that the area swept out by a planet in a certain length of time is always the same. Angular momentum had not been invented in Kepler’s time, and he did not even know the most basic physical facts about the forces at work. He thought of this law as an entirely empirical and unexpectedly simple way of summarizing his data, a rule that succeeded in describing and predicting how the planets sped up and slowed down in their elliptical paths. It is now fairly simple, however, to show that the equal area law amounts to a statement that the planet’s angular momentum stays constant.

There is no simple geometrical rule for the area of a pie wedge cut out of an ellipse, but if we consider a very short time interval, as shown in figure h, the shaded shape swept out by the planet is very nearly a triangle. We do know how to compute the area of a triangle. It is one half the product of the base and the height:

\[ \text{area} = \frac{1}{2}bh. \]

We wish to relate this to angular momentum, which contains the variables $r$ and $v_\perp$. If we consider the sun to be the axis of rotation, then the variable $r$ is identical to the base of the triangle, $r = b$. Referring to the magnified portion of the figure, $v_\perp$ can be related to $h$, because the two right triangles are similar:

\[ \frac{h}{\text{distance traveled}} = \frac{v_\perp}{|v|}. \]

The area can thus be rewritten as

\[ \text{area} = \frac{1}{2}r v_\perp (\text{distance traveled}) \frac{1}{|v|}. \]
The distance traveled equals $|v| \Delta t$, so this simplifies to

$$\text{area} = \frac{1}{2} r v_\perp \Delta t.$$ 

We have found the following relationship between angular momentum and the rate at which area is swept out:

$$L = 2m \frac{\text{area}}{\Delta t}.$$ 

The factor of 2 in front is simply a matter of convention, since any conserved quantity would be an equally valid conserved quantity if you multiplied it by a constant. The factor of $m$ was not relevant to Kepler, who did not know the planets’ masses, and who was only describing the motion of one planet at a time.

We thus find that Kepler’s equal-area law is equivalent to a statement that the planet’s angular momentum remains constant. But wait, why should it remain constant? — the planet is not a closed system, since it is being acted on by the sun’s gravitational force. There are two valid answers. The first is that it is actually the total angular momentum of the sun plus the planet that is conserved. The sun, however, is millions of times more massive than the typical planet, so it accelerates very little in response to the planet’s gravitational force. It is thus a good approximation to say that the sun doesn’t move at all, so that no angular momentum is transferred between it and the planet.

The second answer is that to change the planet’s angular momentum requires not just a force but a force applied in a certain way. Later in this section (starting on page 260) we discuss the transfer of angular momentum by a force, but the basic idea here is that a force directly in toward the axis does not change the angular momentum.

**Discussion Questions**

A Suppose an object is simply traveling in a straight line at constant speed. If we pick some point not on the line and call it the axis of rotation, is area swept out by the object at a constant rate?

B The figure is a strobe photo of a pendulum bob, taken from underneath the pendulum looking straight up. The black string can’t be seen in the photograph. The bob was given a slight sideways push when it was released, so it did not swing in a plane. The bright spot marks the center, i.e., the position the bob would have if it hung straight down at us. Does the bob’s angular momentum appear to remain constant if we consider the center to be the axis of rotation?

4.1.3 **Two theorems about angular momentum**

With plain old momentum, $\mathbf{p}$, we had the freedom to work in any inertial frame of reference we liked. The same object could have different values of momentum in two different frames, if the
Two asteroids collide.

Everyone has a strong tendency to think of the diver as rotating about his own center of mass. However, he is flying in an arc, and he also has angular momentum because of this motion.

This rigid object has angular momentum both because it is spinning about its center of mass and because it is moving through space.

Colliding asteroids described with different axes example 3

Observers on planets A and B both see the two asteroids colliding. The asteroids are of equal mass and their impact speeds are the same. Astronomers on each planet decide to define their own planet as the axis of rotation. Planet A is twice as far from the collision as planet B. The asteroids collide and stick. For simplicity, assume planets A and B are both at rest.

With planet A as the axis, the two asteroids have the same amount of angular momentum, but one has positive angular momentum and the other has negative. Before the collision, the total angular momentum is therefore zero. After the collision, the two asteroids will have stopped moving, and again the total angular momentum is zero. The total angular momentum both before and after the collision is zero, so angular momentum is conserved if you choose planet A as the axis.

The only difference with planet B as axis is that $r$ is smaller by a factor of two, so all the angular momenta are halved. Even though the angular momenta are different than the ones calculated by planet A, angular momentum is still conserved.

The earth spins on its own axis once a day, but simultaneously travels in its circular one-year orbit around the sun, so any given part of it traces out a complicated loopy path. It would seem difficult
to calculate the earth’s angular momentum, but it turns out that there is an intuitively appealing shortcut: we can simply add up the angular momentum due to its spin plus that arising from its center of mass’s circular motion around the sun. This is a special case of the following general theorem:

**The spin theorem:** An object’s angular momentum with respect to some outside axis A can be found by adding up two parts:

1. The first part is the object’s angular momentum found by using its own center of mass as the axis, i.e., the angular momentum the object has because it is spinning.
2. The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

A system with its center of mass at rest example 4

In the special case of an object whose center of mass is at rest, the spin theorem implies that the object’s angular momentum is the same regardless of what axis we choose. (This is an even stronger statement than the choice of axis theorem, which only guarantees that angular momentum is conserved for any given choice of axis, without specifying that it is the same for all such choices.)

Angular momentum of a rigid object example 5

A motorcycle wheel has almost all its mass concentrated at the outside. If the wheel has mass \( m \) and radius \( r \), and the time required for one revolution is \( T \), what is the spin part of its angular momentum?

This is an example of the commonly encountered special case of rigid motion, as opposed to the rotation of a system like a hurricane in which the different parts take different amounts of time to go around. We don’t really have to go through a laborious process of adding up contributions from all the many parts of a wheel, because they are all at about the same distance from the axis, and are all moving around the axis at about the same speed. The velocity is all perpendicular to the spokes,

\[
v_\perp = \frac{\text{circumference}}{T} = \frac{2\pi r}{T}
\]

and the angular momentum of the wheel about its center is

\[
L = mv_\perp r = m\left(\frac{2\pi r}{T}\right)r = 2\pi mr^2/T.
\]

Note that although the factors of \( 2\pi \) in this expression is peculiar to a wheel with its mass concentrated on the rim, the proportionality to \( m/T \) would have been the same for any other rigidly rotating...
object. Although an object with a noncircular shape does not have a radius, it is also true in general that angular momentum is proportional to the square of the object’s size for fixed values of \(m\) and \(T\). For instance doubling an object’s size doubles both the \(v_\perp\) and \(r\) factors in the contribution of each of its parts to the total angular momentum, resulting in an overall factor of four increase.

### 4.1.4 Torque

Force is the rate of transfer of momentum. The corresponding quantity in the case of angular momentum is called torque (rhymes with “fork”). Where force tells us how hard we are pushing or pulling on something, torque indicates how hard we are twisting on it. Torque is represented by the Greek letter tau, \(\tau\), and the rate of change of an object’s angular momentum equals the total torque acting on it,

\[
\tau_{\text{total}} = \frac{dL}{dt}.
\]

As with force and momentum, it often happens that angular momentum recedes into the background and we focus our interest on the torques. The torque-focused point of view is exemplified by the fact that many scientifically untrained but mechanically apt people know all about torque, but none of them have heard of angular momentum. Car enthusiasts eagerly compare engines’ torques, and there is a tool called a torque wrench which allows one to apply a desired amount of torque to a screw and avoid overtightening it.

**Torque distinguished from force**

Of course a force is necessary in order to create a torque — you can’t twist a screw without pushing on the wrench — but force and torque are two different things. One distinction between them is direction. We use positive and negative signs to represent forces in the two possible directions along a line. The direction of a torque, however, is clockwise or counterclockwise, not a linear direction.

The other difference between torque and force is a matter of leverage. A given force applied at a door’s knob will change the door’s angular momentum twice as rapidly as the same force applied halfway between the knob and the hinge. The same amount of force produces different amounts of torque in these two cases.

It’s possible to have a zero total torque with a nonzero total force. An airplane with four jet engines would be designed so that their forces are balanced on the left and right. Their forces are all in the same direction, but the clockwise torques of two of the engines are canceled by the counterclockwise torques of the other two, giving zero total torque.

Conversely we can have zero total force and nonzero total torque. A merry-go-round’s engine needs to supply a nonzero torque on it to bring it up to speed, but there is zero total force on it. If there
was not zero total force on it, its center of mass would accelerate!

**Relationship between force and torque**

How do we calculate the amount of torque produced by a given force? Since it depends on leverage, we should expect it to depend on the distance between the axis and the point of application of the force. I’ll work out an equation relating torque to force for a particular very simple situation, and give a more rigorous derivation on page 290, after developing some mathematical techniques that dramatically shorten and simplify the proof.

Consider a pointlike object which is initially at rest at a distance $r$ from the axis we have chosen for defining angular momentum. We first observe that a force directly inward or outward, along the line connecting the axis to the object, does not impart any angular momentum to the object.

A force perpendicular to the line connecting the axis and the object does, however, make the object pick up angular momentum. Newton’s second law gives

$$a = F/m,$$

and using $a = dv/dt$ we find the velocity the object acquires after a time $dt$,

$$dv = F dt/m.$$

We’re trying to relate force to a change in angular momentum, so we multiply both sides of the equation by $mr$ to give

$$m dv r = F dt r$$

$$dL = F dt r.$$

Dividing by $dt$ gives the torque:

$$\frac{dL}{dt} = Fr$$

$$\tau = Fr.$$

If a force acts at an angle other than 0 or 90° with respect to the line joining the object and the axis, it would be only the component of the force perpendicular to the line that would produce a torque,

$$\tau = F_\perp r.$$

Although this result was proved under a simplified set of circumstances, it is more generally valid:\(^2\)

**Relationship between force and torque:** The rate at which a force transfers angular momentum to an object, i.e., the torque produced by the force, is given by

$$|\tau| = r|F_\perp|,$$

\(^2\)A proof is given in example 28 on page 290.
where \( r \) is the distance from the axis to the point of application of the force, and \( F_\perp \) is the component of the force that is perpendicular to the line joining the axis to the point of application.

The equation is stated with absolute value signs because the positive and negative signs of force and torque indicate different things, so there is no useful relationship between them. The sign of the torque must be found by physical inspection of the case at hand.

From the equation, we see that the units of torque can be written as newtons multiplied by meters. Metric torque wrenches are calibrated in N\( \cdot \)m, but American ones use foot-pounds, which is also a unit of distance multiplied by a unit of force. We know from our study of mechanical work that newtons multiplied by meters equal joules, but torque is a completely different quantity from work, and nobody writes torques with units of joules, even though it would be technically correct.

**Self-check.**

**Self-check A**

Compare the magnitudes and signs of the four torques shown in figure p. 

> Answer, p. 1056

**How torque depends on the direction of the force** example 6

> How can the torque applied to the wrench in the figure be expressed in terms of \( r \), \( |F| \), and the angle \( \theta \)?

> The force vector and its \( F_\perp \) component form the hypotenuse and one leg of a right triangle,

and the interior angle opposite to \( F_\perp \) equals \( \theta \). The absolute value of \( F_\perp \) can thus be expressed as

\[
F_\perp = |F| \sin \theta,
\]

leading to

\[
|\tau| = r|F| \sin \theta.
\]
Sometimes torque can be more neatly visualized in terms of the quantity $r_\perp$ shown in the figure on the left, which gives us a third way of expressing the relationship between torque and force:

$$|\tau| = r_\perp |F|.$$  

Of course you wouldn’t want to go and memorize all three equations for torque. Starting from any one of them you could easily derive the other two using trigonometry. Familiarizing yourself with them can however clue you in to easier avenues of attack on certain problems.

**The torque due to gravity**

Up until now we’ve been thinking in terms of a force that acts at a single point on an object, such as the force of your hand on the wrench. This is of course an approximation, and for an extremely realistic calculation of your hand’s torque on the wrench you might need to add up the torques exerted by each square millimeter where your skin touches the wrench. This is seldom necessary. But in the case of a gravitational force, there is never any single point at which the force is applied. Our planet is exerting a separate tug on every brick in the Leaning Tower of Pisa, and the total gravitational torque on the tower is the sum of the torques contributed by all the little forces. Luckily there is a trick that allows us to avoid such a massive calculation. It turns out that for purposes of computing the total gravitational torque on an object, you can get the right answer by just pretending that the whole gravitational force acts at the object’s center of mass.

**Gravitational torque on an outstretched arm** example 7

▷ Your arm has a mass of 3.0 kg, and its center of mass is 30 cm from your shoulder. What is the gravitational torque on your arm when it is stretched out horizontally to one side, taking the shoulder to be the axis?

▷ The total gravitational force acting on your arm is

$$|F| = (3.0 \text{ kg})(9.8 \text{ m/s}^2) = 29 \text{ N}.$$  

For the purpose of calculating the gravitational torque, we can treat the force as if it acted at the arm’s center of mass. The force is straight down, which is perpendicular to the line connecting the shoulder to the center of mass, so

$$F_\perp = |F| = 29 \text{ N}.$$  

Continuing to pretend that the force acts at the center of the arm, $r$ equals 30 cm = 0.30 m, so the torque is

$$\tau = r F_\perp = 9 \text{ N} \cdot \text{m}.$$
Discussion Questions

A This series of discussion questions deals with past students’ incorrect reasoning about the following problem.

Suppose a comet is at the point in its orbit shown in the figure. The only force on the comet is the sun’s gravitational force. Throughout the question, define all torques and angular momenta using the sun as the axis.

(1) Is the sun producing a nonzero torque on the comet? Explain.

(2) Is the comet’s angular momentum increasing, decreasing, or staying the same? Explain.

Explain what is wrong with the following answers. In some cases, the answer is correct, but the reasoning leading up to it is wrong.

(a) Incorrect answer to part (1): “Yes, because the sun is exerting a force on the comet, and the comet is a certain distance from the sun.”

(b) Incorrect answer to part (1): “No, because the torques cancel out.”

(c) Incorrect answer to part (2): “Increasing, because the comet is speeding up.”

B You whirl a rock over your head on the end of a string, and gradually pull in the string, eventually cutting the radius in half. What happens to the rock’s angular momentum? What changes occur in its speed, the time required for one revolution, and its acceleration? Why might the string break?

C A helicopter has, in addition to the huge fan blades on top, a smaller propeller mounted on the tail that rotates in a vertical plane. Why?

D Which claw hammer would make it easier to get the nail out of the wood if the same force was applied in the same direction?

E The photo shows an amusement park ride whose two cars rotate in opposite directions. Why is this a good design?

4.1.5 Applications to statics

In chapter 2 I defined equilibrium as a situation where the interaction energy is minimized. This is the same as a condition of zero total force, or constant momentum. Thus a car is in equilibrium not just when it is parked but also when it is cruising down a straight road with constant momentum.

Likewise there are many cases where a system is not closed but maintains constant angular momentum. When a merry-go-round is running at constant angular momentum, the engine’s torque is
being canceled by the torque due to friction.

It’s not enough for a boat not to sink — we’d also like to avoid having it capsize. For this reason, we now redefine equilibrium as follows.

When an object has constant momentum and constant angular momentum, we say that it is in equilibrium. Again, this is a scientific redefinition of the common English word, since in ordinary speech nobody would describe a car spinning out on an icy road as being in equilibrium.

Very commonly, however, we are interested in cases where an object is not only in equilibrium but also at rest, and this corresponds more closely to the usual meaning of the word. Statics is the branch of physics concerned with problems such as these.

Solving statics problems is now simply a matter of applying and combining some things you already know:

- You know the behaviors of the various types of forces, for example that a frictional force is always parallel to the surface of contact.
- You know about vector addition of forces. It is the vector sum of the forces that must equal zero to produce equilibrium.
- You know about torque. The total torque acting on an object must be zero if it is to be in equilibrium.
- You know that the choice of axis is arbitrary, so you can make a choice of axis that makes the problem easy to solve.

In general, this type of problem could involve four equations in four unknowns: three equations that say the force components add up to zero, and one equation that says the total torque is zero. Most cases you’ll encounter will not be this complicated. In the example below, only the equation for zero total torque is required in order to get an answer.

*A flagpole example 8

A 10-kg flagpole is being held up by a lightweight horizontal cable, and is propped against the foot of a wall as shown in the figure. If the cable is only capable of supporting a tension of 70 N, how great can the angle $\alpha$ be without breaking the cable?

All three objects in the figure are supposed to be in equilibrium: the pole, the cable, and the wall. Whichever of the three objects we pick to investigate, all the forces and torques on it have to cancel out. It is not particularly helpful to analyze the forces and torques on the wall, since it has forces on it from the ground that are not given and that we don’t want to find. We could study the forces and torques on the cable, but that doesn’t let us use the given information about the pole. The object we need to analyze is the pole.
The pole has three forces on it, each of which may also result in a torque: (1) the gravitational force, (2) the cable’s force, and (3) the wall’s force.

We are free to define an axis of rotation at any point we wish, and it is helpful to define it to lie at the bottom end of the pole, since by that definition the wall’s force on the pole is applied at \( r = 0 \) and thus makes no torque on the pole. This is good, because we don’t know what the wall’s force on the pole is, and we are not trying to find it.

With this choice of axis, there are two nonzero torques on the pole, a counterclockwise torque from the cable and a clockwise torque from gravity. Choosing to represent counterclockwise torques as positive numbers, and using the equation \( |\tau| = r|F| \sin \theta \), we have

\[
|\tau|_{\text{cable}} = r_{\text{cable}}|F_{\text{cable}}| \sin \theta_{\text{cable}} - r_{\text{grav}}|F_{\text{grav}}| \sin \theta_{\text{grav}} = 0.
\]

A little geometry gives \( \theta_{\text{cable}} = 90^\circ - \alpha \) and \( \theta_{\text{grav}} = \alpha \), so

\[
|\tau|_{\text{cable}} = r_{\text{cable}}|F_{\text{cable}}| \sin(90^\circ - \alpha) - r_{\text{grav}}|F_{\text{grav}}| \sin \alpha = 0.
\]

The gravitational force can be considered as acting at the pole’s center of mass, i.e., at its geometrical center, so \( r_{\text{cable}} \) is twice \( r_{\text{grav}} \), and we can simplify the equation to read

\[
2|F_{\text{cable}}| \sin(90^\circ - \alpha) - |F_{\text{grav}}| \sin \alpha = 0.
\]

These are all quantities we were given, except for \( \alpha \), which is the angle we want to find. To solve for \( \alpha \) we need to use the trig identity \( \sin(90^\circ - x) = \cos x \),

\[
2|F_{\text{cable}}| \cos \alpha - |F_{\text{grav}}| \sin \alpha = 0,
\]

which allows us to find

\[
\tan \alpha = 2\frac{|F_{\text{cable}}|}{|F_{\text{grav}}|},
\]

\[
\alpha = \tan^{-1} \left( 2\frac{|F_{\text{cable}}|}{|F_{\text{grav}}|} \right)
\]

\[
= \tan^{-1} \left( 2 \times \frac{70 \text{ N}}{98 \text{ N}} \right)
\]

\[
= 55^\circ.
\]
Example 9.

Stable and unstable equilibria.

The dancer’s equilibrium is unstable. If she didn’t constantly make tiny adjustments, she would tip over.

Art! example 9

The abstract sculpture shown in figure x contains a cube of mass \( m \) and sides of length \( b \). The cube rests on top of a cylinder, which is off-center by a distance \( a \). Find the tension in the cable.

There are four forces on the cube: a gravitational force \( mg \), the force \( F_T \) from the cable, the upward normal force from the cylinder, \( F_N \), and the horizontal static frictional force from the cylinder, \( F_s \).

The total force on the cube in the vertical direction is zero:

\[
F_N - mg = 0.
\]

As our axis for defining torques, let’s choose the center of the cube. The cable’s torque is counterclockwise, the torque due to \( F_N \) clockwise. Letting counterclockwise torques be positive, and using the convenient equation \( \tau = r_\perp F \), we find the equation for the total torque:

\[
bF_T - aF_N = 0.
\]

We could also write down the equation saying that the total horizontal force is zero, but that would bring in the cylinder’s frictional force on the cube, which we don’t know and don’t need to find. We already have two equations in the two unknowns \( F_T \) and \( F_N \), so there’s no need to make it into three equations in three unknowns. Solving the first equation for \( F_N = mg \), we then substitute into the second equation to eliminate \( F_N \), and solve for \( F_T = (a/b)mg \).

As a check, our result makes sense when \( a = 0 \); the cube is balanced on the cylinder, so the cable goes slack.

Why is one equilibrium stable and another unstable? Try pushing your own nose to the left or the right. If you push it a millimeter to the left, it responds with a gentle force to the right. If you push it a centimeter to the left, its force on your finger becomes much stronger. The defining characteristic of a stable equilibrium is that the farther the object is moved away from equilibrium, the stronger the force is that tries to bring it back.

The opposite is true for an unstable equilibrium. In the top figure, the ball resting on the round hill theoretically has zero total force on it when it is exactly at the top. But in reality the total force will not be exactly zero, and the ball will begin to move off to one side. Once it has moved, the net force on the ball is greater than it was, and it accelerates more rapidly. In an unstable equilibrium, the farther the object gets from equilibrium, the stronger the force that pushes it farther from equilibrium.

This idea can be rephrased in terms of energy. The difference between the stable and unstable equilibria shown in figure y is that in the stable equilibrium, the energy is at a minimum, and moving
to either side of equilibrium will increase it, whereas the unstable equilibrium represents a maximum.

Note that we are using the term “stable” in a weaker sense than in ordinary speech. A domino standing upright is stable in the sense we are using, since it will not spontaneously fall over in response to a sneeze from across the room or the vibration from a passing truck. We would only call it unstable in the technical sense if it could be toppled by any force, no matter how small. In everyday usage, of course, it would be considered unstable, since the force required to topple it is so small.

An application of calculus example 10

Nancy Neutron is living in a uranium nucleus that is undergoing fission. Nancy’s nuclear energy as a function of position can be approximated by $U = x^4 - x^2$, where all the units and numerical constants have been suppressed for simplicity. Use calculus to locate the equilibrium points, and determine whether they are stable or unstable.

The equilibrium points occur where the $U$ is at a minimum or maximum, and minima and maxima occur where the derivative (which equals minus the force on Nancy) is zero. This derivative is $\frac{dU}{dx} = 4x^3 - 2x$, and setting it equal to zero, we have $x = 0, \pm 1/\sqrt{2}$. Minima occur where the second derivative is positive, and maxima where it is negative. The second derivative is $12x^2 - 2$, which is negative at $x = 0$ (unstable) and positive at $x = \pm 1/\sqrt{2}$ (stable). Interpretation: the graph of $U$ is shaped like a rounded letter ‘W,’ with the two troughs representing the two halves of the splitting nucleus. Nancy is going to have to decide which half she wants to go with.

4.1.6 Proof of Kepler’s elliptical orbit law

Kepler determined purely empirically that the planets’ orbits were ellipses, without understanding the underlying reason in terms of physical law. Newton’s proof of this fact based on his laws of motion and law of gravity was considered his crowning achievement both by him and by his contemporaries, because it showed that the same physical laws could be used to analyze both the heavens and the earth. Newton’s proof was very lengthy, but by applying the more recent concepts of conservation of energy and angular momentum we can carry out the proof quite simply and succinctly. This subsection can be skipped without losing the continuity of the text.

The basic idea of the proof is that we want to describe the shape of the planet’s orbit with an equation, and then show that this equation is exactly the one that represents an ellipse. Newton’s original proof had to be very complicated because it was based directly on his laws of motion, which include time as a variable. To make any statement about the shape of the orbit, he had to eliminate time
from his equations, leaving only space variables. But conservation laws tell us that certain things don’t change over time, so they have already had time eliminated from them.

There are many ways of representing a curve by an equation, of which the most familiar is \( y = ax + b \) for a line in two dimensions. It would be perfectly possible to describe a planet’s orbit using an \( x\)-\( y \) equation like this, but remember that we are applying conservation of angular momentum, and the space variables that occur in the equation for angular momentum are the distance from the axis, \( r \), and the angle between the velocity vector and the \( r \) vector, which we will call \( \varphi \). The planet will have \( \varphi = 90^\circ \) when it is moving perpendicular to the \( r \) vector, i.e., at the moments when it is at its smallest or greatest distances from the sun. When \( \varphi \) is less than \( 90^\circ \) the planet is approaching the sun, and when it is greater than \( 90^\circ \) it is receding from it. Describing a curve with an \( r\)-\( \varphi \) equation is like telling a driver in a parking lot a certain rule for what direction to steer based on the distance from a certain streetlight in the middle of the lot.

The proof is broken into the three parts for easier digestion. The first part is a simple and intuitively reasonable geometrical fact about ellipses, whose proof we relegate to the caption of figure ac; you will not be missing much if you merely absorb the result without reading the proof.

(1) If we use one of the two foci of an ellipse as an axis for defining the variables \( r \) and \( \varphi \), then the angle between the tangent line and the line drawn to the other focus is the same as \( \varphi \), i.e., the two angles labeled \( \varphi \) in the figure are in fact equal.

The other two parts form the meat of our proof. We state the results first and then prove them.

(2) A planet, moving under the influence of the sun’s gravity with less than the energy required to escape, obeys an equation of the form
\[
\sin \varphi = \frac{1}{\sqrt{-pr^2 + qr}},
\]
where \( p \) and \( q \) are positive constants that depend on the planet’s energy and angular momentum and \( p \) is greater than zero.

(3) A curve is an ellipse if and only if its \( r\)-\( \varphi \) equation is of the form
\[
\sin \varphi = \frac{1}{\sqrt{-pr^2 + qr}},
\]
where \( p \) and \( q \) are positive constants that depend on the size and shape of the ellipse.

Proof of part (2)

The component of the planet’s velocity vector that is perpendicular to the \( r \) vector is \( v_\perp = v \sin \varphi \), so conservation of angular
momentum tells us that $L = mrv \sin \varphi$ is a constant. Since the planet’s mass is a constant, this is the same as the condition

$$rv \sin \varphi = \text{constant}.$$  

Conservation of energy gives

$$\frac{1}{2}mv^2 - G\frac{Mm}{r} = \text{constant}.$$  

We solve the first equation for $v$ and plug into the second equation to eliminate $v$. Straightforward algebra then leads to the equation claimed above, with the constant $p$ being positive because of our assumption that the planet’s energy is insufficient to escape from the sun, i.e., its total energy is negative.

**Proof of part (3)**

We define the quantities $\alpha$, $d$, and $s$ as shown in figure ad. The law of cosines gives

$$d^2 = r^2 + s^2 - 2rs \cos \alpha.$$  

Using $\alpha = 180^\circ - 2\varphi$ and the trigonometric identities $\cos(180^\circ - x) = -\cos x$ and $\cos 2x = 1 - 2 \sin^2 x$, we can rewrite this as

$$d^2 = r^2 + s^2 - 2rs \left(2\sin^2 \varphi - 1\right).$$  

Straightforward algebra transforms this into

$$\sin \varphi = \sqrt{\frac{(r + s)^2 - d^2}{4rs}}.$$  

Since $r + s$ is constant, the top of the fraction is constant, and the denominator can be rewritten as $4rs = 4r(\text{constant} - r)$, which is equivalent to the desired form.
4.2 Rigid-body rotation

4.2.1 Kinematics

When a rigid object rotates, every part of it (every atom) moves in a circle, covering the same angle in the same amount of time, a. Every atom has a different velocity vector, b. Since all the velocities are different, we can’t measure the speed of rotation of the top by giving a single velocity. We can, however, specify its speed of rotation consistently in terms of angle per unit time. Let the position of some reference point on the top be denoted by its angle $\theta$, measured in a circle around the axis. For reasons that will become more apparent shortly, we measure all our angles in radians. Then the change in the angular position of any point on the top can be written as $d\theta$, and all parts of the top have the same value of $d\theta$ over a certain time interval $dt$. We define the angular velocity, $\omega$ (Greek omega),

$$\omega = \frac{d\theta}{dt},$$

[definition of angular velocity; $\theta$ in units of radians]

which is similar to, but not the same as, the quantity $\omega$ we defined earlier to describe vibrations. The relationship between $\omega$ and $t$ is exactly analogous to that between $x$ and $t$ for the motion of a particle through space.

**self-check B**

If two different people chose two different reference points on the top in order to define $\theta=0$, how would their $\theta$-$t$ graphs differ? What effect would this have on the angular velocities?  

Answer, p. 1056

The angular velocity has units of radians per second, rad/s. However, radians are not really units at all. The radian measure of an angle is defined, as the length of the circular arc it makes, divided by the radius of the circle. Dividing one length by another gives a unitless quantity, so anything with units of radians is really unitless. We can therefore simplify the units of angular velocity, and call them inverse seconds, s$^{-1}$.

*A 78-rpm record example 11

▷ In the early 20th century, the standard format for music recordings was a plastic disk that held a single song and rotated at 78 rpm (revolutions per minute). What was the angular velocity of such a disk?

▷ If we measure angles in units of revolutions and time in units of minutes, then 78 rpm is the angular velocity. Using standard physics units of radians/second, however, we have

$$\frac{78 \text{ revolutions}}{1 \text{ minute}} \times \frac{2\pi \text{ radians}}{1 \text{ revolution}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} = 8.2 \text{ s}^{-1}.$$
In the absence of any torque, a rigid body will rotate indefinitely with the same angular velocity. If the angular velocity is changing because of a torque, we define an angular acceleration,

\[ \alpha = \frac{d\omega}{dt} \]  

[definition of angular acceleration]

The symbol is the Greek letter alpha. The units of this quantity are \( \text{rad/s}^2 \), or simply \( \text{s}^{-2} \).

The mathematical relationship between \( \omega \) and \( \theta \) is the same as the one between \( v \) and \( x \), and similarly for \( \alpha \) and \( a \). We can thus make a system of analogies, \( c \), and recycle all the familiar kinematic equations for constant-acceleration motion.

**The synodic period**

example 12

Mars takes nearly twice as long as the Earth to complete an orbit. If the two planets are alongside one another on a certain day, then one year later, Earth will be back at the same place, but Mars will have moved on, and it will take more time for Earth to finish catching up. Angular velocities add and subtract, just as velocity vectors do. If the two planets’ angular velocities are \( \omega_1 \) and \( \omega_2 \), then the angular velocity of one relative to the other is \( \omega_1 - \omega_2 \). The corresponding period, \( 1/(1/T_1 - 1/T_2) \) is known as the synodic period.

**A neutron star**

example 13

A neutron star is initially observed to be rotating with an angular velocity of \( 2.0 \, \text{s}^{-1} \), determined via the radio pulses it emits. If its angular acceleration is a constant \( -1.0 \times 10^{-8} \, \text{s}^{-2} \), how many rotations will it complete before it stops? (In reality, the angular acceleration is not always constant; sudden changes often occur, and are referred to as “starquakes!”)

The equation \( v_f^2 - v_i^2 = 2a\Delta x \) can be translated into \( \omega_f^2 - \omega_i^2 = 2\alpha\Delta \theta \), giving

\[ \Delta \theta = \frac{(\omega_f^2 - \omega_i^2)}{2\alpha} = \frac{2.0 \times 10^8}{2 \times (-1.0 \times 10^{-8})} \]

\[ = 3.2 \times 10^7 \, \text{rotations} \]

4.2.2 Relations between angular quantities and motion of a point

It is often necessary to be able to relate the angular quantities to the motion of a particular point on the rotating object. As we develop these, we will encounter the first example where the advantages of radians over degrees become apparent.

The speed at which a point on the object moves depends on both the object’s angular velocity \( \omega \) and the point’s distance \( r \) from the
axis. We adopt a coordinate system, \( d \), with an inward (radial) axis and a tangential axis. The length of the infinitesimal circular arc \( ds \) traveled by the point in a time interval \( dt \) is related to \( d\theta \) by the definition of radian measure, \( d\theta = ds/r \), where positive and negative values of \( ds \) represent the two possible directions of motion along the tangential axis. We then have \( v_t = ds/dt = r \, d\theta/dt = \omega r \), or

\[
v_t = \omega r. \quad \text{[tangential velocity of a point at a distance } r \text{ from the axis of rotation]}
\]

The radial component is zero, since the point is not moving inward or outward,

\[
v_r = 0. \quad \text{[radial velocity of a point at a distance } r \text{ from the axis of rotation]}
\]

Note that we had to use the definition of radian measure in this derivation. Suppose instead we had used units of degrees for our angles and degrees per second for angular velocities. The relationship between \( d\theta \text{ degrees} \) and \( ds \) is \( d\theta \text{ degrees} = (360/2\pi)s/r \), where the extra conversion factor of \((360/2\pi)\) comes from that fact that there are 360 degrees in a full circle, which is equivalent to \( 2\pi \) radians. The equation for \( v_t \) would then have been \( v_t = (2\pi/360)(\omega \text{ degrees per second})(r) \), which would have been much messier. Simplicity, then, is the reason for using radians rather than degrees; by using radians we avoid infecting all our equations with annoying conversion factors.

Since the velocity of a point on the object is directly proportional to the angular velocity, you might expect that its acceleration would be directly proportional to the angular acceleration. This is not true, however. Even if the angular acceleration is zero, i.e., if the object is rotating at constant angular velocity, every point on it will have an acceleration vector directed toward the axis, \( e \). As derived on page 213, the magnitude of this acceleration is

\[
a_r = \omega^2 r. \quad \text{[radial acceleration of a point at a distance } r \text{ from the axis]}
\]

For the tangential component, any change in the angular velocity \( d\omega \) will lead to a change \( d\omega \cdot r \) in the tangential velocity, so it is easily shown that

\[
a_t = \alpha r. \quad \text{[tangential acceleration of a point at a distance } r \text{ from the axis]}
\]

**Self-check C**

Positive and negative signs of \( \omega \) represent rotation in opposite directions. Why does it therefore make sense physically that \( \omega \) is raised to the first power in the equation for \( v_t \) and to the second power in the one for \( a_r \)?

\[\triangleright \text{Answer, p. 1056}\]
What is your radial acceleration due to the rotation of the earth if you are at the equator?

At the equator, your distance from the Earth’s rotation axis is the same as the radius of the spherical Earth, \( 6.4 \times 10^6 \) m. Your angular velocity is

\[
\omega = \frac{2\pi \text{ radians}}{1 \text{ day}} = 7.3 \times 10^{-5} \text{ s}^{-1},
\]

which gives an acceleration of

\[
a_r = \omega^2 r = 0.034 \text{ m/s}^2.
\]

The angular velocity was a very small number, but the radius was a very big number. Squaring a very small number, however, gives a very very small number, so the \( \omega^2 \) factor “wins,” and the final result is small.

If you’re standing on a bathroom scale, this small acceleration is provided by the imbalance between the downward force of gravity and the slightly weaker upward normal force of the scale on your foot. The scale reading is therefore a little lower than it should be.

4.2.3 Dynamics

If we want to connect all this kinematics to anything dynamical, we need to see how it relates to torque and angular momentum. Our strategy will be to tackle angular momentum first, since angular momentum relates to motion, and to use the additive property of angular momentum: the angular momentum of a system of particles equals the sum of the angular momenta of all the individual particles. The angular momentum of one particle within our rigidly rotating object, \( L = m v \perp r \), can be rewritten as \( L = r p \sin \theta \), where \( r \) and \( p \) are the magnitudes of the particle’s \( r \) and momentum vectors, and \( \theta \) is the angle between these two vectors. (The \( r \) vector points outward perpendicularly from the axis to the particle’s position in space.) In rigid-body rotation the angle \( \theta \) is 90°, so we have simply \( L = rp \). Relating this to angular velocity, we have \( L = rp = (r)(mv) = (r)(m\omega r) = mr^2 \omega \). The particle’s contribution to the total angular momentum is proportional to \( \omega \), with a proportionality constant \( mr^2 \). We refer to \( mr^2 \) as the particle’s contribution to the object’s total moment of inertia, \( I \), where “moment” is used in the sense of “important,” as in “momentous” — a bigger value of \( I \) tells us the particle is more important for determining the total angular momentum. The total moment of inertia
Analogies between rotational and linear quantities.

Example 15

\[ I = \sum m_i r_i^2, \]  
[definition of the moment of inertia;  
for rigid-body rotation in a plane;  
r is the distance  
from the axis, measured perpendicular to the axis]

The angular momentum of a rigidly rotating body is then

\[ L = I \omega. \]  
[angular momentum of  
rigid-body rotation in a plane]

Since torque is defined as \( \frac{dL}{dt} \), and a rigid body has a constant  
moment of inertia, we have

\[ \tau = \frac{dL}{dt} = I \frac{d\omega}{dt} = I \alpha, \]

\[ \tau = I \alpha, \]  
[relationship between torque and  
angular acceleration for rigid-body rotation in a plane]

which is analogous to \( F = ma \).

The complete system of analogies between linear motion and  
rigid-body rotation is given in figure f.

A barbell  
example 15

▷ The barbell shown in figure g consists of two small, dense, massive  
balls at the ends of a very light rod. The balls have masses of  
2.0 kg and 1.0 kg, and the length of the rod is 3.0 m. Find  
the moment of inertia of the rod (1) for rotation about its center  
of mass, and (2) for rotation about the center of the more  
massive ball.

▷ (1) The ball’s center of mass lies 1/3 of the way from the greater  
mass to the lesser mass, i.e., 1.0 m from one and 2.0 m from the  
other. Since the balls are small, we approximate them as if they  
were two pointlike particles. The moment of inertia is

\[ I = (2.0 \text{ kg})(1.0 \text{ m})^2 + (1.0 \text{ kg})(2.0 \text{ m})^2 \]

\[ = 2.0 \text{ kg} \cdot \text{m}^2 + 4.0 \text{ kg} \cdot \text{m}^2 \]

\[ = 6.0 \text{ kg} \cdot \text{m}^2 \]

Perhaps counterintuitively, the less massive ball contributes far  
more to the moment of inertia.

(2) The big ball theoretically contributes a little bit to the moment  
of inertia, since essentially none of its atoms are exactly at \( r=0 \).  
However, since the balls are said to be small and dense, we assume  
all the big ball’s atoms are so close to the axis that we can ignore  
their small contributions to the total moment of inertia:

\[ I = (1.0 \text{ kg})(3.0 \text{ m})^2 \]

\[ = 9.0 \text{ kg} \cdot \text{m}^2 \]

This example shows that the moment of inertia depends on the  
choice of axis. For example, it is easier to wiggle a pen about its  
center than about one end.
The parallel axis theorem

Generalizing the previous example, suppose we pick any axis parallel to axis 1, but offset from it by a distance \( h \). Part (2) of the previous example then corresponds to the special case of \( h = -1.0 \) m (negative being to the left). What is the moment of inertia about this new axis?

The big ball’s distance from the new axis is \((1.0 \text{ m})+h\), and the small one’s is \((2.0 \text{ m})-h\). The new moment of inertia is

\[
I = (2.0 \text{ kg})[(1.0 \text{ m})+h]^2 + (1.0 \text{ kg})[(2.0 \text{ m}) - h]^2
\]

\[
= 6.0 \text{ kg} \cdot \text{m}^2 + (4.0 \text{ kg} \cdot \text{m})h - (4.0 \text{ kg} \cdot \text{m})h + (3.0 \text{ kg})h^2.
\]

The constant term is the same as the moment of inertia about the center-of-mass axis, the first-order terms cancel out, and the third term is just the total mass multiplied by \( h^2 \). The interested reader will have no difficulty in generalizing this to any set of particles (problem 38, p. 302), resulting in the parallel axis theorem: If an object of total mass \( M \) rotates about a line at a distance \( h \) from its center of mass, then its moment of inertia equals \( I_{cm} + Mh^2 \), where \( I_{cm} \) is the moment of inertia for rotation about a parallel line through the center of mass.

Scaling of the moment of inertia

(1) Suppose two objects have the same mass and the same shape, but one is less dense, and larger by a factor \( k \). How do their moments of inertia compare?

(2) What if the densities are equal rather than the masses?

(1) This is like increasing all the distances between atoms by a factor \( k \). All the \( r \)’s become greater by this factor, so the moment of inertia is increased by a factor of \( k^2 \).

(2) This introduces an increase in mass by a factor of \( k^3 \), so the moment of inertia of the bigger object is greater by a factor of \( k^5 \).

4.2.4 Iterated integrals

In various places in this book, starting with subsection 4.2.5, we’ll come across integrals stuck inside other integrals. These are known as iterated integrals, or double integrals, triple integrals, etc. Similar concepts crop up all the time even when you’re not doing calculus, so let’s start by imagining such an example. Suppose you want to count how many squares there are on a chess board, and you don’t know how to multiply eight times eight. You could start from the upper left, count eight squares across, then continue with the second row, and so on, until you how counted every square, giving the result of 64. In slightly more formal mathematical language, we could write the following recipe: for each row, \( r \), from 1 to 8, consider the columns, \( c \), from 1 to 8, and add one to the count for
each one of them. Using the sigma notation, this becomes

\[ \sum_{r=1}^{8} \sum_{c=1}^{8} 1. \]

If you’re familiar with computer programming, then you can think of this as a sum that could be calculated using a loop nested inside another loop. To evaluate the result (again, assuming we don’t know how to multiply, so we have to use brute force), we can first evaluate the inside sum, which equals 8, giving

\[ \sum_{r=1}^{8} 8. \]

Notice how the “dummy” variable \( c \) has disappeared. Finally we do the outside sum, over \( r \), and find the result of 64.

Now imagine doing the same thing with the pixels on a TV screen. The electron beam sweeps across the screen, painting the pixels in each row, one at a time. This is really no different than the example of the chess board, but because the pixels are so small, you normally think of the image on a TV screen as continuous rather than discrete. This is the idea of an integral in calculus. Suppose we want to find the area of a rectangle of width \( a \) and height \( b \), and we don’t know that we can just multiply to get the area \( ab \). The brute force way to do this is to break up the rectangle into a grid of infinitesimally small squares, each having width \( dx \) and height \( dy \), and therefore the infinitesimal area \( dA = dx \, dy \). For convenience, we’ll imagine that the rectangle’s lower left corner is at the origin. Then the area is given by this integral:

\[
\text{area} = \int_{y=0}^{b} \int_{x=0}^{a} \, dA \\
= \int_{y=0}^{b} \int_{x=0}^{a} \, dx \, dy
\]

Notice how the leftmost integral sign, over \( y \), and the rightmost differential, \( dy \), act like bookends, or the pieces of bread on a sandwich. Inside them, we have the integral sign that runs over \( x \), and the differential \( dx \) that matches it on the right. Finally, on the innermost layer, we’d normally have the thing we’re integrating, but here’s it’s 1, so I’ve omitted it. Writing the lower limits of the integrals with \( x = \) and \( y = \) helps to keep it straight which integral goes with which.
differential. The result is

\[
\text{area} = \int_{y=0}^{b} \int_{x=0}^{a} dA \\
= \int_{y=0}^{b} \int_{x=0}^{a} dx \, dy \\
= \int_{y=0}^{b} \left( \int_{x=0}^{a} dx \right) dy \\
= \int_{y=0}^{b} a \, dy \\
= a \int_{y=0}^{b} dy \\
= ab.
\]

**Area of a triangle example 18**

Find the area of a 45-45-90 right triangle having legs \(a\).

Let the triangle’s hypotenuse run from the origin to the point \((a,a)\), and let its legs run from the origin to \((0,a)\), and then to \((a,a)\). In other words, the triangle sits on top of its hypotenuse. Then the integral can be set up the same way as the one before, but for a particular value of \(y\), values of \(x\) only run from 0 (on the \(y\) axis) to \(y\) (on the hypotenuse). We then have

\[
\text{area} = \int_{y=0}^{a} \int_{x=0}^{y} dA \\
= \int_{y=0}^{a} \int_{x=0}^{y} dx \, dy \\
= \int_{y=0}^{a} \left( \int_{x=0}^{y} dx \right) dy \\
= \int_{y=0}^{a} y \, dy \\
= \frac{1}{2} a^2
\]

Note that in this example, because the upper end of the \(x\) values depends on the value of \(y\), it makes a difference which order we do the integrals in. The \(x\) integral has to be on the inside, and we have to do it first.

**Volume of a cube example 19**

Find the volume of a cube with sides of length \(a\).

This is a three-dimensional example, so we’ll have integrals nested three deep, and the thing we’re integrating is the volume \(dV = dx \, dy \, dz\).
volume = \int_{x=0}^{a} \int_{y=0}^{a} \int_{z=0}^{a} dx \, dy \, dz
= \int_{y=0}^{a} \int_{x=0}^{a} a \, dy \, dz
= a \int_{z=0}^{a} \int_{y=0}^{a} dy \, dz
= a \int_{z=0}^{a} a \, dz
= a^3

\textbf{Area of a circle example 20}

▷ Find the area of a circle.

▷ To make it easy, let’s find the area of a semicircle and then double it. Let the circle’s radius be \( r \), and let it be centered on the origin and bounded below by the \( x \) axis. Then the curved edge is given by the equation \( r^2 = x^2 + y^2 \), or \( y = \sqrt{r^2 - x^2} \). Since the \( y \) integral’s limit depends on \( x \), the \( x \) integral has to be on the outside. The area is

\[
\text{area} = \int_{x=-r}^{r} \int_{y=0}^{\sqrt{r^2 - x^2}} dy \, dx
= \int_{x=-r}^{r} \sqrt{r^2 - x^2} \, dx
= r \int_{x=-r}^{r} \sqrt{1 - (x/r)^2} \, dx.
\]

Substituting \( u = x/r \),

\[
\text{area} = r^2 \int_{u=-1}^{1} \sqrt{1 - u^2} \, du
\]

The definite integral equals \( \pi \), as you can find using a trig substitution or simply by looking it up in a table, and the result is, as expected, \( \pi r^2 / 2 \) for the area of the semicircle.

4.2.5 Finding moments of inertia by integration

When calculating the moment of inertia of an ordinary-sized object with perhaps \( 10^{26} \) atoms, it would be impossible to do an actual sum over atoms, even with the world’s fastest supercomputer. Calculus, however, offers a tool, the integral, for breaking a sum down to infinitely many small parts. If we don’t worry about the existence of atoms, then we can use an integral to compute a moment
of inertia as if the object was smooth and continuous throughout, rather than granular at the atomic level. Of course this granularity typically has a negligible effect on the result unless the object is itself an individual molecule. This subsection consists of three examples of how to do such a computation, at three distinct levels of mathematical complication.

**Moment of inertia of a thin rod**

What is the moment of inertia of a thin rod of mass $M$ and length $L$ about a line perpendicular to the rod and passing through its center? We generalize the discrete sum

$$ I = \sum m_i r_i^2 $$

to a continuous one,

$$ I = \int r^2 \, dm $$

$$ = \int_{-L/2}^{L/2} x^2 \frac{M}{L} \, dx \quad [r = |x|, \text{ so } r^2 = x^2] $$

$$ = \frac{1}{12} ML^2 $$

In this example the object was one-dimensional, which made the math simple. The next example shows a strategy that can be used to simplify the math for objects that are three-dimensional, but possess some kind of symmetry.

**Moment of inertia of a disk**

What is the moment of inertia of a disk of radius $b$, thickness $t$, and mass $M$, for rotation about its central axis?

We break the disk down into concentric circular rings of thickness $d\,r$. Since all the mass in a given circular slice has essentially the same value of $r$ (ranging only from $r$ to $r + d\,r$), the slice’s contribution to the total moment of inertia is simply $r^2 \, dm$. We then have

$$ I = \int r^2 \, dm $$

$$ = \int r^2 \rho \, dV, $$

where $V = \pi b^2 t$ is the total volume, $\rho = M/V = M/\pi b^2 t$ is the density, and the volume of one slice can be calculated as the volume enclosed by its outer surface minus the volume enclosed by its inner surface, $dV = \pi(r + dr)^2t - \pi r^2 t = 2\pi tr \, dr$.

$$ I = \int_0^b r^2 \frac{M}{\pi b^2 t} 2\pi t \, r \, dr $$

$$ = \frac{1}{2}Mb^2. $$
In the most general case where there is no symmetry about the rotation axis, we must use iterated integrals, as discussed in subsection 4.2.4. The example of the disk possessed two types of symmetry with respect to the rotation axis: (1) the disk is the same when rotated through any angle about the axis, and (2) all slices perpendicular to the axis are the same. These two symmetries reduced the number of layers of integrals from three to one. The following example possesses only one symmetry, of type (2), and we simply set it up as a triple integral. You may not have seen multiple integrals yet in a math course. If so, just skim this example.

**Moment of inertia of a cube**

What is the moment of inertia of a cube of side \( b \), for rotation about an axis that passes through its center and is parallel to four of its faces? Let the origin be at the center of the cube, and let \( x \) be the rotation axis.

\[
I = \int r^2 \, dm
= \rho \int r^2 \, dV
= \rho \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} (y^2 + z^2) \, dx \, dy \, dz
= \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} (y^2 + z^2) \, dy \, dz
\]

The fact that the last step is a trivial integral results from the symmetry of the problem. The integrand of the remaining double integral breaks down into two terms, each of which depends on only one of the variables, so we break it into two integrals,

\[
I = \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} y^2 \, dy \, dz + \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} z^2 \, dy \, dz
\]

which we know have identical results. We therefore only need to evaluate one of them and double the result:

\[
I = 2 \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} z^2 \, dy \, dz
= 2 \rho b^2 \int_{-b/2}^{b/2} z^2 \, dz
= \frac{1}{6} \rho b^5
= \frac{1}{6} Mb^2
\]

Figure h shows the moments of inertia of some shapes, which were evaluated with techniques like these.
Example 22.

Moments of inertia of some geometric shapes.

The hammer throw example 21

- In the men’s Olympic hammer throw, a steel ball of radius 6.1 cm is swung on the end of a wire of length 1.22 m. What fraction of the ball’s angular momentum comes from its rotation, as opposed to its motion through space?

- It’s always important to solve problems symbolically first, and plug in numbers only at the end, so let the radius of the ball be \( b \), and the length of the wire \( \ell \). If the time the ball takes to go once around the circle is \( T \), then this is also the time it takes to revolve once around its own axis. Its speed is \( v = \frac{2\pi\ell}{T} \), so its angular momentum due to its motion through space is \( mv\ell = \frac{2\pi m\ell^2}{T} \). Its angular momentum due to its rotation around its own center is \( \left(\frac{4\pi}{5}\right)mb^2/T \). The ratio of these two angular momenta is \( \left(\frac{2}{5}\right)(b/\ell)^2 = 1.0 \times 10^{-3} \). The angular momentum due to the ball’s spin is extremely small.

Toppling a rod example 22

- A rod of length \( b \) and mass \( m \) stands upright. We want to strike the rod at the bottom, causing it to fall and land flat. Find the momentum, \( p \), that should be delivered, in terms of \( m \), \( b \), and \( g \). Can this really be done without having the rod scrape on the floor?

- This is a nice example of a question that can very nearly be answered based only on units. Since the three variables, \( m \), \( b \), and \( g \), all have different units, they can’t be added or subtracted. The only way to combine them mathematically is by multiplication or division. Multiplying one of them by itself is exponentiation, so in general we expect that the answer must be of the form

\[
p = Am^j b^k g^l,
\]

where \( A \), \( j \), \( k \), and \( l \) are unitless constants. The result has to have units of \( \text{kg} \cdot \text{m/s} \). To get kilograms to the first power, we need

\[
j = 1,
\]
meters to the first power requires
\[ k + l = 1, \]
and seconds to the power \(-1\) implies
\[ l = 1/2. \]

We find \(j = 1\), \(k = 1/2\), and \(l = 1/2\), so the solution must be of the form
\[ p = Am\sqrt{bg}. \]

Note that no physics was required!

Consideration of units, however, won’t help us to find the unitless constant \(A\). Let \(t\) be the time the rod takes to fall, so that \((1/2)gt^2 = b/2\). If the rod is going to land exactly on its side, then the number of revolutions it completes while in the air must be 1/4, or 3/4, or 5/4, \ldots, but all the possibilities greater than 1/4 would cause the head of the rod to collide with the floor prematurely. The rod must therefore rotate at a rate that would cause it to complete a full rotation in a time \(T = 4t\), and it has angular momentum \(L = (\pi/6)mb^2/T\).

The momentum lost by the object striking the rod is \(p\), and by conservation of momentum, this is the amount of momentum, in the horizontal direction, that the rod acquires. In other words, the rod will fly forward a little. However, this has no effect on the solution to the problem. More importantly, the object striking the rod loses angular momentum \(bp/2\), which is also transferred to the rod. Equating this to the expression above for \(L\), we find \(p = (\pi/12)m\sqrt{bg}\).

Finally, we need to know whether this can really be done without having the foot of the rod scrape on the floor. The figure shows that the answer is no for this rod of finite width, but it appears that the answer would be yes for a sufficiently thin rod. This is analyzed further in homework problem 37 on page 301.
4.3 Angular momentum in three dimensions

Conservation of angular momentum produces some surprising phenomena when extended to three dimensions. Try the following experiment, for example. Take off your shoe, and toss it in to the air, making it spin along its long (toe-to-heel) axis. You should observe a nice steady pattern of rotation. The same happens when you spin the shoe about its shortest (top-to-bottom) axis. But something unexpected happens when you spin it about its third (left-to-right) axis, which is intermediate in length between the other two. Instead of a steady pattern of rotation, you will observe something more complicated, with the shoe changing its orientation with respect to the rotation axis.

4.3.1 Rigid-body kinematics in three dimensions

How do we generalize rigid-body kinematics to three dimensions? When we wanted to generalize the kinematics of a moving particle to three dimensions, we made the numbers \( r, v, \) and \( a \) into vectors \( \mathbf{r}, \mathbf{v}, \) and \( \mathbf{a} \). This worked because these quantities all obeyed the same laws of vector addition. For instance, one of the laws of vector addition is that, just like addition of numbers, vector addition gives the same result regardless of the order of the two quantities being added. Thus you can step sideways 1 m to the right and then step forward 1 m, and the end result is the same as if you stepped forward first and then to the side. In other words, it didn’t matter whether you took \( \Delta \mathbf{r}_1 + \Delta \mathbf{r}_2 \) or \( \Delta \mathbf{r}_2 + \Delta \mathbf{r}_1 \). In math this is called the commutative property of addition.

Angular motion, unfortunately doesn’t have this property, as shown in figure a. Doing a rotation about the \( x \) axis and then
about $y$ gives one result, while doing them in the opposite order gives a different result. These operations don’t “commute,” i.e., it makes a difference what order you do them in.

This means that there is in general no possible way to construct a $\Delta \theta$ vector. However, if you try doing the operations shown in figure a using small rotation, say about 10 degrees instead of 90, you’ll find that the result is nearly the same regardless of what order you use; small rotations are very nearly commutative. Not only that, but the result of the two 10-degree rotations is about the same as a single, somewhat larger, rotation about an axis that lies symmetrically at between the $x$ and $y$ axes at 45 degree angles to each one. This is exactly what we would expect if the two small rotations did act like vectors whose directions were along the axis of rotation. We therefore define a $d\theta$ vector whose magnitude is the amount of rotation in units of radians, and whose direction is along the axis of rotation. Actually this definition is ambiguous, because there it could point in either direction along the axis. We therefore use a right-hand rule as shown in figure b to define the direction of the $d\theta$ vector, and the $\omega$ vector, $\omega = \frac{d\theta}{dt}$, based on it. Aliens on planet Tammyfaye may decide to define it using their left hands rather than their right, but as long as they keep their scientific literature separate from ours, there is no problem. When entering a physics exam, always be sure to write a large warning note on your left hand in magic marker so that you won’t be tempted to use it for the right-hand rule while keeping your pen in your right.

**self-check D**

Use the right-hand rule to determine the directions of the $\omega$ vectors in each rotation shown in figures a/1 through a/5. ▷ Answer, p. 1056

Because the vector relationships among $d\theta$, $\omega$, and $\alpha$ are strictly analogous to the ones involving $dr$, $v$, and $a$ (with the proviso that we avoid describing large rotations using $\Delta \theta$ vectors), any operation in $r$-$v$-$a$ vector kinematics has an exact analog in $\theta$-$\omega$-$\alpha$ kinematics.

### Result of successive 10-degree rotations example 23

▷ What is the result of two successive (positive) 10-degree rotations about the $x$ and $y$ axes? That is, what single rotation about a single axis would be equivalent to executing these in succession?

▷ The result is only going to be approximate, since 10 degrees is not an infinitesimally small angle, and we are not told in what order the rotations occur. To some approximation, however, we can add the $\Delta \theta$ vectors in exactly the same way we would add $\Delta r$ vectors, so we have

$$
\Delta \theta \approx \Delta \theta_1 + \Delta \theta_2
\approx (10 \text{ degrees})\hat{x} + (10 \text{ degrees})\hat{y}.
$$

This is a vector with a magnitude of $\sqrt{(10 \text{ deg})^2 + (10 \text{ deg})^2} =$

---

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14 deg, and it points along an axis midway between the $x$ and $y$ axes.

4.3.2 Angular momentum in three dimensions

The vector cross product

In order to expand our system of three-dimensional kinematics to include dynamics, we will have to generalize equations like $v_t = \omega r$, $\tau = rF \sin \theta_F$, and $L = rp \sin \theta_p$, each of which involves three quantities that we have either already defined as vectors or that we want to redefine as vectors. Although the first one appears to differ from the others in its form, it could just as well be rewritten as $v_t = \omega r \sin \theta_r$, since $\theta_r = 90^\circ$, and $\sin 90^\circ = 1$.

It thus appears that we have discovered something general about the physically useful way to relate three vectors in a multiplicative way: the magnitude of the result always seems to be proportional to the product of the magnitudes of the two vectors being “multiplied,” and also to the sine of the angle between them.

Is this pattern just an accident? Actually the sine factor has a very important physical property: it goes to zero when the two vectors are parallel. This is a Good Thing. The generalization of angular momentum into a three-dimensional vector, for example, is presumably going to describe not just the clockwise or counterclockwise nature of the motion but also from which direction we would have to view the motion so that it was clockwise or counterclockwise. (A clock’s hands go counterclockwise as seen from behind the clock, and don’t rotate at all as seen from above or to the side.) Now suppose a particle is moving directly away from the origin, so that its $r$ and $p$ vectors are parallel. It is not going around the origin from any point of view, so its angular momentum vector had better be zero.

Thinking in a slightly more abstract way, we would expect the angular momentum vector to point perpendicular to the plane of motion, just as the angular velocity vector points perpendicular to the plane of motion. The plane of motion is the plane containing both $r$ and $p$, if we place the two vectors tail-to-tail. But if $r$ and $p$ are parallel and are placed tail-to-tail, then there are infinitely many planes containing them both. To pick one of these planes in preference to the others would violate the symmetry of space, since they should all be equally good. Thus the zero-if-parallel property is a necessary consequence of the underlying symmetry of the laws of physics.

The following definition of a kind of vector multiplication is consistent with everything we’ve seen so far, and on p. 1024 we’ll prove that the definition is unique, i.e., if we believe in the symmetry of space, it is essentially the only way of defining the multiplication of
two vectors to produce a third vector:

**Definition of the vector cross product:**

The cross product \( \mathbf{A} \times \mathbf{B} \) of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is defined as follows:

1. Its magnitude is defined by \( |\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin \theta_{AB} \), where \( \theta_{AB} \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \) when they are placed tail-to-tail.
2. Its direction is along the line perpendicular to both \( \mathbf{A} \) and \( \mathbf{B} \). Of the two such directions, it is the one that obeys the right-hand rule shown in figure c.

The name “cross product” refers to the symbol, and distinguishes it from the dot product, which acts on two vectors but produces a scalar.

Although the vector cross-product has nearly all the properties of numerical multiplication, e.g., \( \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \), it lacks the usual property of commutativity. Try applying the right-hand rule to find the direction of the vector cross product \( \mathbf{B} \times \mathbf{A} \) using the two vectors shown in the figure. This requires starting with a flattened hand with the four fingers pointing along \( \mathbf{B} \), and then curling the hand so that the fingers point along \( \mathbf{A} \). The only possible way to do this is to point your thumb toward the floor, in the opposite direction. Thus for the vector cross product we have

\[ \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \]

a property known as anticommutativity. The vector cross product is also not associative, i.e., \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \) is usually not the same as \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \).

A geometric interpretation of the cross product, d, is that if both \( \mathbf{A} \) and \( \mathbf{B} \) are vectors with units of distance, then the magnitude of their cross product can be interpreted as the area of the parallelogram they form when placed tail-to-tail.

A useful expression for the components of the vector cross product in terms of the components of the two vectors being multiplied is as follows:

\[
\begin{align*}
(A \times B)_x &= A_y B_z - B_y A_z \\
(A \times B)_y &= A_z B_x - B_z A_x \\
(A \times B)_z &= A_x B_y - B_x A_y
\end{align*}
\]

I’ll prove later that these expressions are equivalent to the previous definition of the cross product. Although they may appear formidable, they have a simple structure: the subscripts on the right are the other two besides the one on the left, and each equation is related to the preceding one by a cyclic change in the subscripts, e. If the subscripts were not treated in some completely symmetric
manner like this, then the definition would provide some way to distinguish one axis from another, which would violate the symmetry of space.

**self-check E**
Show that the component equations are consistent with the rule \( \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \).

**Angular momentum in three dimensions**

In terms of the vector cross product, we have:

\[
\mathbf{v} = \omega \times \mathbf{r} \\
\mathbf{L} = \mathbf{r} \times \mathbf{p} \\
\tau = \mathbf{r} \times \mathbf{F}
\]

But wait, how do we know these equations are even correct? For instance, how do we know that the quantity defined by \( \mathbf{r} \times \mathbf{p} \) is in fact conserved? Well, just as we saw on page 216 that the dot product is unique (i.e., can only be defined in one way while observing rotational invariance), the cross product is also unique, as proved on page 1024. If \( \mathbf{r} \times \mathbf{p} \) was not conserved, then there could not be any generally conserved quantity that would reduce to our old definition of angular momentum in the special case of plane rotation. This doesn’t prove conservation of angular momentum — only experiments can prove that — but it does prove that if angular momentum is conserved in three dimensions, there is only one possible way to generalize from two dimensions to three.

**Angular momentum of a spinning top example 24**

As an illustration, we consider the angular momentum of a spinning top. Figures f and g show the use of the vector cross product to determine the contribution of a representative atom to the total angular momentum. Since every other atom’s angular momentum vector will be in the same direction, this will also be the direction of the total angular momentum of the top. This happens to be rigid-body rotation, and perhaps not surprisingly, the angular momentum vector is along the same direction as the angular velocity vector.

Three important points are illustrated by this example: (1) When we do the full three-dimensional treatment of angular momentum, the “axis” from which we measure the position vectors is just an arbitrarily chosen point. If this had not been rigid-body rotation, we would not even have been able to identify a single line about which every atom circled. (2) Starting from figure f, we had to rearrange the vectors to get them tail-to-tail before applying the right-hand rule. If we had attempted to apply the right-hand rule to figure f, the direction of the result would have been exactly the opposite of the correct answer. (3) The equation \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \) cannot be applied all at once to an entire system of particles. The total
momentum of the top is zero, which would give an erroneous result of zero angular momentum (never mind the fact that \( \mathbf{r} \) is not well defined for the top as a whole).

Doing the right-hand rule like this requires some practice. I urge you to make models like g out of rolled up pieces of paper and to practice with the model in various orientations until it becomes natural.

\[ \text{Precession} \text{ example 25} \]

Figure h shows a counterintuitive example of the concepts we’ve been discussing. One expects the torque due to gravity to cause the top to flop down. Instead, the top remains spinning in the horizontal plane, but its axis of rotation starts moving in the direction shown by the shaded arrow. This phenomenon is called precession. Figure i shows that the torque due to gravity is out of the page. (Actually we should add up all the torques on all the atoms in the top, but the qualitative result is the same.) Since torque is the rate of change of angular momentum, \( \tau = \frac{d\mathbf{L}}{dt} \), the \( \Delta \mathbf{L} \) vector must be in the same direction as the torque (division by a positive scalar doesn’t change the direction of the vector). As shown in j, this causes the angular momentum vector to twist in space without changing its magnitude.

For similar reasons, the Earth’s axis precesses once every 26,000 years (although not through a great circle, since the angle between the axis and the force isn’t 90 degrees as in figure h). This precession is due to a torque exerted by the moon. If the Earth was a perfect sphere, there could be no precession effect due to symmetry. However, the Earth’s own rotation causes it to be slightly flattened (oblate) relative to a perfect sphere, giving it “love handles” on which the moon’s gravity can act. The moon’s gravity on the nearer side of the equatorial bulge is stronger, so the torques do not cancel out perfectly. Presently the earth’s axis very nearly lines up with the star Polaris, but in 12,000 years, the pole star will be Vega instead.

\[ \text{The frisbee} \text{ example 26} \]

The flow of the air over a flying frisbee generates lift, and the lift at the front and back of the frisbee isn’t necessarily balanced. If you throw a frisbee without rotating it, as if you were shooting a basketball with two hands, you’ll find that it pitches, i.e., its nose goes either up or down. When I do this with my frisbee, it goes nose down, which apparently means that the lift at the back of the disc is greater than the lift at the front. The two torques are unbalanced, resulting in a total torque that points to the left.

The way you actually throw a frisbee is with one hand, putting a lot of spin on it. If you throw backhand, which is how most people first learn to do it, the angular momentum vector points down (assuming you’re right-handed). On my frisbee, the aerodynamic
torque to the left would therefore tend to make the angular mo-
mentum vector precess in the clockwise direction as seen by the
thrower. This would cause the disc to roll to the right, and there-
fore follow a curved trajectory. Some specialized discs, used in
the sport of disc golf, are actually designed intentionally to show
this behavior; they’re known as “understable” discs. However, the
typical frisbee that most people play with is designed to be stable:
as the disc rolls to one side, the airflow around it is altered in way
that tends to bring the disc back into level flight. Such a disc will
therefore tend to fly in a straight line, provided that it is thrown
with enough angular momentum.

Finding a cross product by components example 27

What is the torque produced by a force given by \( \hat{x} + 2\hat{y} + 3\hat{z} \) (in
units of Newtons) acting on a point whose radius vector is \( 4\hat{x} + 5\hat{y} \)
(in meters)?

It’s helpful to make a table of the components as shown in the
figure. The results are

\[
\begin{align*}
\tau_x &= r_y F_z - F_y r_z = 15 \text{ N} \cdot \text{m} \\
\tau_y &= r_z F_x - F_z r_x = -12 \text{ N} \cdot \text{m} \\
\tau_z &= r_x F_y - F_x r_y = 3 \text{ N} \cdot \text{m}
\end{align*}
\]

Torque and angular momentum example 28

In this example, we prove explicitly the consistency of the equa-
tions involving torque and angular momentum that we proved
above based purely on symmetry. Starting from the definition of
torque, we have

\[
\tau = \frac{dL}{dt} = \frac{d}{dt} \sum_i r_i \times p_i = \sum_i \frac{d}{dt}(r_i \times p_i).
\]

The derivative of a cross product can be evaluated in the same
way as the derivative of an ordinary scalar product:

\[
\tau = \sum_i \left[ (\frac{dr_i}{dt} \times p_i) + (r_i \times \frac{dp_i}{dt}) \right]
\]

The first term is zero for each particle, since the velocity vector is
parallel to the momentum vector. The derivative appearing in the
second term is the force acting on the particle, so

\[
\tau = \sum_i r_i \times F_i,
\]

which is the relationship we set out to prove.
4.3.3 Rigid-body dynamics in three dimensions

The student who is not madly in love with mathematics may wish to skip the rest of this section after absorbing the statement that, for a typical, asymmetric object, the angular momentum vector and the angular velocity vector need not be parallel. That is, only for a body that possesses symmetry about the rotation axis is it true that $L = I\omega$ (the rotational equivalent of $p = m\mathbf{v}$) for some scalar $I$.

Let’s evaluate the angular momentum of a rigidly rotating system of particles:

$$L = \sum r_i \times p_i$$
$$= \sum m_i r_i \times v_i$$
$$= \sum m_i r_i \times (\omega \times r_i)$$

An important mathematical skill is to know when to give up and back off. This is a complicated expression, and there is no reason to expect it to simplify and, for example, take the form of a scalar multiplied by $\omega$. Instead we examine its general characteristics. If we expanded it using the equation that gives the components of a vector cross product, every term would have one of the $\omega$ components raised to the first power, multiplied by a bunch of other stuff. The most general possible form for the result is

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z,$$

which you may recognize as a case of matrix multiplication. The moment of inertia is not a scalar, and not a three-component vector. It is a matrix specified by nine numbers, called its matrix elements.

The elements of the moment of inertia matrix will depend on our choice of a coordinate system. In general, there will be some special coordinate system, in which the matrix has a simple diagonal form:

$$L_x = I_{xx}\omega_x$$
$$L_y = I_{yy}\omega_y$$
$$L_z = I_{zz}\omega_z.$$}

The three special axes that cause this simplification are called the principal axes of the object, and the corresponding coordinate system is the principal axis system. For symmetric shapes such as a rectangular box or an ellipsoid, the principal axes lie along the intersections of the three symmetry planes, but even an asymmetric body has principal axes.
We can also generalize the plane-rotation equation $K = (1/2)I\omega^2$ to three dimensions as follows:

$$K = \sum_i \frac{1}{2} m_i v_i^2$$
$$= \frac{1}{2} \sum_i m_i (\omega \times r_i) \cdot (\omega \times r_i)$$

We want an equation involving the moment of inertia, and this has some evident similarities to the sum we originally wrote down for the moment of inertia. To massage it into the right shape, we need the vector identity $(A \times B) \cdot C = (B \times C) \cdot A$, which we state without proof. We then write

$$K = \frac{1}{2} \sum_i m_i [r_i \times (\omega \times r_i)] \cdot \omega$$
$$= \frac{1}{2} \omega \cdot \sum_i m_i r_i \times (\omega \times r_i)$$
$$= \frac{1}{2} \mathbf{L} \cdot \omega$$

As a reward for all this hard work, let’s analyze the problem of the spinning shoe that I posed at the beginning of the chapter. The three rotation axes referred to there are approximately the principal axes of the shoe. While the shoe is in the air, no external torques are acting on it, so its angular momentum vector must be constant in magnitude and direction. Its kinetic energy is also constant. That’s in the room’s frame of reference, however. The principal axis frame is attached to the shoe, and tumbles madly along with it. In the principal axis frame, the kinetic energy and the magnitude of the angular momentum stay constant, but the actual direction of the angular momentum need not stay fixed (as you saw in the case of rotation that was initially about the intermediate-length axis). Constant $|\mathbf{L}|$ gives

$$L_x^2 + L_y^2 + L_z^2 = \text{constant}.$$  

In the principal axis frame, it’s easy to solve for the components of $\omega$ in terms of the components of $\mathbf{L}$, so we eliminate $\omega$ from the expression $2K = \mathbf{L} \cdot \omega$, giving

$$\frac{1}{I_{xx}}L_x^2 + \frac{1}{I_{yy}}L_y^2 + \frac{1}{I_{zz}}L_z^2 = \text{constant \#2}.$$  

The first equation is the equation of a sphere in the three-dimensional space occupied by the angular momentum vector, while the second one is the equation of an ellipsoid. The top figure corresponds to the case of rotation about the shortest axis, which has the greatest moment of inertia element. The intersection of the two
surfaces consists only of the two points at the front and back of the sphere. The angular momentum is confined to one of these points, and can’t change its direction, i.e., its orientation with respect to the principal axis system, which is another way of saying that the shoe can’t change its orientation with respect to the angular momentum vector. In the bottom figure, the shoe is rotating about the longest axis. Now the angular momentum vector is trapped at one of the two points on the right or left. In the case of rotation about the axis with the intermediate moment of inertia element, however, the intersection of the sphere and the ellipsoid is not just a pair of isolated points but the curve shown with the dashed line. The relative orientation of the shoe and the angular momentum vector can and will change.

One application of the moment of inertia tensor is to video games that simulate car racing or flying airplanes.

One more exotic example has to do with nuclear physics. Although you have probably visualized atomic nuclei as nothing more than featureless points, or perhaps tiny spheres, they are often ellipsoids with one long axis and two shorter, equal ones. Although a spinning nucleus normally gets rid of its angular momentum via gamma ray emission within a period of time on the order of picoseconds, it may happen that a deformed nucleus gets into a state in which has a large angular momentum is along its long axis, which is a very stable mode of rotation. Such states can live for seconds or even years! (There is more to the story — this is the topic on which I wrote my Ph.D. thesis — but the basic insight applies even though the full treatment requires fancy quantum mechanics.)

Our analysis has so far assumed that the kinetic energy of rotation energy can’t be converted into other forms of energy such as heat, sound, or vibration. When this assumption fails, then rotation about the axis of least moment of inertia becomes unstable, and will eventually convert itself into rotation about the axis whose moment of inertia is greatest. This happened to the U.S.’s first artificial satellite, Explorer I, launched in 1958. Note the long, floppy antennas, which tended to dissipate kinetic energy into vibration. It had been designed to spin about its minimum-moment-of-inertia axis, but almost immediately, as soon as it was in space, it began spinning end over end. It was nevertheless able to carry out its science mission, which didn’t depend on being able to maintain a stable orientation, and it discovered the Van Allen radiation belts.

This chapter is summarized on page 1075. Notation and terminology are tabulated on pages 1066-1067.
Problems

The symbols $\sqrt{\text{, }}, \Box, \text{ etc.}$ are explained on page 303.

1. The figure shows a scale drawing of a pair of pliers being used to crack a nut, with an appropriately reduced centimeter grid. Warning: do not attempt this at home; it is bad manners. If the force required to crack the nut is 300 N, estimate the force required of the person’s hand.

$\Box$ Solution, p. 1039

2. You are trying to loosen a stuck bolt on your RV using a big wrench that is 50 cm long. If you hang from the wrench, and your mass is 55 kg, what is the maximum torque you can exert on the bolt?

3. A physical therapist wants her patient to rehabilitate his injured elbow by laying his arm flat on a table, and then lifting a 2.1 kg mass by bending his elbow. In this situation, the weight is 33 cm from his elbow. He calls her back, complaining that it hurts him to grasp the weight. He asks if he can strap a bigger weight onto his arm, only 17 cm from his elbow. How much mass should she tell him to use so that he will be exerting the same torque? (He is raising his forearm itself, as well as the weight.)

4. An object thrown straight up in the air is momentarily at rest when it reaches the top of its motion. Does that mean that it is in equilibrium at that point? Explain.

5. An object is observed to have constant angular momentum. Can you conclude that no torques are acting on it? Explain. [Based on a problem by Serway and Faughn.]

6. A person of mass $m$ stands on the ball of one foot. Find the tension in the calf muscle and the force exerted by the shinbones on the bones of the foot, in terms of $m, g, a,$ and $b$. For simplicity, assume that all the forces are at 90-degree angles to the foot, i.e., neglect the angle between the foot and the floor.

7. Two pointlike particles have the same momentum vector. Can you conclude that their angular momenta are the same? Explain. [Based on a problem by Serway and Faughn.]

8. The box shown in the figure is being accelerated by pulling on it with the rope.
   (a) Assume the floor is frictionless. What is the maximum force that can be applied without causing the box to tip over?
   $\Box$ Hint, p. 1031
   
   (b) Repeat part a, but now let the coefficient of friction be $\mu$.
   $\Box$
   
   (c) What happens to your answer to part b when the box is sufficiently tall? How do you interpret this?
A uniform ladder of mass $m$ and length $\ell$ leans against a smooth wall, making an angle $\theta$ with respect to the ground. The dirt exerts a normal force and a frictional force on the ladder, producing a force vector with magnitude $F_1$ at an angle $\phi$ with respect to the ground. Since the wall is smooth, it exerts only a normal force on the ladder; let its magnitude be $F_2$.

(a) Explain why $\phi$ must be greater than $\theta$. No math is needed.
(b) Choose any numerical values you like for $m$ and $\ell$, and show that the ladder can be in equilibrium (zero torque and zero total force vector) for $\theta=45.00^\circ$ and $\phi=63.43^\circ$.

Continuing problem 9, find an equation for $\phi$ in terms of $\theta$, and show that $m$ and $L$ do not enter into the equation. Do not assume any numerical values for any of the variables. You will need the trig identity $\sin(a - b) = \sin a \cos b - \sin b \cos a$. (As a numerical check on your result, you may wish to check that the angles given in problem 9b satisfy your equation.)

(a) Find the minimum horizontal force which, applied at the axle, will pull a wheel over a step. Invent algebra symbols for whatever quantities you find to be relevant, and give your answer in symbolic form.
(b) Under what circumstances does your result become infinite? Give a physical interpretation. What happens to your answer when the height of the curb is zero? Does this make sense?

A ball is connected by a string to a vertical post. The ball is set in horizontal motion so that it starts winding the string around the post. Assume that the motion is confined to a horizontal plane, i.e., ignore gravity. Michelle and Astrid are trying to predict the final velocity of the ball when it reaches the post. Michelle says that according to conservation of angular momentum, the ball has to speed up as it approaches the post. Astrid says that according to conservation of energy, the ball has to keep a constant speed. Who is right? [Hint: How is this different from the case where you whirl a rock in a circle on a string and gradually reel in the string?]

In the 1950’s, serious articles began appearing in magazines like Life predicting that world domination would be achieved by the nation that could put nuclear bombs in orbiting space stations, from which they could be dropped at will. In fact it can be quite difficult to get an orbiting object to come down. Let the object have energy $E = K + U$ and angular momentum $L$. Assume that the energy is negative, i.e., the object is moving at less than escape velocity. Show that it can never reach a radius less than

$$r_{\text{min}} = \frac{GMm}{2E} \left(-1 + \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}\right).$$

[Note that both factors are negative, giving a positive result.]
14  (a) The bar of mass $m$ is attached at the wall with a hinge, and is supported on the right by a massless cable. Find the tension, $T$, in the cable in terms of the angle $\theta$.\(\checkmark\)
(b) Interpreting your answer to part a, what would be the best angle to use if we wanted to minimize the strain on the cable?\(\checkmark\)
(c) Again interpreting your answer to part a, for what angles does the result misbehave mathematically? Interpret this physically.\(\checkmark\)

15  (a) The two identical rods are attached to one another with a hinge, and are supported by the two massless cables. Find the angle $\alpha$ in terms of the angle $\beta$, and show that the result is a purely geometric one, independent of the other variables involved.\(\checkmark\)
(b) Using your answer to part a, sketch the configurations for $\beta \to 0$, $\beta = 45^\circ$, and $\beta = 90^\circ$. Do your results make sense intuitively?\(\checkmark\)

16  Two bars of length $\ell$ are connected with a hinge and placed on a frictionless cylinder of radius $r$. (a) Show that the angle $\theta$ shown in the figure is related to the unitless ratio $r/\ell$ by the equation

$$\frac{r}{\ell} = \frac{\cos^2 \theta}{2 \tan \theta}.$$

(b) Discuss the physical behavior of this equation for very large and very small values of $r/\ell$.\(\checkmark\)

17  You wish to determine the mass of a ship in a bottle without taking it out. Show that this can be done with the setup shown in the figure, with a scale supporting the bottle at one end, provided that it is possible to take readings with the ship slid to several different locations. Note that you can’t determine the position of the ship’s center of mass just by looking at it, and likewise for the bottle. In particular, you can’t just say, “position the ship right on top of the fulcrum” or “position it right on top of the balance.”\(\checkmark\)

18  Suppose that we lived in a universe in which Newton’s law of gravity gave an interaction energy proportional to $r^{-6}$, rather than $r^{-1}$. Which, if any, of Kepler’s laws would still be true? Which would be completely false? Which would be different, but in a way that could be calculated with straightforward algebra?\(\checkmark\)
19 Use analogies to find the equivalents of the following equations for rotation in a plane:

\[ KE = \frac{p^2}{2m} \]
\[ \Delta x = v_0 \Delta t + \frac{1}{2} a \Delta t^2 \]
\[ W = F \Delta x \]

Example: \( v = \frac{\Delta x}{\Delta t} \rightarrow \omega = \frac{\Delta \theta}{\Delta t} \)

20 For a one-dimensional harmonic oscillator, the solution to the energy conservation equation,

\[ U + K = \frac{1}{2} k x^2 + \frac{1}{2} m v^2 = \text{constant}, \]

is an oscillation with frequency \( \omega = \sqrt{\frac{k}{m}} \).

Now consider an analogous system consisting of a bar magnet hung from a thread, which acts like a magnetic compass. A normal compass is full of water, so its oscillations are strongly damped, but the magnet-on-a-thread compass has very little friction, and will oscillate repeatedly around its equilibrium direction. The magnetic energy of the bar magnet is

\[ U = -Bm \cos \theta, \]

where \( B \) is a constant that measures the strength of the earth’s magnetic field, \( m \) is a constant that parametrizes the strength of the magnet, and \( \theta \) is the angle, measured in radians, between the bar magnet and magnetic north. The equilibrium occurs at \( \theta = 0 \), which is the minimum of \( U \).

(a) Problem 19 on p. 297 gave some examples of how to construct analogies between rotational and linear motion. Using the same technique, translate the equation defining the linear quantity \( k \) to one that defines an analogous angular one \( \kappa \) (Greek letter kappa). Applying this to the present example, find an expression for \( \kappa \). (Assume the thread is so thin that its stiffness does not have any significant effect compared to earth’s magnetic field.) \( \checkmark \)

(b) Find the frequency of the compass’s vibrations. \( \checkmark \)

21 (a) Find the angular velocities of the earth’s rotation and of the earth’s motion around the sun. \( \checkmark \)

(b) Which motion involves the greater acceleration?
22 The sun turns on its axis once every 26.0 days. Its mass is $2.0 \times 10^{30}$ kg and its radius is $7.0 \times 10^{8}$ m. Assume it is a rigid sphere of uniform density.

(a) What is the sun’s angular momentum? 

In a few billion years, astrophysicists predict that the sun will use up all its sources of nuclear energy, and will collapse into a ball of exotic, dense matter known as a white dwarf. Assume that its radius becomes $5.8 \times 10^6$ m (similar to the size of the Earth.) Assume it does not lose any mass between now and then. (Don’t be fooled by the photo, which makes it look like nearly all of the star was thrown off by the explosion. The visually prominent gas cloud is actually thinner than the best laboratory vacuum ever produced on earth. Certainly a little bit of mass is actually lost, but it is not at all unreasonable to make an approximation of zero loss of mass as we are doing.)

(b) What will its angular momentum be?

(c) How long will it take to turn once on its axis?

23 Give a numerical comparison of the two molecules’ moments of inertia for rotation in the plane of the page about their centers of mass.

24 A yo-yo of total mass $m$ consists of two solid cylinders of radius $R$, connected by a small spindle of negligible mass and radius $r$. The top of the string is held motionless while the string unrolls from the spindle. Show that the acceleration of the yo-yo is $g/(1 + R^2/2r^2)$. [Hint: The acceleration and the tension in the string are unknown. Use $\tau = \Delta L/\Delta t$ and $F = ma$ to determine these two unknowns.]

25 Show that a sphere of radius $R$ that is rolling without slipping has angular momentum and momentum in the ratio $L/p = (2/5)R$.

26 Suppose a bowling ball is initially thrown so that it has no angular momentum at all, i.e., it is initially just sliding down the lane. Eventually kinetic friction will get it spinning fast enough so that it is rolling without slipping. Show that the final velocity of the ball equals $5/7$ of its initial velocity. [Hint: You’ll need the result of problem 25.]

27 Find the angular momentum of a particle whose position is $\mathbf{r} = 3\hat{x} - \hat{y} + \hat{z}$ (in meters) and whose momentum is $\mathbf{p} = -2\hat{x} + \hat{y} + \hat{z}$ (in kg·m/s).
28 Find a vector that is perpendicular to both of the following two vectors:
\[ \hat{x} + 2\hat{y} + 3\hat{z} \]
\[ 4\hat{x} + 5\hat{y} + 6\hat{z} \]

√ □

29 Prove property (3) of the vector cross product from the theorem on page 1024.

□

30 Prove the anticommutative property of the vector cross product, \( \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \), using the expressions for the components of the cross product. Note that giving an example does not constitute a proof of a general rule.

□

31 Find three vectors with which you can demonstrate that the vector cross product need not be associative, i.e., that \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \) need not be the same as \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \).

□

32 Which of the following expressions make sense, and which are nonsense? For those that make sense, indicate whether the result is a vector or a scalar.
(a) \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \)
(b) \( (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \)
(c) \( (\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C} \)

□

33 (a) As suggested in the figure, find the area of the infinitesimal region expressed in polar coordinates as lying between \( r \) and \( r + dr \) and between \( \theta \) and \( \theta + d\theta \).

√ □

(b) Generalize this to find the infinitesimal element of volume in cylindrical coordinates \((r, \theta, z)\), where the Cartesian \( z \) axis is perpendicular to the directions measured by \( r \) and \( \theta \).

√ □

(c) Find the moment of inertia for rotation about its axis of a cone whose mass is \( M \), whose height is \( h \), and whose base has a radius \( b \).

√ □

34 Find the moment of inertia of a solid rectangular box of mass \( M \) and uniform density, whose sides are of length \( a, b, \) and \( c \), for rotation about an axis through its center parallel to the edges of

Problems
length $a$.  

\[ \sqrt{300} \]