where the vectors \( \hat{x}, \hat{y}, \) and \( \hat{z}, \) called the unit vectors, are defined as the vectors that have magnitude equal to 1 and directions lying along the \( x, y, \) and \( z \) axes. In speech, they are referred to as “x-hat,” “y-hat,” and “z-hat.”

A slightly different, and harder to remember, version of this notation is unfortunately more prevalent. In this version, the unit vectors are called \( \hat{i}, \hat{j}, \) and \( \hat{k}: \)

\[
\Delta r = (290 \text{ km})\hat{i} + (230 \text{ km})\hat{j}
\]

Applications to relative motion, momentum, and force

Vector addition is the correct way to generalize the one-dimensional concept of adding velocities in relative motion, as shown in the following example:

\[\text{Velocity vectors in relative motion example 65}\]

\(\triangleright\) You wish to cross a river and arrive at a dock that is directly across from you, but the river’s current will tend to carry you downstream. To compensate, you must steer the boat at an angle. Find the angle \( \theta \), given the magnitude, \(|v_{WL}|\), of the water’s velocity relative to the land, and the maximum speed, \(|v_{BW}|\), of which the boat is capable relative to the water.

\(\triangleright\) The boat’s velocity relative to the land equals the vector sum of its velocity with respect to the water and the water’s velocity with respect to the land,

\[
v_{BL} = v_{BW} + v_{WL}
\]

If the boat is to travel straight across the river, i.e., along the \( y \) axis, then we need to have \( v_{BL,x} = 0 \). This \( x \) component equals the sum of the \( x \) components of the other two vectors,

\[
v_{BL,x} = v_{BW,x} + v_{WL,x}
\]

or

\[
0 = -|v_{BW}| \sin \theta + |v_{WL}|
\]

Solving for \( \theta \), we find

\[
\sin \theta = \frac{|v_{WL}|}{|v_{BW}|}
\]

so

\[
\theta = \sin^{-1} \left( \frac{|v_{WL}|}{v_{BW}} \right)
\]
How to generalize one-dimensional equations example 66

How can the one-dimensional relationships

\[ p_{\text{total}} = m_{\text{total}} v_{\text{cm}} \]

and

\[ x_{\text{cm}} = \frac{\sum_j m_j x_j}{\sum_j m_j} \]

be generalized to three dimensions?

Momentum and velocity are vectors, since they have directions in space. Mass is a scalar. If we rewrite the first equation to show the appropriate quantities notated as vectors,

\[ \mathbf{p}_{\text{total}} = m_{\text{total}} \mathbf{v}_{\text{cm}} \]

we get a valid mathematical operation, the multiplication of a vector by a scalar. Similarly, the second equation becomes

\[ \mathbf{r}_{\text{cm}} = \frac{\sum_j m_j \mathbf{r}_j}{\sum_j m_j} \]

which is also valid. Each term in the sum on top contains a vector multiplied by a scalar, which gives a vector. Adding up all these vectors gives a vector, and dividing by the scalar sum on the bottom gives another vector.

This kind of wave-the-magic-wand-and-write-it-all-in-bold-face technique will always give the right generalization from one dimension to three, provided that the result makes sense mathematically — if you find yourself doing something nonsensical, such as adding a scalar to a vector, then you haven’t found the generalization correctly.

Colliding coins example 67

Take two identical coins, put one down on a piece of paper, and slide the other across the paper, shooting it fairly rapidly so that it hits the target coin off-center. If you trace the initial and final positions of the coins, you can determine the directions of their momentum vectors after the collision. The angle between these vectors is always fairly close to, but a little less than, 90 degrees. Why is this?

Let the velocity vector of the incoming coin be \( \mathbf{a} \), and let the two outgoing velocity vectors be \( \mathbf{b} \) and \( \mathbf{c} \). Since the masses are the same, conservation of momentum amounts to \( \mathbf{a} = \mathbf{b} + \mathbf{c} \), which means that it has to be possible to assemble the three vectors into a triangle. If we assume that no energy is converted into heat and sound, then conservation of energy gives (discarding the common factor of \( m/2 \)) \( a^2 = b^2 + c^2 \) for the magnitudes of the
three vectors. This is the Pythagorean theorem, which will hold only if the three vectors form a right triangle.

The fact that we observe the angle to be somewhat less than 90 degrees shows that the assumption used in the proof is only approximately valid: a little energy is converted into heat and sound. The opposite case would be a collision between two blobs of putty, where the maximum possible amount of energy is converted into heat and sound, the two blobs fly off together, giving an angle of zero between their momentum vectors. The real-life experiment interpolates between the ideal extremes of 0 and 90 degrees, but comes much closer to 90.

Force is a vector, and we add force vectors when more than one force acts on the same object.

Pushing a block up a ramp  

Figure s/1 shows a block being pushed up a frictionless ramp at constant speed by an applied force $F_a$. How much force is required, in terms of the block’s mass, $m$, and the angle of the ramp, $\theta$?

We analyzed this simple machine in example 38 on page 168 using the concept of work. Here we’ll do it using vector addition of forces. Figure s/2 shows the other two forces acting on the block: a normal force, $F_n$, created by the ramp, and the gravitational force, $F_g$. Because the block is being pushed up at constant speed, it has zero acceleration, and the total force on it must be zero. In figure s/3, we position all the force vectors tip-to-tail for addition. Since they have to add up to zero, they must join up without leaving a gap, so they form a triangle. Using trigonometry we find

$$F_a = F_g \sin \theta$$

$$= mg \sin \theta .$$

Buoyancy, again  

In example 10 on page 85, we found that the energy required to raise a cube immersed in a fluid is as if the cube’s mass had been reduced by an amount equal to the mass of the fluid that otherwise would have been in the volume it occupies (Archimedes’ principle). From the energy perspective, this effect occurs because raising the cube allows a certain amount of fluid to move downward, and the decreased gravitational energy of the fluid tends to offset the increased gravitational energy of the cube. The proof given there, however, could not easily be extended to other shapes.

Thinking in terms of force rather than energy, it becomes easier to give a proof that works for any shape. A certain upward force is
needed to support the object in figure t. If this force was applied, then the object would be in equilibrium: the vector sum of all the forces acting on it would be zero. These forces are $F_a$, the upward force just mentioned, $F_g$, the downward force of gravity, and $F_f$, the total force from the fluid:

$$F_a + F_g + F_f = 0$$

Since the fluid is under more pressure at a greater depth, the part of the fluid underneath the object tends to make more force than the part above, so the fluid tends to help support the object.

Now suppose the object was removed, and instantly replaced with an equal volume of fluid. The new fluid would be in equilibrium without any force applied to hold it up, so

$$F_{gf} + F_f = 0$$

where $F_{gf}$, the weight of the fluid, is not the same as $F_g$, the weight of the object, but $F_f$ is the same as before, since the pressure of the surrounding fluid is the same as before at any particular depth. We therefore have

$$F_a = -(F_g - F_{gf})$$

which is Archimedes’ principle in terms of force: the force required to support the object is lessened by an amount equal to the weight of the fluid that would have occupied its volume.

By the way, the word “pressure” that I threw around casually in the preceding example has a precise technical definition: force per unit area. The SI units of pressure are N/m², which can be abbreviated as pascals, 1 Pa = 1 N/m². Atmospheric pressure is about 100 kPa. By applying the equation $F_g + F_f = 0$ to the top and bottom surfaces of a cubical volume of fluid, one can easily prove that the difference in pressure between two different depths is $\Delta P = \rho g \Delta y$. (In physics, “fluid” can refer to either a gas or a liquid.) Pressure is discussed in more detail in chapter 5.

A solar sail example 70

A solar sail, figure u/1, allows a spacecraft to get its thrust without using internal stores of energy or having to carry along mass that it can shove out the back like a rocket. Sunlight strikes the sail and bounces off, transferring momentum to the sail. A working 30-meter-diameter solar sail, Cosmos 1, was built by an American company, and was supposed to be launched into orbit aboard a Russian booster launched from a submarine, but launch attempts in 2001 and 2005 both failed.

In this example, we will calculate the optimal orientation of the sail, assuming that “optimal” means changing the vehicle’s energy as rapidly as possible. For simplicity, we model the complicated shape of the sail’s surface as a disk, seen edge-on in figure u/2, and we assume that the craft is in a nearly circular orbit
around the sun, hence the 90-degree angle between the direction of motion and the incoming sunlight. We assume that the sail is 100% reflective. The orientation of the sail is specified using the angle $\theta$ between the incoming rays of sunlight and the perpendicular to the sail. In other words, $\theta=0$ if the sail is catching the sunlight full-on, while $\theta=90^\circ$ means that the sail is edge-on to the sun.

Conservation of momentum gives

$$p_{\text{light},i} = p_{\text{light},f} + \Delta p_{\text{sail}},$$

where $\Delta p_{\text{sail}}$ is the change in momentum picked up by the sail. Breaking this down into components, we have

$$0 = p_{\text{light},f,x} + \Delta p_{\text{sail},x} \quad \text{and} \quad p_{\text{light},i,y} = p_{\text{light},f,y} + \Delta p_{\text{sail},y}.$$

As in example 53 on page 189, the component of the force that is directly away from the sun (up in figure u/2) doesn’t change the energy of the craft, so we only care about $\Delta p_{\text{sail},x}$, which equals $-p_{\text{light},f,x}$. The outgoing light ray forms an angle of $2\theta$ with the negative $y$ axis, or $270^\circ - 2\theta$ measured counterclockwise from the $x$ axis, so the useful thrust depends on $-\cos(270^\circ - 2\theta) = \sin 2\theta$.

However, this is all assuming a given amount of light strikes the sail. During a certain time period, the amount of sunlight striking the sail depends on the cross-sectional area the sail presents to the sun, which is proportional to $\cos \theta$. For $\theta=90^\circ$, $\cos \theta$ equals zero, since the sail is edge-on to the sun.

Putting together these two factors, the useful thrust is proportional to $\sin 2\theta \cos \theta$, and this quantity is maximized for $\theta \approx 35^\circ$. A counterintuitive fact about this maneuver is that as the spacecraft spirals outward, its total energy (kinetic plus gravitational) increases, but its kinetic energy actually decreases!

A layback example 71

The figure shows a rock climber using a technique called a layback. He can make the normal forces $F_{N1}$ and $F_{N2}$ large, which has the side-effect of increasing the frictional forces $F_{F1}$ and $F_{F2}$, so that he doesn’t slip down due to the gravitational (weight) force $F_{W}$. The purpose of the problem is not to analyze all of this in detail, but simply to practice finding the components of the forces based on their magnitudes. To keep the notation simple, let’s write $F_{N1}$ for $|F_{N1}|$, etc. The crack overhangs by a small, positive angle $\theta \approx 9^\circ$.

In this example, we determine the $x$ component of $F_{N1}$. The other nine components are left as an exercise to the reader (problem 81, p. 236).
The easiest method is the one demonstrated in example 62 on p. 199. Casting vector $N_1$'s shadow on the ground, we can tell that it would point to the left, so its $x$ component is negative. The only two possibilities for its $x$ component are therefore $-F_{N_1} \cos \theta$ or $-F_{N_1} \sin \theta$. We expect this force to have a large $x$ component and a much smaller $y$. Since $\theta$ is small, $\cos \theta \approx 1$, while $\sin \theta$ is small. Therefore the $x$ component must be $-F_{N_1} \cos \theta$.

Discussion Questions

A An object goes from one point in space to another. After it arrives at its destination, how does the magnitude of its $\Delta r$ vector compare with the distance it traveled?

B In several examples, I've dealt with vectors having negative components. Does it make sense as well to talk about negative and positive vectors?

C If you're doing graphical addition of vectors, does it matter which vector you start with and which vector you start from the other vector's tip?

D If you add a vector with magnitude 1 to a vector of magnitude 2, what magnitudes are possible for the vector sum?

E Which of these examples of vector addition are correct, and which are incorrect?

F Is it possible for an airplane to maintain a constant velocity vector but not a constant $|v|$? How about the opposite – a constant $|v|$ but not a constant velocity vector? Explain.

w / Example 71 and problem 81 on p. 236.
New York and Rome are at about the same latitude, so the earth’s rotation carries them both around nearly the same circle. Do the two cities have the same velocity vector (relative to the center of the earth)? If not, is there any way for two cities to have the same velocity vector?

The figure shows a roller coaster car rolling down and then up under the influence of gravity. Sketch the car’s velocity vectors and acceleration vectors. Pick an interesting point in the motion and sketch a set of force vectors acting on the car whose vector sum could have resulted in the right acceleration vector.

The following is a question commonly asked by students:

“Why does the force vector always have to point in the same direction as the acceleration vector? What if you suddenly decide to change your force on an object, so that your force is no longer pointing in the same direction that the object is accelerating?”

What misunderstanding is demonstrated by this question? Suppose, for example, a spacecraft is blasting its rear main engines while moving forward, then suddenly begins firing its sideways maneuvering rocket as well. What does the student think Newton’s laws are predicting?

Debug the following incorrect solutions to this vector addition problem.

Problem: Freddi Fish™ swims 5.0 km northeast, and then 12.0 km in the direction 55 degrees west of south. How far does she end up from her starting point, and in what direction is she from her starting point?

Incorrect solution #1: 
\[ 5.0 \text{ km} + 12.0 \text{ km} = 17.0 \text{ km} \]

Incorrect solution #2: 
\[ \sqrt{(5.0 \text{ km})^2 + (12.0 \text{ km})^2} = 13.0 \text{ km} \]

Incorrect solution #3: 
Let \( A \) and \( B \) be her two \( \Delta r \) vectors, and let \( C = A + B \). Then

\[ A_x = (5.0 \text{ km}) \cos 45^\circ = 3.5 \text{ km} \]
\[ B_x = (12.0 \text{ km}) \cos 55^\circ = 6.9 \text{ km} \]
\[ A_y = (5.0 \text{ km}) \sin 45^\circ = 3.5 \text{ km} \]
\[ B_y = (12.0 \text{ km}) \sin 55^\circ = 9.8 \text{ km} \]
\[ C_x = A_x + B_x = 10.4 \text{ km} \]
\[ C_y = A_y + B_y = 13.3 \text{ km} \]
\[ |C| = \sqrt{C_x^2 + C_y^2} = 16.9 \text{ km} \]
\[ \text{direction} = \tan^{-1}(13.3/10.4) = 52^\circ \text{ north of east} \]
Incorrect solution #4:  
(same notation as above)

\[
A_x = (5.0 \text{ km}) \cos 45^\circ = 3.5 \text{ km} \\
B_x = -(12.0 \text{ km}) \cos 55^\circ = -6.9 \text{ km} \\
A_y = (5.0 \text{ km}) \sin 45^\circ = 3.5 \text{ km} \\
B_y = -(12.0 \text{ km}) \sin 55^\circ = -9.8 \text{ km} \\
C_x = A_x + B_x \\
\quad = -3.4 \text{ km} \\
C_y = A_y + B_y \\
\quad = -6.3 \text{ km} \\
|C| = \sqrt{C_x^2 + C_y^2} \\
\quad = 7.2 \text{ km} \\
direction = \tan^{-1}(-6.3/-3.4) \\
\quad = 62^\circ \text{ north of east}
\]

Incorrect solution #5:  
(same notation as above)

\[
A_x = (5.0 \text{ km}) \cos 45^\circ = 3.5 \text{ km} \\
B_x = -(12.0 \text{ km}) \sin 55^\circ = -9.8 \text{ km} \\
A_y = (5.0 \text{ km}) \sin 45^\circ = 3.5 \text{ km} \\
B_y = -(12.0 \text{ km}) \cos 55^\circ = -6.9 \text{ km} \\
C_x = A_x + B_x \\
\quad = -6.3 \text{ km} \\
C_y = A_y + B_y \\
\quad = -3.4 \text{ km} \\
|C| = \sqrt{C_x^2 + C_y^2} \\
\quad = 7.2 \text{ km} \\
direction = \tan^{-1}(-3.4/-6.3) \\
\quad = 28^\circ \text{ north of east}
\]

3.4.4 Calculus with vectors

Differentiation

In one dimension, we define the velocity as the derivative of the position with respect to time, and we can think of the derivative as what we get when we calculate \(\Delta x/\Delta t\) for very short time intervals. The quantity \(\Delta x = x_f - x_i\) is calculated by subtraction. In three dimensions, \(x\) becomes \(\mathbf{r}\), and the \(\Delta \mathbf{r}\) vector is calculated by vector subtraction, \(\Delta \mathbf{r} = \mathbf{r}_f - \mathbf{r}_i\). Vector subtraction is defined component by component, so when we take the derivative of a vector, this means we end up taking the derivative component by component,

\[
v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}
\]
This figure shows an intuitive justification for the fact proved mathematically in the example, that the direction of the force and acceleration in circular motion is inward. The heptagon, 2, is a better approximation to a circle than the triangle, 1. To make an infinitely good approximation to circular motion, we would need to use an infinitely large number of infinitesimal taps, which would amount to a steady inward force.

or

\[
\frac{\mathbf{d}r}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}
\]

All of this reasoning applies equally well to any derivative of a vector, so for instance we can take the second derivative,

\[
a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt}
\]

or

\[
\frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{x} + \frac{dv_y}{dt}\hat{y} + \frac{dv_z}{dt}\hat{z}
\]

A counterintuitive consequence of this is that the acceleration vector does not need to be in the same direction as the motion. The velocity vector points in the direction of motion, but by Newton’s second law, \( \mathbf{a} = \mathbf{F}/m \), the acceleration vector points in the same direction as the force, not the motion. This is easiest to understand if we take velocity vectors from two different moments in the motion, and visualize subtracting them graphically to make a \( \Delta \mathbf{v} \) vector. The direction of the \( \Delta \mathbf{v} \) vector tells us the direction of the acceleration vector as well, since the derivative \( d\mathbf{v}/dt \) can be approximated as \( \Delta \mathbf{v}/\Delta t \). As shown in figure z/1, a change in the magnitude of the velocity vector implies an acceleration that is in the direction of motion. A change in the direction of the velocity vector produces an acceleration perpendicular to the motion, z/2.

Circular motion

\[\text{example 72}\]

An object moving in a circle of radius \( r \) in the \( x\)-\( y \) plane has

\[
x = r \cos \omega t \quad \text{and} \quad y = r \sin \omega t,
\]

where \( \omega \) is the number of radians traveled per second, and the positive or negative sign indicates whether the motion is clockwise or counterclockwise. What is its acceleration?

The components of the velocity are

\[
\mathbf{v_x} = -\omega r \sin \omega t \quad \text{and} \quad \mathbf{v_y} = \omega r \cos \omega t,
\]

and for the acceleration we have

\[
\mathbf{a_x} = -\omega^2 r \cos \omega t \quad \text{and} \quad \mathbf{a_y} = -\omega^2 r \sin \omega t.
\]

The acceleration vector has cosines and sines in the same places as the \( r \) vector, but with minus signs in front, so it points in the opposite direction, i.e., toward the center of the circle. By Newton’s second law, \( \mathbf{a} = \mathbf{F}/m \), this shows that the force must be inward as well; without this force, the object would fly off straight.
The magnitude of the acceleration is

\[ |\mathbf{a}| = \sqrt{a_x^2 + a_y^2} = \omega^2 r. \]

It makes sense that \( \omega \) is squared, since reversing the sign of \( \omega \) corresponds to reversing the direction of motion, but the acceleration is toward the center of the circle, regardless of whether the motion is clockwise or counterclockwise. This result can also be rewritten in the form

\[ |\mathbf{a}| = \frac{|\mathbf{v}|^2}{r}. \]

Although I’ve relegated the results \( a = \omega^2 r = |\mathbf{v}|^2 / r \) to an example because they are a straightforward corollary of more general principles already developed, they are important and useful enough to record for later use. These results are counterintuitive as well. Until Newton, physicists and laypeople alike had assumed that the planets would need a force to push them forward in their orbits. Figure aa may help to make it more plausible that only an inward force is required. A forward force might be needed in order to cancel out a backward force such as friction, ab, but the total force in the forward-backward direction needs to be exactly zero for constant-speed motion. When you are in a car undergoing circular motion, there is also a strong illusion of an outward force. But what object could be making such a force? The car’s seat makes an inward force on you, not an outward one. There is no object that could be exerting an outward force on your body. In reality, this force is an illusion that comes from our brain’s intuitive efforts to interpret the situation within a noninertial frame of reference. As shown in figure ac, we can describe everything perfectly well in an inertial frame of reference, such as the frame attached to the sidewalk. In such a frame, the bowling ball goes straight because there is no force on it. The wall of the truck’s bed hits the ball, not the other way around.

Integration

An integral is really just a sum of many infinitesimally small terms. Since vector addition is defined in terms of addition of the components, an integral of a vector quantity is found by doing integrals component by component.

Projectile motion example 73

Find the motion of an object whose acceleration vector is constant, for instance a projectile moving under the influence of gravity.

We integrate the acceleration to get the velocity, and then integrate the velocity to get the position as a function of time. Doing
this to the \( x \) component of the acceleration, we find

\[
x = \int \left( \int a_x \, dt \right) \, dt
\]

\[
= \int (a_x t + v_{xo}) \, dt,
\]

where \( v_{xo} \) is a constant of integration, and

\[
x = \frac{1}{2} a_x t^2 + v_{xo} t + x_0.
\]

Similarly, \( y = (1/2) a_y t^2 + v_{yo} t + y_0 \) and \( z = (1/2) a_z t^2 + v_{zo} t + z_0 \). Once one has gained a little confidence, it becomes natural to do the whole thing as a single vector integral,

\[
r = \int \left( \int a \, dt \right) \, dt
\]

\[
= \int (a t + v_0) \, dt
\]

\[
= \frac{1}{2} a t^2 + v_0 t + r_0,
\]

where now the constants of integration are vectors.

**Discussion Questions**

**A** In the game of crack the whip, a line of people stand holding hands, and then they start sweeping out a circle. One person is at the center, and rotates without changing location. At the opposite end is the person who is running the fastest, in a wide circle. In this game, someone always ends up losing their grip and flying off. Suppose the person on the end loses her grip. What path does she follow as she goes flying off? (Assume she is going so fast that she is really just trying to put one foot in front of the other fast enough to keep from falling; she is not able to get any significant horizontal force between her feet and the ground.)

**B** Suppose the person on the outside is still holding on, but feels that she may lose her grip at any moment. What force or forces are acting on her, and in what directions are they? (We are not interested in the vertical forces, which are the earth’s gravitational force pulling down, and the ground’s normal force pushing up.) Make a table in the format shown in subsection 3.2.6.

**C** Suppose the person on the outside is still holding on, but feels that she may lose her grip at any moment. What is wrong with the following analysis of the situation? “The person whose hand she’s holding exerts an inward force on her, and because of Newton’s third law, there’s an equal and opposite force acting outward. That outward force is the one she feels throwing her outward, and the outward force is what might make her go flying off, if it’s strong enough.”

**D** If the only force felt by the person on the outside is an inward force, why doesn’t she go straight in?

**E** In the amusement park ride shown in the figure, the cylinder spins faster and faster until the customer can pick her feet up off the floor without falling. In the old Coney Island version of the ride, the floor actually
dropped out like a trap door, showing the ocean below. (There is also a version in which the whole thing tilts up diagonally, but we’re discussing the version that stays flat.) If there is no outward force acting on her, why does she stick to the wall? Analyze all the forces on her.

F What is an example of circular motion where the inward force is a normal force? What is an example of circular motion where the inward force is friction? What is an example of circular motion where the inward force is the sum of more than one force?

G Does the acceleration vector always change continuously in circular motion? The velocity vector?

H A certain amount of force is needed to provide the acceleration of circular motion. What if we are exerting a force perpendicular to the direction of motion in an attempt to make an object trace a circle of radius \( r \), but the force isn’t as big as \( m|v|^2/r \)?

I Suppose a rotating space station is built that gives its occupants the illusion of ordinary gravity. What happens when a person in the station lets go of a ball? What happens when she throws a ball straight “up” in the air (i.e., towards the center)?

3.4.5 The dot product

How would we generalize the mechanical work equation \( dE = F\, dx \) to three dimensions? Energy is a scalar, but force and distance are vectors, so it might seem at first that the kind of “magic-wand” generalization discussed on page 202 failed here, since we don’t know of any way to multiply two vectors together to get a scalar. Actually, this is Nature giving us a hint that there is such a multiplication operation waiting for us to invent it, and since Nature is simple, we can be assured that this operation will work just fine in any situation where a similar generalization is required.

How should this operation be defined? Let’s consider what we would get by performing this operation on various combinations of the unit vectors \( \mathbf{\hat{x}}, \mathbf{\hat{y}}, \) and \( \mathbf{\hat{z}} \). The conventional notation for the operation is to put a dot, \( \cdot \), between the two vectors, and the operation is therefore called the dot product. Rotational invariance requires that we handle the three coordinate axes in the same way, without giving special treatment to any of them, so we must have \( \mathbf{\hat{x}} \cdot \mathbf{\hat{x}} = \mathbf{\hat{y}} \cdot \mathbf{\hat{y}} = \mathbf{\hat{z}} \cdot \mathbf{\hat{z}} \) and \( \mathbf{\hat{x}} \cdot \mathbf{\hat{y}} = \mathbf{\hat{y}} \cdot \mathbf{\hat{z}} = \mathbf{\hat{z}} \cdot \mathbf{\hat{x}} \). This is supposed to be a way of generalizing ordinary multiplication, so for consistency with the property \( 1 \times 1 = 1 \) of ordinary numbers, the result of multiplying a magnitude-one vector by itself had better be the scalar 1, so \( \mathbf{\hat{x}} \cdot \mathbf{\hat{x}} = \mathbf{\hat{y}} \cdot \mathbf{\hat{y}} = \mathbf{\hat{z}} \cdot \mathbf{\hat{z}} = 1 \). Furthermore, there is no way to satisfy rotational invariance unless we define the mixed products to be zero, \( \mathbf{\hat{x}} \cdot \mathbf{\hat{y}} = \mathbf{\hat{y}} \cdot \mathbf{\hat{z}} = \mathbf{\hat{z}} \cdot \mathbf{\hat{x}} = 0 \); for example, a 90-degree rotation of our frame of reference about the z axis reverses the sign of \( \mathbf{\hat{x}} \cdot \mathbf{\hat{y}} \), but rotational invariance requires that \( \mathbf{\hat{x}} \cdot \mathbf{\hat{y}} \) produce the same result either way, and zero is the only number that stays the same when we reverse its sign. Establishing these six products of unit vectors suffices to define the operation in general, since any
two vectors that we want to multiply can be broken down into components, e.g., \((2\hat{x} + 3\hat{z}) \cdot \hat{z} = 2\hat{x} \cdot \hat{z} + 3\hat{z} \cdot \hat{z} = 0 + 3 = 3\). Thus by requiring rotational invariance and consistency with multiplication of ordinary numbers, we find that there is only one possible way to define a multiplication operation on two vectors that gives a scalar as the result.\(^{17}\) The dot product has all of the properties we normally associate with multiplication, except that there is no “dot division.”

**Dot product in terms of components** \(^{1}\) example 74
If we know the components of any two vectors \(\mathbf{b}\) and \(\mathbf{c}\), we can find their dot product:

\[
\mathbf{b} \cdot \mathbf{c} = (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) \cdot (c_x\hat{x} + c_y\hat{y} + c_z\hat{z}) = b_xc_x + b_yc_y + b_zc_z.
\]

**Magnitude expressed with a dot product** \(^{1}\) example 75
If we take the dot product of any vector \(\mathbf{b}\) with itself, we find

\[
\mathbf{b} \cdot \mathbf{b} = (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) \cdot (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) = b_x^2 + b_y^2 + b_z^2,
\]

so its magnitude can be expressed as

\[
|\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}}.
\]

We will often write \(\mathbf{b}^2\) to mean \(\mathbf{b} \cdot \mathbf{b}\), when the context makes it clear what is intended. For example, we could express kinetic energy as \((1/2)m|\mathbf{v}|^2\), \((1/2)m\mathbf{v} \cdot \mathbf{v}\), or \((1/2)m\mathbf{v}^2\). In the third version, nothing but context tells us that \(\mathbf{v}\) really stands for the magnitude of some vector \(\mathbf{v}\).

**Geometric interpretation** \(^{1}\) example 76
In figure af, vectors \(\mathbf{a}\), \(\mathbf{b}\), and \(\mathbf{c}\) represent the sides of a triangle, and \(\mathbf{a} = \mathbf{b} + \mathbf{c}\). The law of cosines gives

\[
|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta.
\]

Using the result of example 75, we can also write this as

\[
|\mathbf{c}|^2 = \mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b}.
\]

Matching up terms in these two expressions, we find

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.
\]

\(^{17}\)There is, however, a different operation, discussed in the next chapter, which multiplies two vectors to give a vector.
which is a geometric interpretation for the dot product.

The result of example 76 is very useful. It gives us a way to find the angle between two vectors if we know their components. It can be used to show that the dot product of any two perpendicular vectors is zero. It also leads to a nifty proof that the dot product is rotationally invariant — up until now I’ve only proved that if a rotationally invariant product exists, the dot product is it — because angles and lengths aren’t affected by a rotation, so the right side of the equation is rotationally invariant, and therefore so is the left side.

I introduced the whole discussion of the dot product by way of generalizing the equation $dE = F \, dx$ to three dimensions. In terms of a dot product, we have

$$dE = \mathbf{F} \cdot d\mathbf{r}.$$ 

If $\mathbf{F}$ is a constant, integrating both sides gives

$$\Delta E = \mathbf{F} \cdot \Delta \mathbf{r}.$$ 

(If that step seemed like black magic, try writing it out in terms of components.) If the force is perpendicular to the motion, as in figure ag, then the work done is zero. The pack horse is doing work within its own body, but is not doing work on the pack.

### Pushing a lawnmower

**Example 77**

- I push a lawnmower with a force $\mathbf{F} = (110 \text{ N})\hat{x} - (40 \text{ N})\hat{y}$, and the total distance I travel is $(100 \text{ m})\hat{x}$. How much work do I do?

- The dot product is $11000 \text{ N} \cdot \text{m} = 11000 \text{ J}$.

A good application of the dot product is to allow us to write a simple, streamlined proof of separate conservation of the momentum components. (You can skip the proof without losing the continuity of the text.) The argument is a generalization of the one-dimensional proof on page 130, and makes the same assumption about the type of system of particles we’re dealing with. The kinetic energy of one of the particles is $(1/2)m v \cdot v$, and when we transform into a different frame of reference moving with velocity $\mathbf{u}$ relative to the original frame, the one-dimensional rule $v \rightarrow v + u$ turns into vector addition, $\mathbf{v} \rightarrow \mathbf{v} + \mathbf{u}$. In the new frame of reference, the kinetic energy is $(1/2)m(\mathbf{v} + \mathbf{u}) \cdot (\mathbf{v} + \mathbf{u})$. For a system of $n$ particles, we have

$$K = \sum_{j=1}^{n} \frac{1}{2} m_j (\mathbf{v}_j + \mathbf{u}) \cdot (\mathbf{v}_j + \mathbf{u})$$

$$= \frac{1}{2} \left[ \sum_{j=1}^{n} m_j \mathbf{v}_j \cdot \mathbf{v}_j + 2 \sum_{j=1}^{n} m_j \mathbf{v}_j \cdot \mathbf{u} + \sum_{j=1}^{n} m_j \mathbf{u} \cdot \mathbf{u} \right].$$
As in the proof on page 130, the first sum is simply the total kinetic energy in the original frame of reference, and the last sum is a constant, which has no effect on the validity of the conservation law. The middle sum can be rewritten as

\[
2 \sum_{j=1}^{n} m_j v_j \cdot \mathbf{u} = 2 \mathbf{u} \cdot \sum_{j=1}^{n} m_j v_j = 2 \mathbf{u} \cdot \sum_{j=1}^{n} \mathbf{p}_j
\]

so the only way energy can be conserved for all values of \( \mathbf{u} \) is if the vector sum of the momenta is conserved as well.

### 3.4.6 Gradients and line integrals (optional)

This subsection introduces a little bit of vector calculus. It can be omitted without loss of continuity, but the techniques will be needed in our study of electricity and magnetism, and it may be helpful to be exposed to them in easy-to-visualize mechanical contexts before applying them to invisible electrical and magnetic phenomena.

In physics we often deal with fields of force, meaning situations where the force on an object depends on its position. For instance, figure ah could represent a map of the trade winds affecting a sailing ship, or a chart of the gravitational forces experienced by a space probe entering a double-star system. An object moving under the influence of this force will not necessarily be moving in the same direction as the force at every moment. The sailing ship can tack against the wind, due to the force from the water on the keel. The space probe, if it entered from the top of the diagram at high speed, would start to curve around to the right, but its inertia would carry it forward, and it wouldn’t instantly swerve to match the direction of the gravitational force. For convenience, we’ve defined the gravitational field, \( \mathbf{g} \), as the force per unit mass, but that trick only leads to a simplification because the gravitational force on an object is proportional to its mass. Since this subsection is meant to apply to any kind of force, we’ll discuss everything in terms of the actual force vector, \( \mathbf{F} \), in units of newtons.

If an object moves through the field of force along some curved path from point \( \mathbf{r}_1 \) to point \( \mathbf{r}_2 \), the force will do a certain amount of work on it. To calculate this work, we can break the path up into infinitesimally short segments, find the work done along each segment, and add them all up. For an object traveling along a nice straight \( x \) axis, we use the symbol \( dx \) to indicate the length of any infinitesimally short segment. In three dimensions, moving along a curve, each segment is a tiny vector \( d\mathbf{r} = \hat{x} dx + \hat{y} dy + \hat{z} dz \). The work theorem can be expressed as a dot product, so the work done along a segment is \( \mathbf{F} \cdot d\mathbf{r} \). We want to integrate this, but we don’t
know how to integrate with respect to a variable that’s a vector, so let’s define a variable $s$ that indicates the distance traveled so far along the curve, and integrate with respect to it instead. The expression $\mathbf{F} \cdot d\mathbf{r}$ can be rewritten as $|\mathbf{F}| |d\mathbf{r}| \cos \theta$, where $\theta$ is the angle between $\mathbf{F}$ and $d\mathbf{r}$. But $|d\mathbf{r}|$ is simply $ds$, so the amount of work done becomes

$$\Delta E = \int_{r_1}^{r_2} |\mathbf{F}| \cos \theta \, ds.$$ 

Both $\mathbf{F}$ and $\theta$ are functions of $s$. As a matter of notation, it’s cumbersome to have to write the integral like this. Vector notation was designed to eliminate this kind of drudgery. We therefore define the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

as a way of notating this type of integral. The ‘$C$’ refers to the curve along which the object travels. If we don’t know this curve then we typically can’t evaluate the line integral just by knowing the initial and final positions $\mathbf{r}_1$ and $\mathbf{r}_2$.

The basic idea of calculus is that integration undoes differentiation, and vice-versa. In one dimension, we could describe an interaction either in terms of a force or in terms of an interaction energy. We could integrate force with respect to position to find minus the energy, or we could find the force by taking minus the derivative of the energy. In the line integral, position is represented by a vector. What would it mean to take a derivative with respect to a vector? The correct way to generalize the derivative $dU/dx$ to three dimensions is to replace it with the following vector,

$$\frac{dU}{dx} \hat{x} + \frac{dU}{dy} \hat{y} + \frac{dU}{dz} \hat{z},$$

called the gradient of $U$, and written with an upside-down delta like this, $\nabla U$. Each of these three derivatives is really what’s known as a partial derivative. What that means is that when you’re differentiating $U$ with respect to $x$, you’re supposed to treat $y$ and $z$ and constants, and similarly when you do the other two derivatives. To emphasize that a derivative is a partial derivative, it’s customary to write it using the symbol $\partial$ in place of the differential $d$’s. Putting all this notation together, we have

$$\nabla U = \frac{\partial U}{\partial x} \hat{x} + \frac{\partial U}{\partial y} \hat{y} + \frac{\partial U}{\partial z} \hat{z} \quad [\text{definition of the gradient}].$$

The gradient looks scary, but it has a very simple physical interpretation. It’s a vector that points in the direction in which $U$ is
increasing most rapidly, and it tells you how rapidly $U$ is increasing in that direction. For instance, sperm cells in plants and animals find the egg cells by traveling in the direction of the gradient of the concentration of certain hormones. When they reach the location of the strongest hormone concentration, they find their destiny. In terms of the gradient, the force corresponding to a given interaction energy is $F = -\nabla U$.

\textit{Force exerted by a spring \example{78}}

In one dimension, Hooke's law is $U = (1/2)kx^2$. Suppose we tether one end of a spring to a post, but it's free to stretch and swing around in a plane. Let's say its equilibrium length is zero, and let's choose the origin of our coordinate system to be at the post. Rotational invariance requires that its energy only depend on the magnitude of the $r$ vector, not its direction, so in two dimensions we have $U = (1/2)k|r|^2 = (1/2)k(x^2 + y^2)$. The force exerted by the spring is then

\[
F = -\nabla U = -\left(\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y}\right)
= -kx\hat{x} - ky\hat{y}.
\]

The magnitude of this force vector is $k|r|$, and its direction is toward the origin.

\textit{This chapter is summarized on page 952. Notation and terminology are tabulated on pages 945-946.}
Problems

The symbols √, □, etc. are explained on page 237.

1 Derive a formula expressing the kinetic energy of an object in terms of its momentum and mass. √ □

2 Two people in a rowboat wish to move around without causing the boat to move. What should be true about their total momentum? Explain. □

3 A bullet leaves the barrel of a gun with a kinetic energy of 90 J. The gun barrel is 50 cm long. The gun has a mass of 4 kg, the bullet 10 g.
(a) Find the bullet’s final velocity. ✓ □
(b) Find the bullet’s final momentum. ✓ □
(c) Find the momentum of the recoiling gun. ✓ □
(d) Find the kinetic energy of the recoiling gun, and explain why the recoiling gun does not kill the shooter. ✓ □

4 The big difference between the equations for momentum and kinetic energy is that one is proportional to \( v \) and one to \( v^2 \). Both, however, are proportional to \( m \). Suppose someone tells you that there’s a third quantity, funkosity, defined as \( f = m^2 v \), and that funkosity is conserved. How do you know your leg is being pulled?
▷ Solution, p. 937 □

5 A ball of mass 2\( m \) collides head-on with an initially stationary ball of mass \( m \). No kinetic energy is transformed into heat or sound. In what direction is the mass-2\( m \) ball moving after the collision, and how fast is it going compared to its original velocity?
▷ Answer, p. 933 □

6 A very massive object with velocity \( v \) collides head-on with an object at rest whose mass is very small. No kinetic energy is converted into other forms. Prove that the low-mass object recoils with velocity 2\( v \). [Hint: Use the center-of-mass frame of reference.] □

7 A mass \( m \) moving at velocity \( v \) collides with a stationary target having the same mass \( m \). Find the maximum amount of energy that can be released as heat and sound. ✓ □

8 A rocket ejects exhaust with an exhaust velocity \( u \). The rate at which the exhaust mass is used (mass per unit time) is \( b \). We assume that the rocket accelerates in a straight line starting from rest, and that no external forces act on it. Let the rocket’s initial mass (fuel plus the body and payload) be \( m_i \), and \( m_f \) be its final mass, after all the fuel is used up. (a) Find the rocket’s final velocity, \( v \), in terms of \( u \), \( m_i \), and \( m_f \). Neglect the effects of special relativity. (b) A typical exhaust velocity for chemical rocket engines is 4000 m/s. Estimate the initial mass of a rocket that could accelerate a one-ton payload to 10% of the speed of light, and show that this
design won’t work. (For the sake of the estimate, ignore the mass of the fuel tanks. The speed is fairly small compared to \( c \), so it’s not an unreasonable approximation to ignore relativity.)

\[ \sqrt{9} \]

An object is observed to be moving at constant speed along a line. Can you conclude that no forces are acting on it? Explain. [Based on a problem by Serway and Faughn.]

\[ \begin{align*} 
\text{At low speeds, every car’s acceleration is limited by traction, not by the engine’s power. Suppose that at low speeds, a certain car is normally capable of an acceleration of 3 m/s}^2. \text{ If it is towing a trailer with half as much mass as the car itself, what acceleration can it achieve? [Based on a problem from PSSC Physics.]} 
\end{align*} \]

11. (a) Let \( T \) be the maximum tension that an elevator’s cable can withstand without breaking, i.e., the maximum force it can exert. If the motor is programmed to give the car an acceleration \( a \) (\( a > 0 \) is upward), what is the maximum mass that the car can have, including passengers, if the cable is not to break? \( \sqrt{ } \)

(b) Interpret the equation you derived in the special cases of \( a = 0 \) and of a downward acceleration of magnitude \( g \).

12. When the contents of a refrigerator cool down, the changed molecular speeds imply changes in both momentum and energy. Why, then, does a fridge transfer power through its radiator coils, but not force? \( \triangleright \) Solution, p. 937

13. A helicopter of mass \( m \) is taking off vertically. The only forces acting on it are the earth’s gravitational force and the force, \( F_{\text{air}} \), of the air pushing up on the propeller blades.

(a) If the helicopter lifts off at \( t = 0 \), what is its vertical speed at time \( t \)?

(b) Check that the units of your answer to part a make sense.

(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you’ve figured out this mathematical relationship, show that it makes sense physically.

(d) Plug numbers into your equation from part a, using \( m = 2300 \) kg, \( F_{\text{air}} = 27000 \) N, and \( t = 4.0 \) s. \( \sqrt{ } \)

14. A blimp is initially at rest, hovering, when at \( t = 0 \) the pilot turns on the engine driving the propeller. The engine cannot instantly get the propeller going, but the propeller speeds up steadily. The steadily increasing force between the air and the propeller is given by the equation \( F = kt \), where \( k \) is a constant. If the mass of the blimp is \( m \), find its position as a function of time. (Assume that during the period of time you’re dealing with, the blimp is not yet moving fast enough to cause a significant backward force due to
Problem 16. A car is accelerating forward along a straight road. If the force of the road on the car’s wheels, pushing it forward, is a constant 3.0 kN, and the car’s mass is 1000 kg, then how long will the car take to go from 20 m/s to 50 m/s?

Solution, p. 937

Problem 19. A little old lady and a pro football player collide head-on. Compare their forces on each other, and compare their accelerations. Explain.

Problem 17. The earth is attracted to an object with a force equal and opposite to the force of the earth on the object. If this is true, why is it that when you drop an object, the earth does not have an acceleration equal and opposite to that of the object?

Problem 18. When you stand still, there are two forces acting on you, the force of gravity (your weight) and the normal force of the floor pushing up on your feet. Are these forces equal and opposite? Does Newton’s third law relate them to each other? Explain.

Problem 19. Today’s tallest buildings are really not that much taller than the tallest buildings of the 1940’s. One big problem with making an even taller skyscraper is that every elevator needs its own shaft running the whole height of the building. So many elevators are needed to serve the building’s thousands of occupants that the elevator shafts start taking up too much of the space within the building. An alternative is to have elevators that can move both horizontally and vertically: with such a design, many elevator cars can share a few shafts, and they don’t get in each other’s way too much because they can detour around each other. In this design, it becomes impossible to hang the cars from cables, so they would instead have to ride on rails which they grab onto with wheels. Friction would keep them from slipping. The figure shows such a frictional elevator in its vertical travel mode. (The wheels on the bottom are for when it needs to switch to horizontal motion.)

(a) If the coefficient of static friction between rubber and steel is $\mu_s$, and the maximum mass of the car plus its passengers is $M$, how much force must there be pressing each wheel against the rail in order to keep the car from slipping? (Assume the car is not accelerating.)

(b) Show that your result has physically reasonable behavior with respect to $\mu_s$. In other words, if there was less friction, would the wheels need to be pressed more firmly or less firmly? Does your equation behave that way?
20 A tugboat of mass $m$ pulls a ship of mass $M$, accelerating it. Ignore fluid friction acting on their hulls, although there will of course need to be fluid friction acting on the tug’s propellers.
(a) If the force acting on the tug’s propeller is $F$, what is the tension, $T$, in the cable connecting the two ships? \( \triangleright \) Hint, p. 923 \( \checkmark \)
(b) Interpret your answer in the special cases of $M = 0$ and $M = \infty$.

21 Someone tells you she knows of a certain type of Central American earthworm whose skin, when rubbed on polished diamond, has $\mu_k > \mu_s$. Why is this not just empirically unlikely but logically suspect?

22 A uranium atom deep in the earth spits out an alpha particle. An alpha particle is a fragment of an atom. This alpha particle has initial speed $v$, and travels a distance $d$ before stopping in the earth.
(a) Find the force, $F$, from the dirt that stopped the particle, in terms of $v, d$, and its mass, $m$. Don’t plug in any numbers yet. Assume that the force was constant.
(b) Show that your answer has the right units.
(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you’ve figured out this mathematical relationship, show that it makes sense physically.
(d) Evaluate your result for $m = 6.7 \times 10^{-27}$ kg, $v = 2.0 \times 10^4$ km/s, and $d = 0.71$ mm.

23 You are given a large sealed box, and are not allowed to open it. Which of the following experiments measure its mass, and which measure its weight? [Hint: Which experiments would give different results on the moon?]
(a) Put it on a frozen lake, throw a rock at it, and see how fast it scoots away after being hit.
(b) Drop it from a third-floor balcony, and measure how loud the sound is when it hits the ground.
(c) As shown in the figure, connect it with a spring to the wall, and watch it vibrate.
\( \triangleright \) Solution, p. 937

24 While escaping from the palace of the evil Martian emperor, Sally Spacehound jumps from a tower of height $h$ down to the ground. Ordinarily the fall would be fatal, but she fires her blaster rifle straight down, producing an upward force of magnitude $F_B$. This force is insufficient to levitate her, but it does cancel out some of the force of gravity. During the time $t$ that she is falling, Sally is unfortunately exposed to fire from the emperor’s minions, and can’t dodge their shots. Let $m$ be her mass, and $g$ the strength
of gravity on Mars.
(a) Find the time \( t \) in terms of the other variables.
(b) Check the units of your answer to part a.
(c) For sufficiently large values of \( F_B \), your answer to part a becomes nonsense — explain what’s going on.

25 When I cook rice, some of the dry grains always stick to the measuring cup. To get them out, I turn the measuring cup upside-down and hit the “roof” with my hand so that the grains come off of the “ceiling.” (a) Explain why static friction is irrelevant here. (b) Explain why gravity is negligible. (c) Explain why hitting the cup works, and why its success depends on hitting the cup hard enough.

26 A flexible rope of mass \( m \) and length \( L \) slides without friction over the edge of a table. Let \( x \) be the length of the rope that is hanging over the edge at a given moment in time.
(a) Show that \( x \) satisfies the equation of motion \( \frac{d^2 x}{dt^2} = \frac{gx}{L} \).
[Hint: Use \( F = \frac{dp}{dt} \), which allows you to handle the two parts of the rope separately even though mass is moving out of one part and into the other.]
(b) Give a physical explanation for the fact that a larger value of \( x \) on the right-hand side of the equation leads to a greater value of the acceleration on the left side.
(c) When we take the second derivative of the function \( x(t) \) we are supposed to get essentially the same function back again, except for a constant out in front. The function \( e^x \) has the property that it is unchanged by differentiation, so it is reasonable to look for solutions to this problem that are of the form \( x = be^{ct} \), where \( b \) and \( c \) are constants. Show that this does indeed provide a solution for two specific values of \( c \) (and for any value of \( b \)).
(d) Show that the sum of any two solutions to the equation of motion is also a solution.
(e) Find the solution for the case where the rope starts at rest at \( t = 0 \) with some nonzero value of \( x \).

In problems 27-31, analyze the forces using a table in the format shown in section 3.2.6. Analyze the forces in which the italicized object participates.

27 Some people put a spare car key in a little magnetic box that they stick under the chassis of their car. Let’s say that the box is stuck directly underneath a horizontal surface, and the car is parked. (See instructions above.)

28 Analyze two examples of objects at rest relative to the earth that are being kept from falling by forces other than the normal force. Do not use objects in outer space, and do not duplicate problem 27 or 31. (See instructions above.)
29  A person is rowing a boat, with her feet braced. She is doing the part of the stroke that propels the boat, with the ends of the oars in the water (not the part where the oars are out of the water). (See instructions above.)

30  A farmer is in a stall with a cow when the cow decides to press him against the wall, pinning him with his feet off the ground. Analyze the forces in which the farmer participates. (See instructions above.)

31  A propeller plane is cruising east at constant speed and altitude. (See instructions above.)

32  The figure shows a stack of two blocks, sitting on top of a table that is bolted to the floor. All three objects are made from identical wood, with their surfaces finished identically using the same sandpaper. We tap the middle block, giving it an initial velocity $v$ to the right. The tap is executed so rapidly that almost no initial velocity is imparted to the top block.

(a) Find the time that will elapse until the slipping between the top and middle blocks stops. Express your answer in terms of $v$, $m$, $M$, $g$, and the relevant coefficient of friction.

(b) Show that your answer makes sense in terms of units.

(c) Check that your result has the correct behavior when you make $m$ bigger or smaller. Explain. This means that you should discuss the mathematical behavior of the result, and then explain how this corresponds to what would really happen physically.

(d) Similarly, discuss what happens when you make $M$ bigger or smaller.

(e) Similarly, discuss what happens when you make $g$ bigger or smaller.

33  Ginny has a plan. She is going to ride her sled while her dog Foo pulls her, and she holds on to his leash. However, Ginny hasn’t taken physics, so there may be a problem: she may slide right off the sled when Foo starts pulling.

(a) Analyze all the forces in which Ginny participates, making a table as in subsection 3.2.6.

(b) Analyze all the forces in which the sled participates.

(c) The sled has mass $m$, and Ginny has mass $M$. The coefficient of static friction between the sled and the snow is $\mu_1$, and $\mu_2$ is the corresponding quantity for static friction between the sled and her snow pants. Ginny must have a certain minimum mass so that she will not slip off the sled. Find this in terms of the other three variables.

(d) Interpreting your equation from part c, under what conditions will there be no physically realistic solution for $M$? Discuss what this means physically.

34  In each case, identify the force that causes the acceleration, and give its Newton’s-third-law partner. Describe the effect of the
partner force. (a) A swimmer speeds up. (b) A golfer hits the ball off of the tee. (c) An archer fires an arrow. (d) A locomotive slows down.

35 A cop investigating the scene of an accident measures the length $L$ of a car’s skid marks in order to find out its speed $v$ at the beginning of the skid. Express $v$ in terms of $L$ and any other relevant variables.

36 An ice skater builds up some speed, and then coasted across the ice passively in a straight line. (a) Analyze the forces, using a table in the format shown in subsection 3.2.6.
(b) If his initial speed is $v$, and the coefficient of kinetic friction is $\mu_k$, find the maximum theoretical distance he can glide before coming to a stop. Ignore air resistance.
(c) Show that your answer to part b has the right units.
(d) Show that your answer to part b depends on the variables in a way that makes sense physically.
(e) Evaluate your answer numerically for $\mu_k = 0.0046$, and a world-record speed of 14.58 m/s. (The coefficient of friction was measured by De Koning et al., using special skates worn by real speed skaters.)
(f) Comment on whether your answer in part e seems realistic. If it doesn’t, suggest possible reasons why.

37 (a) Using the solution of problem 37 on page 124, predict how the spring constant of a fiber will depend on its length and cross-sectional area.
(b) The constant of proportionality is called the Young’s modulus, $E$, and typical values of the Young’s modulus are about $10^{10}$ to $10^{11}$. What units would the Young’s modulus have in the SI system?

38 This problem depends on the results of problems problem 37 on page 124 and problem 37 from this chapter. When atoms form chemical bonds, it makes sense to talk about the spring constant of the bond as a measure of how “stiff” it is. Of course, there aren’t really little springs — this is just a mechanical model. The purpose of this problem is to estimate the spring constant, $k$, for a single bond in a typical piece of solid matter. Suppose we have a fiber, like a hair or a piece of fishing line, and imagine for simplicity that it is made of atoms of a single element stacked in a cubical manner, as shown in the figure, with a center-to-center spacing $b$. A typical value for $b$ would be about $10^{-10}$ m.
(a) Find an equation for $k$ in terms of $b$, and in terms of the Young’s modulus, $E$, defined in problem 37 and its solution.
(b) Estimate $k$ using the numerical data given in problem 37.
(c) Suppose you could grab one of the atoms in a diatomic molecule like H$_2$ or O$_2$, and let the other atom hang vertically below it. Does
the bond stretch by any appreciable fraction due to gravity?  

39  This problem has been deleted.  

40  Many fish have an organ known as a swim bladder, an air-filled cavity whose main purpose is to control the fish’s buoyancy and allow it to keep from rising or sinking without having to use its muscles. In some fish, however, the swim bladder (or a small extension of it) is linked to the ear and serves the additional purpose of amplifying sound waves. For a typical fish having such an anatomy, the bladder has a resonant frequency of 300 Hz, the bladder’s Q is 3, and the maximum amplification is about a factor of 100 in energy. Over what range of frequencies would the amplification be at least a factor of 50?  

41  An oscillator with sufficiently strong damping has its maximum response at \( \omega = 0 \). Using the result derived on page 916, find the value of Q at which this behavior sets in.

> Hint, p. 923  > Answer, p. 934  

42  An oscillator has \( Q = 6.00 \), and, for convenience, let’s assume \( F_m = 1.00 \), \( \omega_o = 1.00 \), and \( m = 1.00 \). The usual approximations would give

\[
\begin{align*}
\omega_{res} &= \omega_o , \\
A_{res} &= 6.00 , \quad \text{and} \\
\Delta \omega &= 1/6.00 .
\end{align*}
\]

Determine these three quantities numerically using the result derived on page 916, and compare with the approximations.  

43  The apparatus in figure d on page 61 had a natural period of oscillation of 5 hours and 20 minutes. The authors estimated, based on calculations of internal friction in the tungsten wire, that its Q was on the order of \( 10^6 \), but they were unable to measure it empirically because it would have taken years for the amplitude to die down by any measurable amount. Although each aluminum or platinum mass was really moving along an arc of a circle, any actual oscillations caused by a violation of the equivalence of gravitational and inertial mass would have been measured in millions of a degree, so it’s a good approximation to say that each mass’s motion was along a (very short!) straight line segment. We can also treat each mass as if it was oscillating separately from the others. If the principle of equivalence had been violated at the \( 10^{-12} \) level, the limit of their experiment’s sensitivity, the sun’s gravitational force on one of
the 0.4-gram masses would have been about $3 \times 10^{-19}$ N, oscillating with a period of 24 hours due to the rotation of the earth. (We ignore the inertia of the arms, whose total mass was only about 25% of the total mass of the rotating assembly.)

(a) Find the amplitude of the resulting oscillations, and determine the angle to which they would have corresponded, given that the radius of the balance arms was 10 cm. ▷ Answer, p. 934

(b) Show that even if their estimate of $Q$ was wildly wrong, it wouldn’t have affected this result.

44 A firework shoots up into the air, and just before it explodes it has a certain momentum and kinetic energy. What can you say about the momenta and kinetic energies of the pieces immediately after the explosion? [Based on a problem from PSSC Physics.]

▷ Solution, p. 938

45 The figure shows a view from above of a collision about to happen between two air hockey pucks sliding without friction. They have the same speed, $v_i$, before the collision, but the big puck is 2.3 times more massive than the small one. Their sides have sticky stuff on them, so when they collide, they will stick together. At what angle will they emerge from the collision? In addition to giving a numerical answer, please indicate by drawing on the figure how your angle is defined.

▷ Solution, p. 938

46 A learjet traveling due east at 300 mi/hr collides with a jumbo jet which was heading southwest at 150 mi/hr. The jumbo jet’s mass is five times greater than that of the learjet. When they collide, the learjet sticks into the fuselage of the jumbo jet, and they fall to earth together. Their engines stop functioning immediately after the collision. On a map, what will be the direction from the location of the collision to the place where the wreckage hits the ground? (Give an angle.) 

47 (a) A ball is thrown straight up with velocity $v$. Find an equation for the height to which it rises.
(b) Generalize your equation for a ball thrown at an angle $\theta$ above horizontal, in which case its initial velocity components are $v_x = v \cos \theta$ and $v_y = v \sin \theta$.

48 At the 2010 Salinas Lettuce Festival Parade, the Lettuce Queen drops her bouquet while riding on a float moving toward the right. Sketch the shape of its trajectory in her frame of reference, and compare with the shape seen by one of her admirers standing on the sidewalk.

49 Two daredevils, Wendy and Bill, go over Niagara Falls. Wendy sits in an inner tube, and lets the 30 km/hr velocity of the river throw her out horizontally over the falls. Bill paddles a kayak, adding an extra 10 km/hr to his velocity. They go over the edge
of the falls at the same moment, side by side. Ignore air friction. Explain your reasoning.
(a) Who hits the bottom first?
(b) What is the horizontal component of Wendy’s velocity on impact?
(c) What is the horizontal component of Bill’s velocity on impact?
(d) Who is going faster on impact?

50 A baseball pitcher throws a pitch clocked at \(v_x=73.3 \text{ mi/h}\). He throws horizontally. By what amount, \(d\), does the ball drop by the time it reaches home plate, \(L=60.0 \text{ ft}\) away?
(a) First find a symbolic answer in terms of \(L\), \(v_x\), and \(g\).
(b) Plug in and find a numerical answer. Express your answer in units of ft. (Note: 1 ft=12 in, 1 mi=5280 ft, and 1 in=2.54 cm)

![Problem 50.](image)

51 A batter hits a baseball at speed \(v\), at an angle \(\theta\) above horizontal.
(a) Find an equation for the range (horizontal distance to where the ball falls), \(R\), in terms of the relevant variables. Neglect air friction and the height of the ball above the ground when it is hit.
(b) Interpret your equation in the cases of \(\theta=0\) and \(\theta=90^\circ\).
(c) Find the angle that gives the maximum range.

52 In this problem you’ll extend the analysis in problem 51 to include air friction by writing a computer program. For a game played at sea level, the force due to air friction is approximately \((7 \times 10^{-4} \text{ Ns}^2/\text{m}^2)v^2\), in the direction opposite to the motion of the ball. The mass of a baseball is 0.146 kg.
(a) For a ball hit at a speed of 45.0 m/s from a height of 1.0 m, find the optimal angle and the resulting range.
(b) How much farther would the ball fly at the Colorado Rockies’ stadium, where the thinner air gives 18 percent less air friction?
53. If you walk 35 km at an angle 25° counterclockwise from east, and then 22 km at 230° counterclockwise from east, find the distance and direction from your starting point to your destination.

54. The figure shows vectors A and B. As in figure p on p. 200, graphically calculate the following:

\[ A + B, \quad A - B, \quad B - A, \quad -2B, \quad A - 2B \]

No numbers are involved.

55. Phnom Penh is 470 km east and 250 km south of Bangkok. Hanoi is 60 km east and 1030 km north of Phnom Penh.
(a) Choose a coordinate system, and translate these data into \( \Delta x \) and \( \Delta y \) values with the proper plus and minus signs.
(b) Find the components of the \( \Delta r \) vector pointing from Bangkok to Hanoi.

56. Is it possible for a helicopter to have an acceleration due east and a velocity due west? If so, what would be going on? If not, why not?

57. A dinosaur fossil is slowly moving down the slope of a glacier under the influence of wind, rain and gravity. At the same time, the glacier is moving relative to the continent underneath. The dashed lines represent the directions but not the magnitudes of the velocities. Pick a scale, and use graphical addition of vectors to find the magnitude and the direction of the fossil’s velocity relative to the continent. You will need a ruler and protractor.

58. A bird is initially flying horizontally east at 21.1 m/s, but one second later it has changed direction so that it is flying horizontally and 7° north of east, at the same speed. What are the magnitude and direction of its acceleration vector during that one second time interval? (Assume its acceleration was roughly constant.)
59  Your hand presses a block of mass $m$ against a wall with a force $F_H$ acting at an angle $\theta$, as shown in the figure. Find the minimum and maximum possible values of $|F_H|$ that can keep the block stationary, in terms of $m$, $g$, $\theta$, and $\mu_s$, the coefficient of static friction between the block and the wall. Check both your answers in the case of $\theta = 90^\circ$, and interpret the case where the maximum force is infinite.

60  A skier of mass $m$ is coasting down a slope inclined at an angle $\theta$ compared to horizontal. Assume for simplicity that the treatment of kinetic friction given in chapter 5 is appropriate here, although a soft and wet surface actually behaves a little differently. The coefficient of kinetic friction acting between the skis and the snow is $\mu_k$, and in addition the skier experiences an air friction force of magnitude $bv^2$, where $b$ is a constant.

(a) Find the maximum speed that the skier will attain, in terms of the variables $m$, $g$, $\theta$, $\mu_k$, and $b$.

(b) For angles below a certain minimum angle $\theta_{min}$, the equation gives a result that is not mathematically meaningful. Find an equation for $\theta_{min}$, and give a physical explanation of what is happening for $\theta < \theta_{min}$.

61  A gun is aimed horizontally to the west. The gun is fired, and the bullet leaves the muzzle at $t = 0$. The bullet’s position vector as a function of time is $\mathbf{r} = b\hat{x} + ct\hat{y} + dt^2\hat{z}$, where $b$, $c$, and $d$ are positive constants.

(a) What units would $b$, $c$, and $d$ need to have for the equation to make sense?

(b) Find the bullet’s velocity and acceleration as functions of time.

(c) Give physical interpretations of $b$, $c$, $d$, $\hat{x}$, $\hat{y}$, and $\hat{z}$.

62  Annie Oakley, riding north on horseback at 30 mi/hr, shoots her rifle, aiming horizontally and to the northeast. The muzzle speed of the rifle is 140 mi/hr. When the bullet hits a defenseless fuzzy animal, what is its speed of impact? Neglect air resistance, and ignore the vertical motion of the bullet.  

63  A cargo plane has taken off from a tiny airstrip in the Andes, and is climbing at constant speed, at an angle of $\theta = 17^\circ$ with respect to horizontal. Its engines supply a thrust of $F_{thrust} = 200 \text{ kN}$, and the lift from its wings is $F_{lift} = 654 \text{ kN}$. Assume that air resistance (drag) is negligible, so the only forces acting are thrust, lift, and weight. What is its mass, in kg?

64  A wagon is being pulled at constant speed up a slope $\theta$ by a rope that makes an angle $\phi$ with the vertical. (a) Assuming negligible friction, show that the tension in the rope is given by the equation

$$T = \frac{\sin \theta}{\sin(\theta + \phi)}mg,$$
(b) Interpret this equation in the special cases of $\phi = 0$ and $\phi = 180^\circ - \theta$.

65 The angle of repose is the maximum slope on which an object will not slide. On airless, geologically inert bodies like the moon or an asteroid, the only thing that determines whether dust or rubble will stay on a slope is whether the slope is less steep than the angle of repose.

(a) Find an equation for the angle of repose, deciding for yourself what are the relevant variables.
(b) On an asteroid, where $g$ can be thousands of times lower than on Earth, would rubble be able to lie at a steeper angle of repose?

66 When you’re done using an electric mixer, you can get most of the batter off of the beaters by lifting them out of the batter with the motor running at a high enough speed. Let’s imagine, to make things easier to visualize, that we instead have a piece of tape stuck to one of the beaters.

(a) Explain why static friction has no effect on whether or not the tape flies off.
(b) Analyze the forces in which the tape participates, using a table in the format shown in subsection 3.2.6.
(c) Suppose you find that the tape doesn’t fly off when the motor is on a low speed, but at a greater speed, the tape won’t stay on. Why would the greater speed change things? [Hint: If you don’t invoke any law of physics, you haven’t explained it.]

67 Show that the expression $|v|^2/r$ has the units of acceleration.

68 A plane is flown in a loop-the-loop of radius 1.00 km. The plane starts out flying upside-down, straight and level, then begins curving up along the circular loop, and is right-side up when it reaches the top. (The plane may slow down somewhat on the way up.) How fast must the plane be going at the top if the pilot is to experience no force from the seat or the seatbelt while at the top of the loop?

69 Find the angle between the following two vectors:

$$\hat{x} + 2\hat{y} + 3\hat{z}$$
$$4\hat{x} + 5\hat{y} + 6\hat{z}$$

(b) Show that the units of your answer make sense.
(c) Check the physical behavior of your answer in the special cases of $\phi \leq \theta$ and $\theta = 0$, $\phi = 90^\circ$.
(a) We observe that the amplitude of a certain free oscillation decreases from $A_0$ to $A_0/Z$ after $n$ oscillations. Find its $Q$.

(b) The figure is from *Shape memory in Spider draglines*, Emile, Le Floch, and Vollrath, *Nature* 440:621 (2006). Panel 1 shows an electron microscope’s image of a thread of spider silk. In 2, a spider is hanging from such a thread. From an evolutionary point of view, it’s probably a bad thing for the spider if it twists back and forth while hanging like this. (We’re referring to a back-and-forth rotation about the axis of the thread, not a swinging motion like a pendulum.) The authors speculate that such a vibration could make the spider easier for predators to see, and it also seems to me that it would be a bad thing just because the spider wouldn’t be able to control its orientation and do what it was trying to do. Panel 3 shows a graph of such an oscillation, which the authors measured using a video camera and a computer, with a 0.1 g mass hung from it in place of a spider. Compared to human-made fibers such as kevlar or copper wire, the spider thread has an unusual set of properties:

1. It has a low $Q$, so the vibrations damp out quickly.

2. It doesn’t become brittle with repeated twisting as a copper wire would.

3. When twisted, it tends to settle in to a new equilibrium angle, rather than insisting on returning to its original angle. You can see this in panel 2, because although the experimenters initially twisted the wire by 35 degrees, the thread only performed oscillations with an amplitude much smaller than ±35 degrees, settling down to a new equilibrium at 27 degrees.

4. Over much longer time scales (hours), the thread eventually resets itself to its original equilibrium angle (shown as zero degrees on the graph). (The graph reproduced here only shows the motion over a much shorter time scale.) Some human-made materials have this “memory” property as well, but they typically need to be heated in order to make them go back to their original shapes.

Focusing on property number 1, estimate the $Q$ of spider silk from the graph.
Problem 71.

72 A cross-country skier is gliding on a level trail, with negligible friction. Then, when he is at position $x = 0$, the tip of his skis enters a patch of dirt. As he rides onto the dirt, more and more of his weight is being supported by the dirt. The skis have length $\ell$, so if he reached $x = \ell$ without stopping, his weight would be completely on the dirt. This problem deals with the motion for $x < \ell$.

(a) Find the acceleration in terms of $x$, as well as any other relevant constants.

(b) This is a second-order differential equation. You should be able to find the solution simply by thinking about some commonly occurring functions that you know about, and finding two that have the right properties. If these functions are $x = f(t)$ and $x = g(t)$, then the most general solution to the equations of motion will be of the form $x = af + bg$, where $a$ and $b$ are constants to be determined from the initial conditions.

(c) Suppose that the initial velocity $v_0$ at $x = 0$ is such that he stops at $x < \ell$. Find the time until he stops, and show that, counterintuitively, this time is independent of $v_0$. Explain physically why this is true.

73 A microwave oven works by twisting molecules one way and then the other, counterclockwise and then clockwise about their own centers, millions of times a second. If you put an ice cube or a stick of butter in a microwave, you’ll observe that the solid doesn’t heat very quickly, although eventually melting begins in one small spot. Once this spot forms, it grows rapidly, while the rest of the solid remains solid; it appears that a microwave oven heats a liquid much more rapidly than a solid. Explain why this should happen, based on the atomic-level description of heat, solids, and liquids. (See, e.g., figure b on page 110.)

Don’t repeat the following common mistakes:

In a solid, the atoms are packed more tightly and have less space between them. Not true. Ice floats because it’s less dense than water.

In a liquid, the atoms are moving much faster. No, the differ-
ence in average speed between ice at $-1^\circ C$ and water at $1^\circ C$ is only 0.4%.

**Problem 2-16** on page 120 was intended to be solved using conservation of energy. Solve the same problem using Newton’s laws.

**Problem 233**

A bead slides down along a piece of wire that is in the shape of a helix. The helix lies on the surface of a vertical cylinder of radius $r$, and the vertical distance between turns is $d$.

(a) Ordinarily when an object slides downhill under the influence of kinetic friction, the velocity-independence of kinetic friction implies that the acceleration is constant, and therefore there is no limit to the object’s velocity. Explain the physical reason why this argument fails here, so that the bead will in fact have some limiting velocity.

(b) Find the limiting velocity.

(c) Show that your result has the correct behavior in the limit of $r \to \infty$. [Problem by B. Korsunsky]

A person on a bicycle is to coast down a ramp of height $h$ and then pass through a circular loop of radius $r$. What is the smallest value of $h$ for which the cyclist will complete the loop without falling? (Ignore the kinetic energy of the spinning wheels.)
A car accelerates from rest. At low speeds, its acceleration is limited by static friction, so that if we press too hard on the gas, we will “burn rubber” (or, for many newer cars, a computerized traction-control system will override the gas pedal). At higher speeds, the limit on acceleration comes from the power of the engine, which puts a limit on how fast kinetic energy can be developed.

(a) Show that if a force $F$ is applied to an object moving at speed $v$, the power required is given by $P = vF$.

(b) Find the speed $v$ at which we cross over from the first regime described above to the second. At speeds higher than this, the engine does not have enough power to burn rubber. Express your result in terms of the car’s power $P$, its mass $m$, the coefficient of static friction $\mu_s$, and $g$.

(c) Show that your answer to part b has units that make sense.

(d) Show that the dependence of your answer on each of the four variables makes sense physically.

(e) The 2010 Maserati Gran Turismo Convertible has a maximum power of $3.23 \times 10^5$ W (433 horsepower) and a mass (including a 50-kg driver) of $2.03 \times 10^3$ kg. (This power is the maximum the engine can supply at its optimum frequency of 7600 r.p.m. Presumably the automatic transmission is designed so a gear is available in which the engine will be running at very nearly this frequency when the car is moving at $v$.) Rubber on asphalt has $\mu_s \approx 0.9$. Find $v$ for this car. Answer: 18 m/s, or about 40 miles per hour.

(f) Our analysis has neglected air friction, which can probably be approximated as a force proportional to $v^2$. The existence of this force is the reason that the car has a maximum speed, which is 176 miles per hour. To get a feeling for how good an approximation it is to ignore air friction, find what fraction of the engine’s maximum power is being used to overcome air resistance when the car is moving at the speed $v$ found in part e. Answer: 1%.

Two wheels of radius $r$ rotate in the same vertical plane with angular velocities $+\Omega$ and $-\Omega$ about axes that are parallel and at the same height. The wheels touch one another at a point on their circumferences, so that their rotations mesh like gears in a gear train. A board is laid on top of the wheels, so that two friction forces act upon it, one from each wheel. Characterize the three qualitatively different types of motion that the board can exhibit, depending on the initial conditions.
For safety, mountain climbers often wear a climbing harness and tie in to other climbers on a rope team or to anchors such as pitons or snow anchors. When using anchors, the climber usually wants to tie in to more than one, both for extra strength and for redundancy in case one fails. The figure shows such an arrangement, with the climber hanging from a pair of anchors forming a “Y” at an angle $\theta$. The usual advice is to make $\theta < 90^\circ$; for large values of $\theta$, the stress placed on the anchors can be many times greater than the actual load $L$, so that two anchors are actually less safe than one.

(a) Find the force $S$ at each anchor in terms of $L$ and $\theta$.  
(b) Verify that your answer makes sense in the case of $\theta = 0$.
(c) Interpret your answer in the case of $\theta = 180^\circ$.
(d) What is the smallest value of $\theta$ for which $S$ equals or exceeds $L$, so that for larger angles a failure of at least one anchor is more likely than it would have been with a single anchor?

Mountain climbers with masses $m$ and $M$ are roped together while crossing a horizontal glacier when a vertical crevasse opens up under the climber with mass $M$. The climber with mass $m$ drops down on the snow and tries to stop by digging into the snow with the pick of an ice ax. Alas, this story does not have a happy ending, because this doesn’t provide enough friction to stop. Both $m$ and $M$ continue accelerating, with $M$ dropping down into the crevasse and $m$ being dragged across the snow, slowed only by the kinetic friction with coefficient $\mu_k$ acting between the ax and the snow. There is no significant friction between the rope and the lip of the crevasse.

(a) Find the acceleration $a$.
(b) Check the units of your result.
(c) Check the dependence of your equation on the variables. That means that for each variable, you should determine what its effect on $a$ should be physically, and then what your answer from part a says its effect would be mathematically.
Complete example 71 on p. 205 by expressing the remaining nine \( x \) and \( y \) components of the forces in terms of the five magnitudes and the small, positive angle \( \theta \approx 9^\circ \) by which the crack overhangs.

In a well known stunt from circuses and carnivals, a motorcyclist rides around inside a big bowl, gradually speeding up and rising higher. Eventually the cyclist can get up to where the walls of the bowl are vertical. Let’s estimate the conditions under which a running human could do the same thing.

(a) If the runner can run at speed \( v \), and her shoes have a coefficient of static friction \( \mu_s \), what is the maximum radius of the circle? \( \checkmark \)

(b) Show that the units of your answer make sense.

(c) Check that its dependence on the variables makes sense.

(d) Evaluate your result numerically for \( v = 10 \text{ m/s} \) (the speed of an olympic sprinter) and \( \mu_s = 5 \). (This is roughly the highest coefficient of static friction ever achieved for surfaces that are not sticky. The surface has an array of microscopic fibers like a hair brush, and is inspired by the hairs on the feet of a gecko. These assumptions are not necessarily realistic, since the person would have to run at an angle, which would be physically awkward.) \( \checkmark \)
83 Problem 79 discussed a possible correct way of setting up a redundant anchor for mountaineering. The figure for this problem shows an incorrect way of doing it, by arranging the rope in a triangle (which we’ll take to be isosceles). One of the bad things about the triangular arrangement is that it requires more force from the anchors, making them more likely to fail. (a) Using the same notation as in problem 79, find $S$ in terms of $L$ and $\theta$.  
(b) Verify that your answer makes sense in the case of $\theta = 0$, and compare with the correct setup.

84 At a picnic, someone hands you a can of beer. The ground is uneven, and you don’t want to spill your drink. You reason that it will be more stable if you drink some of it first in order to lower its center of mass. How much should you drink in order to make the center of mass as low as possible? [Based on a problem by Walter van B. Roberts and Martin Gardner.]

85 “Big wall” climbing is a specialized type of rock climbing that involves going up tall cliffs such as the ones in Yosemite, usually with the climbers spending at least one night sleeping on a natural ledge or an artificial “portaledge.” In this style of climbing, each pitch of the climb involves strenuously hauling up several heavy bags of gear — a fact that has caused these climbs to be referred to as “vertical ditch digging.” (a) If an 80 kg haul bag has to be pulled up the full length of a 60 m rope, how much work is done? (b) Since it can be difficult to lift 80 kg, a 2:1 pulley is often used. The hauler then lifts the equivalent of 40 kg, but has to pull in 120 m of rope. How much work is done in this case?

Key to symbols:

- easy
- typical
- challenging
- difficult
- very difficult

✓ An answer check is available at www.lightandmatter.com.
Exercises

Exercise 3A: Force and Motion

Equipment:

- 2-meter pieces of butcher paper
- wood blocks with hooks
- string
- masses to put on top of the blocks to increase friction
- spring scales (preferably calibrated in Newtons)

Suppose a person pushes a crate, sliding it across the floor at a certain speed, and then repeats the same thing but at a higher speed. This is essentially the situation you will act out in this exercise. What do you think is different about her force on the crate in the two situations? Discuss this with your group and write down your hypothesis:

1. First you will measure the amount of friction between the wood block and the butcher paper when the wood and paper surfaces are slipping over each other. The idea is to attach a spring scale to the block and then slide the butcher paper under the block while using the scale to keep the block from moving with it. Depending on the amount of force your spring scale was designed to measure, you may need to put an extra mass on top of the block in order to increase the amount of friction. It is a good idea to use long piece of string to attach the block to the spring scale, since otherwise one tends to pull at an angle instead of directly horizontally.

   First measure the amount of friction force when sliding the butcher paper as slowly as possible:

   Now measure the amount of friction force at a significantly higher speed, say 1 meter per second. (If you try to go too fast, the motion is jerky, and it is impossible to get an accurate reading.)

   Discuss your results. Why are we justified in assuming that the string’s force on the block (i.e., the scale reading) is the same amount as the paper’s frictional force on the block?

2. Now try the same thing but with the block moving and the paper standing still. Try two different speeds.

   Do your results agree with your original hypothesis? If not, discuss what’s going on. How does the block “know” how fast to go?
**Exercise 3B: Vibrations**

Equipment:

- air track and carts of two different masses
- springs
- spring scales

Place the cart on the air track and attach springs so that it can vibrate.

1. Test whether the period of vibration depends on amplitude. Try at least two moderate amplitudes, for which the springs do not go slack, and at least one amplitude that is large enough so that they do go slack.

2. Try a cart with a different mass. Does the period change by the expected factor, based on the equation $T = 2\pi \sqrt{m/k}$?

3. Use a spring scale to pull the cart away from equilibrium, and make a graph of force versus position. Is it linear? If so, what is its slope?

4. Test the equation $T = 2\pi \sqrt{m/k}$ numerically.
Exercise 3C: Worksheet on Resonance

1. Compare the oscillator’s energies at A, B, C, and D.

2. Compare the Q values of the two oscillators.

3. Match the x-t graphs in #2 with the amplitude-frequency graphs below.
Exercise D is on the following two pages.
Exercise 3D: Vectors and Motion

Each diagram on page 243 shows the motion of an object in an $x - y$ plane. Each dot is one location of the object at one moment in time. The time interval from one dot to the next is always the same, so you can think of the vector that connects one dot to the next as a $\mathbf{v}$ vector, and subtract to find $\Delta \mathbf{v}$ vectors.

1. Suppose the object in diagram 1 is moving from the top left to the bottom right. Deduce whatever you can about the force acting on it. Does the force always have the same magnitude? The same direction?

Invent a physical situation that this diagram could represent.

What if you reinterpret the diagram, and reverse the object’s direction of motion?

2. What can you deduce about the force that is acting in diagram 2?

Invent a physical situation that diagram 2 could represent.

3. What can you deduce about the force that is acting in diagram 3?

Invent a physical situation.
Chapter 4
Conservation of Angular Momentum

4.1 Angular Momentum In Two Dimensions

4.1.1 Angular momentum

“Sure, and maybe the sun won’t come up tomorrow.” Of course, the sun only appears to go up and down because the earth spins, so the cliche should really refer to the unlikelihood of the earth’s stopping its rotation abruptly during the night. Why can’t it stop? It wouldn’t violate conservation of momentum, because the earth’s rotation doesn’t add anything to its momentum. While California spins in one direction, some equally massive part of India goes the opposite way, canceling its momentum. A halt to Earth’s rotation would entail a drop in kinetic energy, but that energy could simply be converted into some other form, such as heat.

Other examples along these lines are not hard to find. An atom spins at the same rate for billions of years. A high-diver who is rotating when he comes off the board does not need to make any physical effort to continue rotating, and indeed would be unable to stop rotating before he hit the water.

These observations have the hallmarks of a conservation law:

A closed system is involved. Nothing is making an effort to twist the earth, the hydrogen atom, or the high-diver. They are isolated from rotation-changing influences, i.e., they are closed systems.

Something remains unchanged. There appears to be a numerical quantity for measuring rotational motion such that the total amount of that quantity remains constant in a closed system.
Something can be transferred back and forth without changing the total amount. In the photo of the old-fashioned high jump, a, the jumper wants to get his feet out in front of him so he can keep from doing a “face plant” when he lands. Bringing his feet forward would involve a certain quantity of counterclockwise rotation, but he didn’t start out with any rotation when he left the ground. Suppose we consider counterclockwise as positive and clockwise as negative. The only way his legs can acquire some positive rotation is if some other part of his body picks up an equal amount of negative rotation. This is why he swings his arms up behind him, clockwise.

What numerical measure of rotational motion is conserved? Car engines and old-fashioned LP records have speeds of rotation measured in rotations per minute (r.p.m.), but the number of rotations per minute (or per second) is not a conserved quantity. A twirling figure skater, for instance, can pull her arms in to increase her r.p.m.’s. The first section of this chapter deals with the numerical definition of the quantity of rotation that results in a valid conservation law.

When most people think of rotation, they think of a solid object like a wheel rotating in a circle around a fixed point. Examples of this type of rotation, called rigid rotation or rigid-body rotation, include a spinning top, a seated child’s swinging leg, and a helicopter’s spinning propeller. Rotation, however, is a much more general phenomenon, and includes noncircular examples such as a comet in an elliptical orbit around the sun, or a cyclone, in which the core completes a circle more quickly than the outer parts.

If there is a numerical measure of rotational motion that is a conserved quantity, then it must include nonrigid cases like these, since nonrigid rotation can be traded back and forth with rigid rotation. For instance, there is a trick for finding out if an egg is raw or hardboiled. If you spin a hardboiled egg and then stop it briefly with your finger, it stops dead. But if you do the same with a raw egg, it springs back into rotation because the soft interior was still swirling around within the momentarily motionless shell. The pattern of flow of the liquid part is presumably very complex and nonuniform due to the asymmetric shape of the egg and the different consistencies of the yolk and the white, but there is apparently some way to describe the liquid’s total amount of rotation with a single number, of which some percentage is given back to the shell when you release it.

The best strategy is to devise a way of defining the amount of rotation of a single small part of a system. The amount of rotation of a system such as a cyclone will then be defined as the total of all the contributions from its many small parts.

The quest for a conserved quantity of rotation even requires us to broaden the rotation concept to include cases where the motion
doesn’t repeat or even curve around. If you throw a piece of putty at a door, b, the door will recoil and start rotating. The putty was traveling straight, not in a circle, but if there is to be a general conservation law that can cover this situation, it appears that we must describe the putty as having had some “rotation,” which it then gave up to the door. The best way of thinking about it is to attribute rotation to any moving object or part of an object that changes its angle in relation to the axis of rotation. In the putty-and-door example, the hinge of the door is the natural point to think of as an axis, and the putty changes its angle as seen by someone standing at the hinge, c. For this reason, the conserved quantity we are investigating is called *angular momentum*. The symbol for angular momentum can’t be “a” or “m,” since those are used for acceleration and mass, so the letter $L$ is arbitrarily chosen instead.

Imagine a 1 kg blob of putty, thrown at the door at a speed of 1 m/s, which hits the door at a distance of 1 m from the hinge. We define this blob to have 1 unit of angular momentum. When it hits the door, the door will recoil and start rotating. We can use the speed at which the door recoils as a measure of the angular momentum the blob brought in.\(^1\)

Experiments show, not surprisingly, that a 2 kg blob thrown in the same way makes the door rotate twice as fast, so the angular momentum of the putty blob must be proportional to mass,

$$L \propto m$$

Similarly, experiments show that doubling the velocity of the blob will have a doubling effect on the result, so its angular momentum must be proportional to its velocity as well,

$$L \propto mv$$

You have undoubtedly had the experience of approaching a closed door with one of those bar-shaped handles on it and pushing on the wrong side, the side close to the hinges. You feel like an idiot, because you have so little leverage that you can hardly budge the door. The same would be true with the putty blob. Experiments would show that the amount of rotation the blob can give to the door is proportional to the distance, $r$, from the axis of rotation, so angular momentum must be proportional to $r$ as well,

$$L \propto mvr$$

We are almost done, but there is one missing ingredient. We know on grounds of symmetry that a putty ball thrown directly

\(^1\)We assume that the door is much more massive than the blob. Under this assumption, the speed at which the door recoils is much less than the original speed of the blob, so the blob has lost essentially all its angular momentum, and given it to the door.
Only the component of the velocity vector perpendicular to the line connecting the object to the axis should be counted into the definition of angular momentum.

A figure skater pulls in her arms so that she can execute a spin more rapidly.

\[ L = mv_{\perp}r \]

More generally, \( v_{\perp} \) should be thought of as the component of the object’s velocity vector that is perpendicular to the line joining the object to the axis of rotation.

We find that this equation agrees with the definition of the original putty blob as having one unit of angular momentum, and we can now see that the units of angular momentum are \((\text{kg} \cdot \text{m/s}) \cdot \text{m}\), i.e., \(\text{kg} \cdot \text{m}^2/\text{s}\). Summarizing, we have

\[ L = mv_{\perp}r \]

[angular momentum of a particle in two dimensions]

where \( m \) is the particle’s mass, \( v_{\perp} \) is the component of its velocity vector perpendicular to the line joining it to the axis of rotation, and \( r \) is its distance from the axis. (Note that \( r \) is not necessarily the radius of a circle.) Positive and negative signs of angular momentum are used to describe opposite directions of rotation. The angular momentum of a finite-sized object or a system of many objects is found by dividing it up into many small parts, applying the equation to each part, and adding to find the total amount of angular momentum. (As implied by the word “particle,” matter isn’t the only thing that can have angular momentum. Light can also have angular momentum, and the above equation would not apply to light.)

Conservation of angular momentum has been verified over and over again by experiment, and is now believed to be one of the most fundamental principles of physics, along with conservation of mass, energy, and momentum.

A figure skater pulls her arms in.

Example 1

When a figure skater is twirling, there is very little friction between her and the ice, so she is essentially a closed system, and her angular momentum is conserved. If she pulls her arms in, she is decreasing \( r \) for all the atoms in her arms. It would violate conservation of angular momentum if she then continued rotating at the same speed, i.e., taking the same amount of time for each revolution, because her arms’ contributions to her angular momentum would have decreased, and no other part of her would have increased its angular momentum. This is impossible because it would violate conservation of angular momentum. If her total angular momentum is to remain constant, the decrease in \( r \) for her arms must be compensated for by an overall increase in
her rate of rotation. That is, by pulling her arms in, she substantially reduces the time for each rotation.

\textit{Earth's slowing rotation and the receding moon} \hspace{1cm} \textit{example 2}

The earth's rotation is actually slowing down very gradually, with the kinetic energy being dissipated as heat by friction between the land and the tidal bulges raised in the seas by the earth's gravity. Does this mean that angular momentum is not really perfectly conserved? No, it just means that the earth is not quite a closed system by itself. If we consider the earth and moon as a system, then the angular momentum lost by the earth must be gained by the moon somehow. In fact very precise measurements of the distance between the earth and the moon have been carried out by bouncing laser beams off of a mirror left there by astronauts, and these measurements show that the moon is receding from the earth at a rate of 4 centimeters per year! The moon's greater value of $r$ means that it has a greater angular momentum, and the increase turns out to be exactly the amount lost by the earth. In the days of the dinosaurs, the days were significantly shorter, and the moon was closer and appeared bigger in the sky.

But what force is causing the moon to speed up, drawing it out into a larger orbit? It is the gravitational forces of the earth's tidal bulges. In figure \textit{g}, the earth's rotation is counterclockwise (arrow). The moon's gravity creates a bulge on the side near it, because its gravitational pull is stronger there, and an "anti-bulge" on the far side, since its gravity there is weaker. For simplicity, let's focus on the tidal bulge closer to the moon. Its frictional force is trying to slow down the earth's rotation, so its force on the earth's solid crust is toward the bottom of the figure. By Newton's third law, the crust must thus make a force on the bulge which is toward the top of the figure. This causes the bulge to be pulled forward at a slight angle, and the bulge's gravity therefore pulls the moon forward, accelerating its orbital motion about the earth and flinging it outward.

The result would obviously be extremely difficult to calculate directly, and this is one of those situations where a conservation law allows us to make precise quantitative statements about the outcome of a process when the calculation of the process itself would be prohibitively complex.

\textit{Restriction to rotation in a plane}

Is angular momentum a vector, or a scalar? It does have a direction in space, but it's a direction of rotation, not a straight-line direction like the directions of vectors such as velocity or force. It turns out that there is a way of defining angular momentum as a vector, but in this section the examples will be confined to a single plane of rotation, i.e., effectively two-dimensional situations. In this
special case, we can choose to visualize the plane of rotation from one side or the other, and to define clockwise and counterclockwise rotation as having opposite signs of angular momentum. “Effectively” two-dimensional means that we can deal with objects that aren’t flat, as long as the velocity vectors of all their parts lie in a plane.

Discussion Questions

A Conservation of plain old momentum, $p$, can be thought of as the greatly expanded and modified descendant of Galileo’s original principle of inertia, that no force is required to keep an object in motion. The principle of inertia is counterintuitive, and there are many situations in which it appears superficially that a force is needed to maintain motion, as maintained by Aristotle. Think of a situation in which conservation of angular momentum, $L$, also seems to be violated, making it seem incorrectly that something external must act on a closed system to keep its angular momentum from “running down.”

4.1.2 Application to planetary motion

We now discuss the application of conservation of angular momentum to planetary motion, both because of its intrinsic importance and because it is a good way to develop a visual intuition for angular momentum.

Kepler’s law of equal areas states that the area swept out by a planet in a certain length of time is always the same. Angular momentum had not been invented in Kepler’s time, and he did not even know the most basic physical facts about the forces at work. He thought of this law as an entirely empirical and unexpectedly simple way of summarizing his data, a rule that succeeded in describing and predicting how the planets sped up and slowed down in their elliptical paths. It is now fairly simple, however, to show that the equal area law amounts to a statement that the planet’s angular momentum stays constant.

There is no simple geometrical rule for the area of a pie wedge cut out of an ellipse, but if we consider a very short time interval, as shown in figure h, the shaded shape swept out by the planet is very nearly a triangle. We do know how to compute the area of a triangle. It is one half the product of the base and the height:

$$\text{area} = \frac{1}{2}bh.$$

We wish to relate this to angular momentum, which contains the variables $r$ and $v_\perp$. If we consider the sun to be the axis of rotation, then the variable $r$ is identical to the base of the triangle, $r = b$. Referring to the magnified portion of the figure, $v_\perp$ can be related to $h$, because the two right triangles are similar:

$$\frac{h}{\text{distance traveled}} = \frac{v_\perp}{|v|}.$$