Newton’s laws in one dimension:

**Newton’s first law:** If there is no force acting on an object, it stays in the same state of motion.

**Newton’s second law:** The sum of all the forces acting on an object determines the rate at which its momentum changes, \( F_{\text{total}} = \frac{dp}{dt} \).

**Newton’s third law:** Forces occur in opposite pairs. If object A interacts with object B, then A’s force on B and B’s force on A are related by \( F_{AB} = -F_{BA} \).

The second law is the definition of force, which we’ve already encountered.\(^4\) The first law is a special case of the second law — if \( \frac{dp}{dt} \) is zero, then \( p = mv \) is a constant, and since mass is conserved, constant \( p \) implies constant \( v \). The third law is a restatement of conservation of momentum; for two objects interacting, we have constant total momentum, so \( 0 = \frac{d}{dt}(p_A + p_B) = F_{BA} + F_{AB} \).

**\( a = F/m \) example 22**

Many modern textbooks restate Newton’s second law as \( a = F/m \), i.e., as an equation that predicts an object’s acceleration based on the force exerted on it. This is easily derived from Newton’s original form as follows: \( a = \frac{dv}{dt} = \frac{(dp/dt)/m}{m} = \frac{F}{m} \).

**Gravitational force related to g example 23**

As a special case of the previous example, consider an object in free fall, and let the \( x \) axis point down. Then \( a = +g \), and \( F = ma = mg \). For example, the gravitational force on a 1 kg mass at the earth’s surface is about 9.8 N. Even if other forces act on the object, and it isn’t in free fall, the gravitational force on it is still the same, and can still be calculated as \( mg \).

**Changing frames of reference example 24**

Suppose we change from one frame reference into another, which is moving relative to the first one at a constant velocity \( u \). If an object of mass \( m \) is moving at velocity \( v \) (which need not be constant), then the effect is to change its momentum from \( mv \) in one frame to \( mv + mu \) in the other. Force is defined as the derivative of momentum with respect to time, and the derivative of a constant is zero, so adding the constant \( mu \) has no effect on the result. We therefore conclude that observers in different inertial frames of reference agree on forces.

**Using the third law correctly**

If you’ve already accepted Galilean relativity in your heart, then there is nothing really difficult about the first and second laws. The third law, however, is more of a conceptual challenge. The first example 22

---

\(^4\)This is with the benefit of hindsight. At the time, the word “force” already had certain connotations, and people thought they understood what it meant and how to measure it, e.g., by using a spring scale. From their point of view, \( F = dp/dt \) was not a definition but a testable — and controversial! — statement.
Two magnets exert forces on each other.

It doesn't make sense for the man to talk about the woman’s money canceling out his bar tab, because there is no good reason to combine his debts and her assets.

Newton’s third law does not mean that forces always cancel out so that nothing can ever move. If these two ice skaters, initially at rest, push against each other, they will both move.

A more humane and practical experiment is shown in figure d. A large magnet and a small magnet are weighed separately, and then one magnet is hung from the pan of the top balance so that it is directly above the other magnet. There is an attraction between the two magnets, causing the reading on the top scale to increase and the reading on the bottom scale to decrease. The large magnet is more “powerful” in the sense that it can pick up a heavier paperclip from the same distance, so many people have a strong expectation that one scale’s reading will change by a far different amount than the other. Instead, we find that the two changes are equal in magnitude but opposite in direction, so the upward force of the top magnet on the bottom magnet is of the same magnitude as the downward force of the bottom magnet on the top magnet.

To students, it often sounds as though Newton’s third law implies nothing could ever change its motion, since the two equal and opposite forces would always cancel. As illustrated in figure e, the fallacy arises from assuming that we can add things that it doesn’t make sense to add. It only makes sense to add up forces that are acting on the same object, whereas two forces related to each other by Newton’s third law are always acting on two different objects. If two objects are interacting via a force and no other forces are involved, then both objects will accelerate — in opposite directions, as shown in figure f!

Here are some suggestions for avoiding misapplication of Newton’s third law:

1. It always relates exactly two forces, not more.
2. The two forces involve exactly two objects, in the pattern A on B, B on A.
3. The two forces are always of the same type, e.g., friction and friction, or gravity and gravity.

Directions of forces

We’ve already seen that momentum, unlike energy, has a direction in space. Since force is defined in terms of momentum, force also has a direction in space. For motion in one dimension, we have to pick a coordinate system, and given that choice, forces and momenta will be positive or negative. We’ve already used signs to represent directions of forces in Newton’s third law, $F_{AB} = -F_{BA}$.
There is, however, a complication with force that we were able to avoid with momentum. If an object is moving on a line, we're guaranteed that its momentum is in one of two directions: the two directions along the line. But even an object that stays on a line may still be subject to forces that act perpendicularly to the line. For example, suppose a coin is sliding to the right across a table, and let's choose a positive x axis that points to the right. The coin's motion is along a horizontal line, and its momentum is positive and decreasing. Because the momentum is decreasing, its time derivative \( \frac{dp}{dt} \) is negative. This derivative equals the horizontal force of friction \( F_1 \), and its negative sign tells us that this force on the coin is to the left.

But there are also vertical forces on the coin. The Earth exerts a downward gravitational force \( F_2 \) on it, and the table makes an upward force \( F_3 \) that prevents the coin from sinking into the wood. In fact, without these vertical forces the horizontal frictional force wouldn't exist: surfaces don't exert friction against one another unless they are being pressed together.

To avoid mathematical complication, we want to postpone the full three-dimensional treatment of force and momentum until section 3.4. For now, we'll limit ourselves to examples like the coin, in which the motion is confined to a line, and any forces perpendicular to the line cancel each other out.

**Discussion Questions**

**A** Criticize the following incorrect statement:

“If an object is at rest and the total force on it is zero, it stays at rest. There can also be cases where an object is moving and keeps on moving without having any total force on it, but that can only happen when there’s no friction, like in outer space.”

**B** The table gives laser timing data for Ben Johnson’s 100 m dash at the 1987 World Championship in Rome. (His world record was later revoked because he tested positive for steroids.) How does the total force on him change over the duration of the race?

<table>
<thead>
<tr>
<th>Time (m)</th>
<th>Position (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.84</td>
</tr>
<tr>
<td>20</td>
<td>2.86</td>
</tr>
<tr>
<td>30</td>
<td>3.80</td>
</tr>
<tr>
<td>40</td>
<td>4.67</td>
</tr>
<tr>
<td>50</td>
<td>5.53</td>
</tr>
<tr>
<td>60</td>
<td>6.38</td>
</tr>
<tr>
<td>70</td>
<td>7.23</td>
</tr>
<tr>
<td>80</td>
<td>8.10</td>
</tr>
<tr>
<td>90</td>
<td>8.96</td>
</tr>
<tr>
<td>100</td>
<td>9.83</td>
</tr>
</tbody>
</table>

Discussion question B.

**C** You hit a tennis ball against a wall. Explain any and all incorrect ideas in the following description of the physics involved: “According to Newton’s third law, there has to be a force opposite to your force on the ball. The opposite force is the ball’s mass, which resists acceleration, and also air resistance.”

**D** Tam Anh grabs Sarah by the hand and tries to pull her. She tries to remain standing without moving. A student analyzes the situation as follows. “If Tam Anh’s force on Sarah is greater than her force on him, he can get her to move. Otherwise, she’ll be able to stay where she is.” What’s wrong with this analysis?

### 3.2.3 What force is not

Violin teachers have to endure their beginning students’ screeching. A frown appears on the woodwind teacher’s face as she watches...
her student take a breath with an expansion of his ribcage but none in his belly. What makes physics teachers cringe is their students’ verbal statements about forces. Below I have listed several dicta about what force is not.

**Force is not a property of one object.**

A great many of students’ incorrect descriptions of forces could be cured by keeping in mind that a force is an interaction of two objects, not a property of one object.

*Incorrect statement:* “That magnet has a lot of force.”

- If the magnet is one millimeter away from a steel ball bearing, they may exert a very strong attraction on each other, but if they were a meter apart, the force would be virtually undetectable. The magnet’s strength can be rated using certain electrical units (ampere – meters²), but not in units of force.

**Force is not a measure of an object’s motion.**

If force is not a property of a single object, then it cannot be used as a measure of the object’s motion.

*Incorrect statement:* “The freight train rumbled down the tracks with awesome force.”

- Force is not a measure of motion. If the freight train collides with a stalled cement truck, then some awesome forces will occur, but if it hits a fly the force will be small.

**Force is not energy.**

*Incorrect statement:* “How can my chair be making an upward force on my rear end? It has no power!”

- Power is a concept related to energy, e.g., a 100-watt lightbulb uses up 100 joules per second of energy. When you sit in a chair, no energy is used up, so forces can exist between you and the chair without any need for a source of power.

**Force is not stored or used up.**

Because energy can be stored and used up, people think force also can be stored or used up.

*Incorrect statement:* “If you don’t fill up your tank with gas, you’ll run out of force.”

- Energy is what you’ll run out of, not force.

**Forces need not be exerted by living things or machines.**

Transforming energy from one form into another usually requires some kind of living or mechanical mechanism. The concept is not applicable to forces, which are an interaction between objects, not a thing to be transferred or transformed.

*Incorrect statement:* “How can a wooden bench be making an upward force on my rear end? It doesn’t have any springs or anything inside it.”
No springs or other internal mechanisms are required. If the bench
didn’t make any force on you, you would obey Newton’s second law and
fall through it. Evidently it does make a force on you!

A force is the direct cause of a change in motion.

I can click a remote control to make my garage door change from
being at rest to being in motion. My finger’s force on the button,
however, was not the force that acted on the door. When we speak
of a force on an object in physics, we are talking about a force that
acts directly. Similarly, when you pull a reluctant dog along by its
leash, the leash and the dog are making forces on each other, not
your hand and the dog. The dog is not even touching your hand.

self-check B
Which of the following things can be correctly described in terms of
force?

(1) A nuclear submarine is charging ahead at full steam.

(2) A nuclear submarine’s propellers spin in the water.

(3) A nuclear submarine needs to refuel its reactor periodically.

Answer, p. 1059

Discussion Questions

A Criticize the following incorrect statement: “If you shove a book
across a table, friction takes away more and more of its force, until finally
it stops.”

B You hit a tennis ball against a wall. Explain any and all incorrect
ideas in the following description of the physics involved: “The ball gets
some force from you when you hit it, and when it hits the wall, it loses part
of that force, so it doesn’t bounce back as fast. The muscles in your arm
are the only things that a force can come from.”

3.2.4 Forces between solids

Conservation laws are more fundamental than Newton’s laws,
and they apply where Newton’s laws don’t, e.g., to light and to the
internal structure of atoms. However, there are certain problems
that are much easier to solve using Newton’s laws. As a trivial
example, if you drop a rock, it could conserve momentum and energy
by levitating, or by falling in the usual manner. With Newton’s
laws, however, we can reason that $a = F/m$, so the rock must
respond to the gravitational force by accelerating.

Less trivially, suppose a person is hanging onto a rope, and we
want to know if she will slip. Unlike the case of the levitating rock,
here the no-motion solution could be perfectly reasonable if her grip
is strong enough. We know that her hand’s interaction with the rope
is fundamentally an electrical interaction between the atoms in the
surface of her palm and the nearby atoms in the surface of the rope.

\[^5^\text{This pathological solution was first noted on page 83, and discussed in more}
\text{detail on page 1025.}\]
For practical problem-solving, however, this is a case where we're better off forgetting the fundamental classification of interactions at the atomic level and working with a more practical, everyday classification of forces. In this practical scheme, we have three types of forces that can occur between solid objects in contact:

<table>
<thead>
<tr>
<th>Force Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal force, $F_n$</td>
<td>perpendicular to the surface of contact, and prevents objects from passing through each other by becoming as strong as necessary (up to the point where the objects break). “Normal” means perpendicular.</td>
</tr>
<tr>
<td>Static friction, $F_s$</td>
<td>parallel to the surface of contact, and prevents the surfaces from starting to slip by becoming as strong as necessary, up to a maximum value of $F_{s,max}$. “Static” means not moving, i.e., not slipping.</td>
</tr>
<tr>
<td>Kinetic friction, $F_k$</td>
<td>parallel to the surface of contact, and tends to slow down any slippage once it starts. “Kinetic” means moving, i.e., slipping.</td>
</tr>
</tbody>
</table>

**self-check C**

Can a frictionless surface exert a normal force? Can a frictional force exist without a normal force?

If you put a coin on this page, which is horizontal, gravity pulls down on the coin, but the atoms in the paper and the coin repel each other electrically, and the atoms are compressed until the repulsion becomes strong enough to stop the downward motion of the coin. We describe this complicated and invisible atomic process by saying that the paper makes an upward normal force on the coin, and the coin makes a downward normal force on the paper. (The two normal forces are related by Newton’s third law. In fact, Newton’s third law only relates forces that are of the same type.)

If you now tilt the book a little, static friction keeps the coin from slipping. The picture at the microscopic level is even more complicated than the previous description of the normal force. One model is to think of the tiny bumps and depressions in the coin as settling into the similar irregularities in the paper. This model predicts that rougher surfaces should have more friction, which is sometimes true but not always. Two very smooth, clean glass surfaces or very well finished machined metal surfaces may actually stick better than rougher surfaces would, the probable explanation being that there is some kind of chemical bonding going on, and the smoother surfaces allow more atoms to be in contact.

Finally, as you tilt the book more and more, there comes a point where static friction reaches its maximum value. The surfaces be-
come unstuck, and the coin begins to slide over the paper. Kinetic friction slows down this slipping motion significantly. In terms of energy, kinetic friction is converting mechanical energy into heat, just like when you rub your hands together to keep warm. One model of kinetic friction is that the tiny irregularities in the two surfaces bump against each other, causing vibrations whose energy rapidly converts to heat and sound — you can hear this sound if you rub your fingers together near your ear.

For dry surfaces, experiments show that the following equations usually work fairly well:

\[ F_{s,\text{max}} \approx \mu_s F_n, \]

and

\[ F_k \approx \mu_k F_n, \]

where \( \mu_s \), the coefficient of static friction, and \( \mu_k \), the coefficient of kinetic friction, are constants that depend on the properties of the two surfaces, such as what they’re made of and how rough they are.

**Self-check D**
1. When a baseball player slides in to a base, is the friction static, or kinetic?
2. A mattress stays on the roof of a slowly accelerating car. Is the friction static, or kinetic?
3. Does static friction create heat? Kinetic friction?  
   ▶ Answer, p. 1059
Maximum acceleration of a car  

- Rubber on asphalt gives $\mu_k \approx 0.4$ and $\mu_s \approx 0.6$. What is the upper limit on a car’s acceleration on a flat road, assuming that the engine has plenty of power and that air friction is negligible?

- This isn’t a flying car, so we don’t expect it to accelerate vertically. The vertical forces acting on the car should cancel out. The earth makes a downward gravitational force on the car whose absolute value is $mg$, so the road apparently makes an upward normal force of the same magnitude, $F_n = mg$.

Now what about the horizontal motion? As is always true, the coefficient of static friction is greater than the coefficient of kinetic friction, so the maximum acceleration is obtained with static friction, i.e., the driver should try not to burn rubber. The maximum force of static friction is $F_{\text{s, max}} = \mu_s F_n = \mu_s mg$. The maximum acceleration is $a = F_s/m = \mu_s g \approx 6 \text{ m/s}^2$. This is true regardless of how big the tires are, since the experimentally determined relationship $F_{\text{s, max}} = \mu_s F_n$ is independent of surface area.

Self-check E

Find the direction of each of the forces in figure k.  

Answer, p. 1059

Locomotives  

Looking at a picture of a locomotive, l, we notice two obvious things that are different from an automobile. Where a car typically has two drive wheels, a locomotive normally has many — ten in this example. (Some also have smaller, unpowered wheels in front of and behind the drive wheels, but this example doesn’t.) Also, cars these days are generally built to be as light as possible for their size, whereas locomotives are very massive, and no effort seems to be made to keep their weight low. (The steam locomotive in the photo is from about 1900, but this is true even for modern diesel and electric trains.)

The reason locomotives are built to be so heavy is for traction. The upward normal force of the rails on the wheels, $F_N$, cancels the downward force of gravity, $F_W$, so ignoring plus and minus signs, these two forces are equal in absolute value, $F_N = F_W$. 

1. The cliff’s normal force on the climber’s feet.  
2. The track’s static frictional force on the wheel of the accelerating dragster.  
3. The ball’s normal force on the bat.
Given this amount of normal force, the maximum force of static friction is \( F_s = \mu_s F_N = \mu_s F_W \). This static frictional force, of the rails pushing forward on the wheels, is the only force that can accelerate the train, pull it uphill, or cancel out the force of air resistance while cruising at constant speed. The coefficient of static friction for steel on steel is about 1/4, so no locomotive can pull with a force greater than about 1/4 of its own weight. If the engine is capable of supplying more than that amount of force, the result will simply be to break static friction and spin the wheels.

The reason this is all so different from the situation with a car is that a car isn’t pulling something else. If you put extra weight in a car, you improve the traction, but you also increase the inertia of the car, and make it just as hard to accelerate. In a train, the inertia is almost all in the cars being pulled, not in the locomotive.

The other fact we have to explain is the large number of driving wheels. First, we have to realize that increasing the number of driving wheels neither increases nor decreases the total amount of static friction, because static friction is independent of the amount of surface area in contact. (The reason four-wheel-drive is good in a car is that if one or more of the wheels is slipping on ice or in mud, the other wheels may still have traction. This isn’t typically an issue for a train, since all the wheels experience the same conditions.) The advantage of having more driving wheels on a train is that it allows us to increase the weight of the locomotive without crushing the rails, or damaging bridges.

### 3.2.5 Fluid friction

Try to drive a nail into a waterfall and you will be confronted with the main difference between solid friction and fluid friction. Fluid friction is purely kinetic; there is no static fluid friction. The nail in the waterfall may tend to get dragged along by the water flowing past it, but it does not stick in the water. The same is true for gases such as air: recall that we are using the word “fluid” to include both gases and liquids.

Unlike kinetic friction between solids, fluid friction increases rapidly with velocity. It also depends on the shape of the object, which is why a fighter jet is more streamlined than a Model T. For objects of the same shape but different sizes, fluid friction typically scales up with the cross-sectional area of the object, which is one of the main reasons that an SUV gets worse mileage on the freeway.
Discussion Question

A Criticize the following analysis: “A book is sitting on a table. I shove it, overcoming static friction. Then it slows down until it has less force than static friction, and it stops.”

3.2.6 Analysis of forces

Newton’s first and second laws deal with the total of all the forces exerted on a specific object, so it is very important to be able to figure out what forces there are. Once you have focused your attention on one object and listed the forces on it, it is also helpful to describe all the corresponding forces that must exist according to Newton’s third law. We refer to this as “analyzing the forces” in which the object participates.

<table>
<thead>
<tr>
<th>Force acting on barge</th>
<th>Force related to it by Newton’s third law</th>
</tr>
</thead>
<tbody>
<tr>
<td>ropes’ forward normal forces on barge</td>
<td>barge’s backward normal force on ropes</td>
</tr>
<tr>
<td>water’s backward fluid friction force on barge</td>
<td>barge’s forward fluid friction force on water</td>
</tr>
<tr>
<td>planet earth’s downward gravitational force on barge</td>
<td>barge’s upward gravitational force on earth</td>
</tr>
<tr>
<td>water’s upward “floating” force on barge</td>
<td>barge’s downward “floating” force on water</td>
</tr>
</tbody>
</table>

Here I’ve used the word “floating” force as an example of a sensible invented term for a type of force not classified on the tree in the previous section. A more formal technical term would be “hydrostatic force.” Note how the pairs of forces are all structured as “A’s force on B, B’s force on A”: ropes on barge and barge on ropes; water on barge and barge on water. Because all the forces in the left column are forces acting on the barge, all the forces in the right column are forces being exerted by the barge, which is why each entry in the column begins with “barge.”

Often you may be unsure whether you have missed one of the forces. Here are three strategies for checking your list:

1. See what physical result would come from the forces you’ve found so far. Suppose, for instance, that you’d forgotten the “floating” force on the barge in the example above. Looking at the forces you’d found, you would have found that there was a downward gravitational force on the barge which was not canceled by any upward force. The barge isn’t supposed to sink, so you know you need to find a fourth, upward force.

2. Whenever one solid object touches another, there will be a normal force, and possibly also a frictional force; check for both.

3. Make a drawing of the object, and draw a dashed boundary line around it that separates it from its environment. Look for points on the boundary where other objects come in contact with your object. This strategy guarantees that you’ll find every contact force that acts on the object, although it won’t
help you to find non-contact forces.

The following is another example in which we can profit by checking against our physical intuition for what should be happening.

<table>
<thead>
<tr>
<th>force acting on Cindy</th>
<th>force related to it by Newton’s third law</th>
</tr>
</thead>
<tbody>
<tr>
<td>planet earth’s downward gravitational force on Cindy</td>
<td>Cindy’s upward gravitational force on earth</td>
</tr>
<tr>
<td>ropes upward frictional force on Cindy (her hand)</td>
<td>Cindy’s downward frictional force on the rope</td>
</tr>
<tr>
<td>cliff’s rightward normal force on Cindy</td>
<td>Cindy’s leftward normal force on the cliff</td>
</tr>
</tbody>
</table>

The two vertical forces cancel, which is what they should be doing if she is to go down at a constant rate. The only horizontal force on her is the cliff’s force, which is not canceled by any other force, and which therefore will produce an acceleration of Cindy to the right. This makes sense, since she is hopping off. (This solution is a little oversimplified, because the rope is slanting, so it also applies a small leftward force to Cindy. As she flies out to the right, the slant of the rope will increase, pulling her back in more strongly.)

I believe that constructing the type of table described in this section is the best method for beginning students. Most textbooks, however, prescribe a pictorial way of showing all the forces acting on an object. Such a picture is called a free-body diagram. It should not be a big problem if a future physics professor expects you to be able to draw such diagrams, because the conceptual reasoning is the same. You simply draw a picture of the object, with arrows representing the forces that are acting on it. Arrows representing contact forces are drawn from the point of contact, noncontact forces from the center of mass. Free-body diagrams do not show the equal and opposite forces exerted by the object itself.

Discussion Questions

A When you fire a gun, the exploding gases push outward in all directions, causing the bullet to accelerate down the barrel. What Newton’s-third-law pairs are involved? [Hint: Remember that the gases themselves are an object.]

B In the example of the barge going down the canal, I referred to a “floating” or “hydrostatic” force that keeps the boat from sinking. If you were adding a new branch on the force-classification tree to represent this force, where would it go?
A pool ball is rebounding from the side of the pool table. Analyze the forces in which the ball participates during the short time when it is in contact with the side of the table.

The earth’s gravitational force on you, i.e., your weight, is always equal to $mg$, where $m$ is your mass. So why can you get a shovel to go deeper into the ground by jumping onto it? Just because you’re jumping, that doesn’t mean your mass or weight is any greater, does it?

3.2.7 Transmission of forces by low-mass objects

You’re walking your dog. The dog wants to go faster than you do, and the leash is taut. Does Newton’s third law guarantee that your force on your end of the leash is equal and opposite to the dog’s force on its end? If they’re not exactly equal, is there any reason why they should be approximately equal?

If there was no leash between you, and you were in direct contact with the dog, then Newton’s third law would apply, but Newton’s third law cannot relate your force on the leash to the dog’s force on the leash, because that would involve three separate objects. Newton’s third law only says that your force on the leash is equal and opposite to the leash’s force on you,

$$F_{yL} = -F_{Ly},$$

and that the dog’s force on the leash is equal and opposite to its force on the dog

$$F_{dL} = -F_{Ld}.$$  

Still, we have a strong intuitive expectation that whatever force we make on our end of the leash is transmitted to the dog, and vice-versa. We can analyze the situation by concentrating on the forces that act on the leash, $F_{dL}$ and $F_{yL}$. According to Newton’s second law, these relate to the leash’s mass and acceleration:

$$F_{dL} + F_{yL} = m_La_L.$$  

The leash is far less massive than any of the other objects involved, and if $m_L$ is very small, then apparently the total force on the leash is also very small, $F_{dL} + F_{yL} \approx 0$, and therefore

$$F_{dL} \approx -F_{yL}.$$  

Thus even though Newton’s third law does not apply directly to these two forces, we can approximate the low-mass leash as if it was not intervening between you and the dog. It’s at least approximately as if you and the dog were acting directly on each other, in which case Newton’s third law would have applied.

In general, low-mass objects can be treated approximately as if they simply transmitted forces from one object to another. This can be true for strings, ropes, and cords, and also for rigid objects such as rods and sticks.
If we imagine dividing a taut rope up into small segments, then any segment has forces pulling outward on it at each end. If the rope is of negligible mass, then all the forces equal $+T$ or $-T$, where $T$, the tension, is a single number.

If you look at a piece of string under a magnifying glass as you pull on the ends more and more strongly, you will see the fibers straightening and becoming taut. Different parts of the string are apparently exerting forces on each other. For instance, if we think of the two halves of the string as two objects, then each half is exerting a force on the other half. If we imagine the string as consisting of many small parts, then each segment is transmitting a force to the next segment, and if the string has very little mass, then all the forces are equal in magnitude. We refer to the magnitude of the forces as the tension in the string, $T$.

The term “tension” refers only to internal forces within the string. If the string makes forces on objects at its ends, then those forces are typically normal or frictional forces (example 29).

### Types of force made by ropes example 29

- Analyze the forces in figures o/1 and o/2.
- In all cases, a rope can only make “pulling” forces, i.e., forces that are parallel to its own length and that are toward itself, not away from itself. You can’t push with a rope!

In o/1, the rope passes through a type of hook, called a carabiner, used in rock climbing and mountaineering. Since the rope can only pull along its own length, the direction of its force on the carabiner must be down and to the right. This is perpendicular to the surface of contact, so the force is a normal force.

<table>
<thead>
<tr>
<th>Force acting on carabiner</th>
<th>Force related to it by Newton’s third law</th>
</tr>
</thead>
<tbody>
<tr>
<td>rope’s normal force on carabiner</td>
<td>carabiner’s normal force on rope</td>
</tr>
</tbody>
</table>

(There are presumably other forces acting on the carabiner from other hardware above it.)

In figure o/2, the rope can only exert a net force at its end that is parallel to itself and in the pulling direction, so its force on the hand is down and to the left. This is parallel to the surface of contact, so it must be a frictional force. If the rope isn’t slipping through the hand, we have static friction. Friction can’t exist with-
out normal forces. These forces are perpendicular to the surface of contact. For simplicity, we show only two pairs of these normal forces, as if the hand were a pair of pliers.

<table>
<thead>
<tr>
<th>force acting on person</th>
<th>force related to it by Newton’s third law</th>
</tr>
</thead>
<tbody>
<tr>
<td>rope’s static frictional force on person</td>
<td>person’s static frictional force on rope</td>
</tr>
<tr>
<td>rope’s normal force on person</td>
<td>person’s normal force on rope</td>
</tr>
<tr>
<td>rope’s normal force on person</td>
<td>person’s normal force on rope</td>
</tr>
</tbody>
</table>

(There are presumably other forces acting on the person as well, such as gravity.)

If a rope goes over a pulley or around some other object, then the tension throughout the rope is approximately equal so long as the pulley has negligible mass and there is not too much friction. A rod or stick can be treated in much the same way as a string, but it is possible to have either compression or tension.

**Discussion Question**

A  When you step on the gas pedal, is your foot’s force being transmitted in the sense of the word used in this section?

### 3.2.8 Work

**Energy transferred to a particle**

To change the kinetic energy, \( K = (1/2)mv^2 \), of a particle moving in one dimension, we must change its velocity. That will entail a change in its momentum, \( p = mv \), as well, and since force is the rate of transfer of momentum, we conclude that the only way to change a particle’s kinetic energy is to apply a force.\(^6\) A force in the same direction as the motion speeds it up, increasing the kinetic energy, while a force in the opposite direction slows it down.

Consider an infinitesimal time interval during which the particle moves an infinitesimal distance \( dx \), and its kinetic energy changes by \( dK \). In one dimension, we represent the direction of the force and the direction of the motion with positive and negative signs for \( F \) and \( dx \), so the relationship among the signs can be summarized as follows:

\(^6\)The converse isn’t true, because kinetic energy doesn’t depend on the direction of motion, but momentum does. We can change a particle’s momentum without changing its energy, as when a pool ball bounces off a bumper, reversing the sign of \( p \).
\[
\begin{array}{c|c|c}
F > 0 & dx > 0 & dK > 0 \\
F < 0 & dx < 0 & dK > 0 \\
F > 0 & dx < 0 & dK < 0 \\
F < 0 & dx > 0 & dK < 0 \\
\end{array}
\]

This looks exactly like the rule for determining the sign of a product, and we can easily show using the chain rule that this is indeed a multiplicative relationship:

\[
dK = \frac{dK}{dv} \frac{dv}{dt} \frac{dt}{dx} \quad \text{[chain rule]}
\]

\[
= (mv)(a)(1/v) \, dx
\]

\[
= ma \, dx
\]

\[
= F \, dx \quad \text{[Newton’s second law]}
\]

We can verify that force multiplied by distance has units of energy:

\[
N \cdot m = \frac{kg \cdot m}{s} \times m
\]

\[
= kg \cdot m^2/s^2
\]

\[
= J
\]

**A TV picture tube example 30**

▷ At the back of a typical TV’s picture tube, electrical forces accelerate each electron to an energy of \(5 \times 10^{-16} \text{ J}\) over a distance of about 1 cm. How much force is applied to a single electron? (Assume the force is constant.) What is the corresponding acceleration?

▷ Integrating

\[
dK = F \, dx,
\]

we find

\[
K_f - K_i = F(x_f - x_i)
\]

or

\[
\Delta K = F \Delta x.
\]

The force is

\[
F = \frac{\Delta K}{\Delta x}
\]

\[
= \frac{5 \times 10^{-16} \text{ J}}{0.01 \text{ m}}
\]

\[
= 5 \times 10^{-14} \text{ N}.
\]
This may not sound like an impressive force, but it’s enough to supply an electron with a spectacular acceleration. Looking up the mass of an electron on p. 1072, we find

\[
a = F/m = 5 \times 10^{16} \text{ m/s}^2.
\]

**An air gun example 31**

An airgun, figure p, uses compressed air to accelerate a pellet. As the pellet moves from \( x_1 \) to \( x_2 \), the air decompresses, so the force is not constant. Using methods from chapter 5, one can show that the air’s force on the pellet is given by \( F = bx^{-7/5} \). A typical high-end airgun used for competitive target shooting has

\[
x_1 = 0.046 \text{ m}, \quad x_2 = 0.41 \text{ m},
\]

and

\[
b = 4.4 \text{ N} \cdot \text{m}^{7/5}.
\]

What is the kinetic energy of the pellet when it leaves the muzzle? (Assume friction is negligible.)

Since the force isn’t constant, it would be incorrect to do \( F = \Delta K/\Delta x \). Integrating both sides of the equation \( dK = F \, dx \), we have

\[
\Delta K = \int_{x_1}^{x_2} F \, dx = -\frac{5b}{2} \left( x_2^{-2/5} - x_1^{-2/5} \right) = 22 \text{ J}
\]

In general, when energy is transferred by a force,\(^7\) we use the term *work* to refer to the amount of energy transferred. This is different from the way the word is used in ordinary speech. If you stand for a long time holding a bag of cement, you get tired, and everyone will agree that you’ve worked hard, but you haven’t changed the energy of the cement, so according to the definition of the physics term, you haven’t done any work on the bag. There has been an energy transformation inside your body, of chemical energy into heat, but this just means that one part of your body did positive work (lost energy) while another part did a corresponding amount of negative work (gained energy).

\(^7\)The part of the definition about “by a force” is meant to exclude the transfer of energy by heat conduction, as when a stove heats soup.
Work in general

I derived the expression $F \, dx$ for one particular type of kinetic-energy transfer, the work done in accelerating a particle, and then defined work as a more general term. Is the equation correct for other types of work as well? For example, if a force lifts a mass $m$ against the resistance of gravity at constant velocity, the increase in the mass’s gravitational energy is $d(mgy) = mg \, dy = F \, dy$, so again the equation works, but this still doesn’t prove that the equation is always correct as a way of calculating energy transfers.

Imagine a black box, containing a gasoline-powered engine, which is designed to reel in a steel cable, exerting a certain force $F$. For simplicity, we imagine that this force is always constant, so we can talk about $\Delta x$ rather than an infinitesimal $\, dx$. If this black box is used to accelerate a particle (or any mass without internal structure), and no other forces act on the particle, then the original derivation applies, and the work done by the box is $W = F \Delta x$. Since $F$ is constant, the box will run out of gas after reeling in a certain amount of cable $\Delta x$. The chemical energy inside the box has decreased by $-W$, while the mass being accelerated has gained $W$ worth of kinetic energy.

Now what if we use the black box to pull a plow? The energy increase in the outside world is of a different type than before; it takes the forms of (1) the gravitational energy of the dirt that has been lifted out to the sides of the furrow, (2) frictional heating of the dirt and the plowshare, and (3) the energy needed to break up the dirt clods (a form of electrical energy involving the attractions among the atoms in the clod). The box, however, only communicates with the outside world via the hole through which its cable passes. The amount of chemical energy lost by the gasoline can therefore only depend on $F$ and $\Delta x$, so it is the same $-W$ as when the box was being used to accelerate a mass, and thus by conservation of energy, the work done on the outside world is again $W$.

This is starting to sound like a proof that the force-times-distance method is always correct, but there was one subtle assumption involved, which was that the force was exerted at one point (the end of the cable, in the black box example). Real life often isn’t like that. For example, a cyclist exerts forces on both pedals at once. Serious cyclists use toe-clips, and the conventional wisdom is that one should use equal amounts of force on the upstroke and downstroke, to make full use of both sets of muscles. This is a two-dimensional example, since the pedals go in circles. We’re only discussing one-dimensional motion right now, so let’s just pretend that the upstroke and down...

---

8 “Black box” is a traditional engineering term for a device whose inner workings we don’t care about.

9 For conceptual simplicity, we ignore the transfer of heat energy to the outside world via the exhaust and radiator. In reality, the sum of these energies plus the useful kinetic energy transferred would equal $W$. 

Section 3.2 Force in one dimension
stroke are both executed in straight lines. Since the forces are in opposite directions, one is positive and one is negative. The cyclist’s total force on the crank set is zero, but the work done isn’t zero. We have to add the work done by each stroke, \( W = F_1 \Delta x_1 + F_2 \Delta x_2 \). (I’m pretending that both forces are constant, so we don’t have to do integrals.) Both terms are positive; one is a positive number multiplied by a positive number, while the other is a negative times a negative.

This might not seem like a big deal — just remember not to use the total force — but there are many situations where the total force is all we can measure. The ultimate example is heat conduction. Heat conduction is not supposed to be counted as a form of work, since it occurs without a force. But at the atomic level, there are forces, and work is done by one atom on another. When you hold a hot potato in your hand, the transfer of heat energy through your skin takes place with a total force that’s extremely close to zero. At the atomic level, atoms in your skin are interacting electrically with atoms in the potato, but the attractions and repulsions add up to zero total force. It’s just like the cyclist’s feet acting on the pedals, but with zillions of forces involved instead of two. There is no practical way to measure all the individual forces, and therefore we can’t calculate the total energy transferred.

To summarize, \( \sum F_j \, dx_j \) is a correct way of calculating work, where \( F_j \) is the individual force acting on particle \( j \), which moves a distance \( dx_j \). However, this is only useful if you can identify all the individual forces and determine the distance moved at each point of contact. For convenience, I’ll refer to this as the work theorem. (It doesn’t have a standard name.)

There is, however, something useful we can do with the total force. We can use it to calculate the part of the work done on an object that consists of a change in the kinetic energy it has due to the motion of its center of mass. The proof is essentially the same as the proof on p. 165, except that now we don’t assume the force is acting on a single particle, so we have to be a little more delicate. Let the object consist of \( n \) particles. Its total kinetic energy is \( K = \sum_{j=1}^{n} (1/2)m_j v_j^2 \), but this is what we’ve already realized can’t be calculated using the total force. The kinetic energy it has due to motion of its center of mass is

\[
K_{cm} = \frac{1}{2} m_{total} v_{cm}^2.
\]

Figure r shows some examples of the distinction between \( K_{cm} \) and
Differentiating $K_{cm}$, we have

$$dK_{cm} = m_{total}v_{cm} dv_{cm}$$

$$= m_{total}v_{cm} \frac{dv_{cm}}{dt} \frac{dt}{dx_{cm}} \, dx_{cm} \quad [\text{chain rule}]$$

$$= m_{total} \frac{dv_{cm}}{dt} \, dx_{cm} \quad [\frac{dt}{dx_{cm}} = \frac{1}{v_{cm}}]$$

$$= \frac{dp_{total}}{dt} \, dx_{cm} \quad [p_{total} = m_{total}v_{cm}]$$

$$= F_{total} \, dx_{cm}$$

I’ll call this the kinetic energy theorem — like the work theorem, it has no standard name.

**An ice skater pushing off from a wall** example 32

The kinetic energy theorem tells us how to calculate the skater’s kinetic energy if we know the amount of force and the distance her center of mass travels while she is pushing off.

The work theorem tells us that the wall does no work on the skater, since the point of contact isn’t moving. This makes sense, because the wall does not have any source of energy.

**Absorbing an impact without recoiling?** example 33

▷ Is it possible to absorb an impact without recoiling? For instance, if a ping-pong ball hits a brick wall, does the wall “give” at all?

▷ There will always be a recoil. In the example proposed, the wall will surely have some energy transferred to it in the form of heat and vibration. The work theorem tells us that we can only have an energy transfer if the distance traveled by the point of contact is nonzero.

**Dragging a refrigerator at constant velocity** example 34

The fridge’s momentum is constant, so there is no net momentum transfer, and the total force on it must be zero: your force is canceling the floor’s kinetic frictional force. The kinetic energy theorem is therefore true but useless. It tells us that there is zero total force on the refrigerator, and that the refrigerator’s kinetic energy doesn’t change.

The work theorem tells us that the work you do equals your hand’s force on the refrigerator multiplied by the distance traveled. Since we know the floor has no source of energy, the only way for the floor and refrigerator to gain energy is from the work you do. We can thus calculate the total heat dissipated by friction in the refrigerator and the floor.

Note that there is no way to find how much of the heat is dissipated in the floor and how much in the refrigerator.
Accelerating a cart

If you push on a cart and accelerate it, there are two forces acting on the cart: your hand's force, and the static frictional force of the ground pushing on the wheels in the opposite direction.

Applying the work theorem to your force tells us how to calculate the work you do.

Applying the work theorem to the floor's force tells us that the floor does no work on the cart. There is no motion at the point of contact, because the atoms in the floor are not moving. (The atoms in the surface of the wheel are also momentarily at rest when they touch the floor.) This makes sense, because the floor does not have any source of energy.

The kinetic energy theorem refers to the total force, and because the floor's backward force cancels part of your force, the total force is less than your force. This tells us that only part of your work goes into the kinetic energy associated with the forward motion of the cart's center of mass. The rest goes into rotation of the wheels.

Discussion Questions

A Criticize the following incorrect statement: “A force doesn’t do any work unless it’s causing the object to move.”

B To stop your car, you must first have time to react, and then it takes some time for the car to slow down. Both of these times contribute to the distance you will travel before you can stop. The figure shows how the average stopping distance increases with speed. Because the stopping distance increases more and more rapidly as you go faster, the rule of one car length per 10 m.p.h. of speed is not conservative enough at high speeds. In terms of work and kinetic energy, what is the reason for the more rapid increase at high speeds?
Conservation of energy provided the necessary tools for analyzing some mechanical systems, such as the seesaw on page 85 and the pulley arrangements of the homework problems on page 122, but we could only analyze those machines by computing the total energy of the system. That approach wouldn’t work for systems like the biceps/forearm machine on page 85, or the one in figure t, where the energy content of the person’s body is impossible to compute directly. Even though the seesaw and the biceps/forearm system were clearly just two different forms of the lever, we had no way to treat them both on the same footing. We can now successfully attack such problems using the work and kinetic energy theorems.

\[ \text{Distance covered before reacting} \quad \text{per 10 mph} \quad \text{Actual stopping distance} \]

\[ \text{Constant tension around a pulley} \quad \text{example 36} \]

In figure t, what is the relationship between the force applied by the person’s hand and the force exerted on the block?

If we assume the rope and the pulley are ideal, i.e., frictionless and massless, then there is no way for them to absorb or release energy, so the work done by the hand must be the same as the work done on the block. Since the hand and the block move the same distance, the work theorem tells us the two forces are the same.

Similar arguments provide an alternative justification for the statement made in section 3.2.7 that show that an idealized rope exerts the same force, the tension, anywhere it’s attached to something, and the same amount of force is also exerted by each segment of the rope on the neighboring segments. Going around an ideal pulley also has no effect on the tension.
This is an example of a simple machine, which is any mechanical system that manipulates forces to do work. This particular machine reverses the direction of the motion, but doesn’t change the force or the speed of motion.

**A mechanical advantage** example 37

The idealized pulley in figure u has negligible mass, so its kinetic energy is zero, and the kinetic energy theorem tells us that the total force on it is zero. We know, as in the preceding example, that the two forces pulling it to the right are equal to each other, so the force on the left must be twice as strong. This simple machine doubles the applied force, and we refer to this ratio as a *mechanical advantage* (M.A.) of 2. There’s no such thing as a free lunch, however; the distance traveled by the load is cut in half, and there is no increase in the amount of work done.

**Inclined plane and wedge** example 38

In figure v, the force applied by the hand is equal to the one applied to the load, but there is a mechanical advantage compared to the force that would have been required to lift the load straight up. The distance traveled up the inclined plane is greater by a factor of $1/\sin \theta$, so by the work theorem, the force is smaller by a factor of $\sin \theta$, and we have $M.A. = 1/\sin \theta$. The wedge, w, is similar.

**Archimedes’ screw** example 39

In one revolution, the crank travels a distance $2\pi b$, and the water rises by a height $h$. The mechanical advantage is $2\pi b/h$.

### 3.2.10 Force related to interaction energy

In section 2.3, we saw that there were two equivalent ways of looking at gravity, the gravitational field and the gravitational energy. They were related by the equation $dU = mg \, dr$, so if we knew the field, we could find the energy by integration, $U = \int mg \, dr$, and if we knew the energy, we could find the field by differentiation, $g = \left(1/m\right) dU/\, dr$.

The same approach can be applied to other interactions, for example a mass on a spring. The main difference is that only in gravitational interactions does the strength of the interaction depend on the mass of the object, so in general, it doesn’t make sense to separate out the factor of $m$ as in the equation $dU = mg \, dr$. Since $F = mg$ is the gravitational force, we can rewrite the equation in the more suggestive form $dU = F \, dr$. This form no longer refers to gravity specifically, and can be applied much more generally. The only remaining detail is that I’ve been fairly cavalier about positive and negative signs up until now. That wasn’t such a big problem for gravitational interactions, since gravity is always attractive, but it requires more careful treatment for nongravitational forces, where we don’t necessarily know the direction of the force in advance, and
we need to use positive and negative signs carefully for the direction of the force.

In general, suppose that forces are acting on a particle — we can think of them as coming from other objects that are “off stage” — and that the interaction between the particle and the off-stage objects can be characterized by an interaction energy, $U$, which depends only on the particle’s position, $x$. Using the kinetic energy theorem, we have $dK = F \, dx$. (It’s not necessary to write $K_{cm}$, since a particle can’t have any other kind of kinetic energy.) Conservation of energy tells us $dK + dU = 0$, so the relationship between force and interaction energy is $dU = -F \, dx$, or

$$F = -\frac{dU}{dx} \quad \text{[relationship between force and interaction energy]}.$$ 

**Force exerted by a spring example 40**

▷ A mass is attached to the end of a spring, and the energy of the spring is $U = (1/2)kx^2$, where $x$ is the position of the mass, and $x = 0$ is defined to be the equilibrium position. What is the force the spring exerts on the mass? Interpret the sign of the result.

▷ Differentiating, we find

$$F = -\frac{dU}{dx} = -kx.$$ 

If $x$ is positive, then the force is negative, i.e., it acts so as to bring the mass back to equilibrium, and similarly for $x < 0$ we have $F > 0$.

Most books do the $F = -kx$ form before the $U = (1/2)kx^2$ form, and call it Hooke’s law. Neither form is really more fundamental than the other — we can always get from one to the other by integrating or differentiating.

**Newton’s law of gravity example 41**

▷ Given the equation $U = -Gm_1m_2/r$ for the energy of gravitational interactions, find the corresponding equation for the gravitational force on mass $m_2$. Interpret the positive and negative signs.

▷ We have to be a little careful here, because we’ve been taking $r$ to be positive by definition, whereas the position, $x$, of mass $m_2$ could be positive or negative, depending on which side of $m_1$ it’s on.

For positive $x$, we have $r = x$, and differentiation gives

$$F = -\frac{dU}{dx} = -Gm_1m_2/x^2.$$
As in the preceding example, we have $F < 0$ when $x$ is positive, because the object is being attracted back toward $x = 0$.

When $x$ is negative, the relationship between $r$ and $x$ becomes $r = -x$, and the result for the force is the same as before, but with a minus sign. We can combine the two equations by writing

$$|F| = \frac{Gm_1 m_2}{r^2},$$

and this is the form traditionally known as Newton’s law of gravity. As in the preceding example, the $U$ and $F$ equations contain equivalent information, and neither is more fundamental than the other.

### Equilibrium example 42

I previously described the condition for equilibrium as a local maximum or minimum of $U$. A differentiable function has a zero derivative at its extrema, and we can now relate this directly to force: zero force acts on an object when it is at equilibrium.
3.3 Resonance

Resonance is a phenomenon in which an oscillator responds most strongly to a driving force that matches its own natural frequency of vibration. For example, suppose a child is on a playground swing with a natural frequency of 1 Hz. That is, if you pull the child away from equilibrium, release her, and then stop doing anything for a while, she’ll oscillate at 1 Hz. If there was no friction, as we assumed in section 2.5, then the sum of her gravitational and kinetic energy would remain constant, and the amplitude would be exactly the same from one oscillation to the next. However, friction is going to convert these forms of energy into heat, so her oscillations would gradually die out. To keep this from happening, you might give her a push once per cycle, i.e., the frequency of your pushes would be 1 Hz, which is the same as the swing’s natural frequency. As long as you stay in rhythm, the swing responds quite well. If you start the swing from rest, and then give pushes at 1 Hz, the swing’s amplitude rapidly builds up, as in figure a, until after a while it reaches a steady state in which friction removes just as much energy as you put in over the course of one cycle.

self-check F

In figure a, compare the amplitude of the cycle immediately following the first push to the amplitude after the second. Compare the energies as well.

What will happen if you try pushing at 2 Hz? Your first push puts in some momentum, $p$, but your second push happens after only half a cycle, when the swing is coming right back at you, with momentum $-p$! The momentum transfer from the second push is exactly enough to stop the swing. The result is a very weak, and not very sinusoidal, motion, b.

Making the math easy

This is a simple and physically transparent example of resonance: the swing responds most strongly if you match its natural rhythm. However, it has some characteristics that are mathematically ugly and possibly unrealistic. The quick, hard pushes are known as impulse forces, c, and they lead to an $x$-$t$ graph that has nondifferentiable kinks. Impulsive forces like this are not only badly behaved mathematically, they are usually undesirable in practical terms. In a car engine, for example, the engineers work very hard to make the force on the pistons change smoothly, to avoid excessive vibration. Throughout the rest of this section, we’ll assume a driving force that is sinusoidal, d, i.e., one whose $F$-$t$ graph is either a sine function or a function that differs from a sine wave in phase, such as a cosine. The force is positive for half of each cycle and negative for the other half, i.e., there is both pushing and pulling. Sinusoidal functions have many nice mathematical characteristics (we can differentiate and integrate them, and the sum of sinusoidal functions...
that have the same frequency is a sinusoidal function), and they are also used in many practical situations. For instance, my garage door zapper sends out a sinusoidal radio wave, and the receiver is tuned to resonance with it.

A second mathematical issue that I glossed over in the swing example was how friction behaves. In section 3.2.4, about forces between solids, the empirical equation for kinetic friction was independent of velocity. Fluid friction, on the other hand, is velocity-dependent. For a child on a swing, fluid friction is the most important form of friction, and is approximately proportional to \( v^2 \). In still other situations, e.g., with a low-density gas or friction between solid surfaces that have been lubricated with a fluid such as oil, we may find that the frictional force has some other dependence on velocity, perhaps being proportional to \( v \), or having some other complicated velocity dependence that can’t even be expressed with a simple equation. It would be extremely complicated to have to treat all of these different possibilities in complete generality, so for the rest of this section, we’ll assume friction proportional to velocity

\[
F = -bv,
\]

simply because the resulting equations happen to be the easiest to solve. Even when the friction doesn’t behave in exactly this way, many of our results may still be at least qualitatively correct.

3.3.1 Damped, free motion

Numerical treatment

An oscillator that has friction is referred to as damped. Let’s use numerical techniques to find the motion of a damped oscillator that is released away from equilibrium, but experiences no driving force after that. We can expect that the motion will consist of oscillations that gradually die out.

In section 2.5, we simulated the undamped case using our tried and true Python function based on conservation of energy. Now, however, that approach becomes a little awkward, because it involves splitting up the path to be traveled into \( n \) tiny segments, but in the presence of damping, each swing is a little shorter than the last one, and we don’t know in advance exactly how far the oscillation will get before turning around. An easier technique here is to use force rather than energy. Newton’s second law, \( a = F/m \), gives \( a = (-kx - bv)/m \), where we’ve made use of the result of example 40 for the force exerted by the spring. This becomes a little prettier if we rewrite it in the form

\[
ma + bv + kx = 0,
\]

which gives symmetric treatment to three terms involving \( x \) and its first and second derivatives, \( v \) and \( a \). Now instead of calculating
the time $\Delta t = \Delta x/v$ required to move a predetermined distance $\Delta x$, we pick $\Delta t$ and determine the distance traveled in that time, $\Delta x = v\Delta t$. Also, we can no longer update $v$ based on conservation of energy, since we don’t have any easy way to keep track of how much mechanical energy has been changed into heat energy. Instead, we recalculate the velocity using $\Delta v = a\Delta t$.

```python
import math

# chosen to give a period of 1 second
k=39.4784
m=1.
# chosen to make the results simple
b=0.211
x=1.
v=0.
t=0.
dt=.01
n=1000
for j in range(n):
x=x+v*dt
a=(-k*x-b*v)/m
if (v>0) and (v+a*dt<0) :
    print("turnaround at t=",t," , x=",x)
v=v+a*dt
t=t+dt
```

```
turnaround at t= 0.99 , x= 0.899919262445
turnaround at t= 1.99 , x= 0.809844934046
turnaround at t= 2.99 , x= 0.728777519477
turnaround at t= 3.99 , x= 0.655817260033
turnaround at t= 4.99 , x= 0.590154191135
turnaround at t= 5.99 , x= 0.531059189965
turnaround at t= 6.99 , x= 0.477875914756
turnaround at t= 7.99 , x= 0.430013546991
turnaround at t= 8.99 , x= 0.386940256644
turnaround at t= 9.99 , x= 0.348177318484
```

The spring constant, $k = 4\pi = 39.4784$ N/m, is designed so that if the undamped equation $f = (1/2\pi)\sqrt{k/m}$ was still true, the frequency would be 1 Hz. We start by noting that the addition of a small amount of damping doesn’t seem to have changed the period at all, or at least not to within the accuracy of the calculation.\footnote{This subroutine isn’t as accurate a way of calculating the period as the energy-based one we used in the undamped case, since it only checks whether the mass turned around at some point during the time interval $\Delta t$.} You can check for yourself, however, that a large value of $b$, say 5 N·s/m, does change the period significantly.

We release the mass from $x = 1$ m, and after one cycle, it only comes back to about $x = 0.9$ m. I chose $b = 0.211$ N·s/m by fiddling

\[\text{Section 3.3 Rezonance}\]
around until I got this result, since a decrease of exactly 10% is easy to discuss. Notice how the amplitude after two cycles is about 0.81 m, i.e., 1 m times 0.9\(^2\): the amplitude has again dropped by exactly 10%. This pattern continues as long as the simulation runs, e.g., for the last two cycles, we have 0.34818/0.38694=0.89982, or almost exactly 0.9 again. It might have seemed capricious when I chose to use the unrealistic equation \( F = -bv \), but this is the payoff. Only with \(-bv\) friction do we get this kind of mathematically simple exponential decay.

Because the decay is exponential, it never dies out completely; this is different from the behavior we would have had with Coulomb friction, which does make objects grind completely to a stop at some point. With friction that acts like \( F = -bv \), \( v \) gets smaller as the oscillations get smaller. The smaller and smaller force then causes them to die out at a rate that is slower and slower.

**Analytic treatment**

Taking advantage of this unexpectedly simple result, let’s find an analytic solution for the motion. The numerical output suggests that we assume a solution of the form

\[ x = Ae^{-ct} \sin(\omega_f t + \delta), \]

where the unknown constants \( \omega_f \) and \( c \) will presumably be related to \( m \), \( b \), and \( k \). The constant \( c \) indicates how quickly the oscillations die out. The constant \( \omega_f \) is, as before, defined as 2\( \pi \) times the frequency, with the subscript \( f \) to indicate a free (undriven) solution. All our equations will come out much simpler if we use \( \omega s \) everywhere instead of \( f s \) from now on, and, as physicists often do, I’ll generally use the word “frequency” to refer to \( \omega \) when the context makes it clear what I’m talking about. The phase angle \( \delta \) has no real physical significance, since we can define \( t = 0 \) to be any moment in time we like.

**Self-check G**

In figure f, which graph has the greater value of \( c \)?  

> Answer, p. 1059

The factor \( A \) for the initial amplitude can also be omitted without loss of generality, since the equation we’re trying to solve, \( ma + bv + kx = 0 \), is linear. That is, \( v \) and \( a \) are the first and second derivatives of \( x \), and the derivative of \( Ax \) is simply \( A \) times the derivative of \( x \). Thus, if \( x(t) \) is a solution of the equation, then multiplying it by a constant gives an equally valid solution. This is another place where we see that a damping force proportional to \( v \) is the easiest to handle mathematically. For a damping force proportional to \( v^2 \), for example, we would have had to solve the equation \( ma + bv^2 + kx = 0 \), which is nonlinear.

For the purpose of determining \( \omega_f \) and \( c \), the most general form we need to consider is therefore \( x = e^{-ct} \sin \omega_f t \), whose first and
A damped sine wave is compared with an undamped one, with \( m \) and \( k \) kept the same and only \( b \) changed. Second derivatives are 

\[
v = e^{-ct} (-c \sin \omega_f t + \omega \cos \omega_f t)
\]

and

\[
a = e^{-ct} \left( c^2 \sin \omega_f t - 2\omega_f c \cos \omega_f t - \omega_f^2 \sin \omega_f t \right)
\]

Plugging these into the equation \( ma + bv + kx = 0 \) and setting the sine and cosine parts equal to zero gives, after some tedious algebra,

\[
c = \frac{b}{2m}
\]

and

\[
\omega_f = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.
\]

Intuitively, we expect friction to “slow down” the motion, as when we ride a bike into a big patch of mud. “Slow down,” however, could have more than one meaning here. It could mean that the oscillator would take more time to complete each cycle, or it could mean that as time went on, the oscillations would die out, thus giving smaller velocities.

Our mathematical results show that both of these things happen. The first equation says that \( c \), which indicates how quickly the oscillations damp out, is directly related to \( b \), the strength of the damping.

The second equation, for the frequency, can be compared with the result from page 118 of \( \sqrt{k/m} \) for the undamped system. Let’s refer to this now as \( \omega_o \), to distinguish it from the actual frequency \( \omega_f \) of the free oscillations when damping is present. The result for \( \omega_f \) will be less than \( \omega_o \), due to the presence of the \( b^2/4m^2 \) term. This tells us that the addition of friction to the system does increase the time required for each cycle. However, it is very common for the \( b^2/4m^2 \) term to be negligible, so that \( \omega_f \approx \omega_o \).

Figure g shows an example. The damping here is quite strong: after only one cycle of oscillation, the amplitude has already been reduced by a factor of 2, corresponding to a factor of 4 in energy. However, the frequency of the damped oscillator is only about 1% lower than that of the undamped one; after five periods, the accumulated lag is just barely visible in the offsetting of the arrows. We can see that extremely strong damping — even stronger than this — would have been necessary in order to make \( \omega_f \approx \omega_o \) a poor approximation.

### 3.3.2 The quality factor

It’s usually impractical to measure \( b \) directly and determine \( c \) from the equation \( c = b/2m \). For a child on a swing, measuring \( b \) would require putting the child in a wind tunnel! It’s usually much easier to characterize the amount of damping by observing the actual damped oscillations and seeing how many cycles it takes for the mechanical energy to decrease by a certain factor. The unitless
### Summary of Notation

- **k**  spring constant
- **m**  mass of the oscillator
- **b**  sets the amount of damping, \( F = -bv \)
- **T**  period
- **f**  frequency, \( 1/T \)
- **\( \omega \)** (Greek letter omega), angular frequency, \( 2\pi f \), often referred to simply as “frequency”
- **\( \omega_0 \)**  frequency the oscillator would have without damping, \( \sqrt{k/m} \)
- **\( \omega_f \)**  frequency of the free vibrations
- **c**  sets the time scale for the exponential decay envelope \( e^{-ct} \) of the free vibrations
- **\( F_m \)**  strength of the driving force, which is assumed to vary sinusoidally with frequency \( \omega \)
- **A**  amplitude of the steady-state response
- **\( \delta \)**  phase angle of the steady-state response

### Quality factor, \( Q \)

The quality factor, \( Q \), is defined as \( Q = \omega_0 / 2c \), and in the limit of weak damping, where \( \omega \approx \omega_0 \), this can be interpreted as the number of cycles required for the mechanical energy to fall off by a factor of \( e^{2\pi} = 535.49 \ldots \)

Using this new quantity, we can rewrite the equation for the frequency of damped oscillations in the slightly more elegant form

\[
\omega_f = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}
\]

#### self-check H

What if we wanted to make a simpler definition of \( Q \), as the number of oscillations required for the vibrations to die out completely, rather than the number required for the energy to fall off by this obscure factor?

Answer, p. 1060

---

**A graph example 43**

The damped motion in figure g has \( Q \approx 4.5 \), giving \( \sqrt{1 - \frac{1}{4Q^2}} \approx 0.99 \), as claimed at the end of the preceding subsection.

**Exponential decay in a trumpet example 44**

The vibrations of the air column inside a trumpet have a \( Q \) of about 10. This means that even after the trumpet player stops blowing, the note will keep sounding for a short time. If the player suddenly stops blowing, how will the sound intensity 20 cycles later compare with the sound intensity while she was still blowing?

The trumpet’s \( Q \) is 10, so after 10 cycles the energy will have fallen off by a factor of 535. After another 10 cycles we lose another factor of 535, so the sound intensity is reduced by a factor of \( 535 \times 535 = 2.9 \times 10^5 \).

The decay of a musical sound is part of what gives it its character, and a good musical instrument should have the right \( Q \), but the \( Q \) that is considered desirable is different for different instruments. A guitar is meant to keep on sounding for a long time after a string has been plucked, and might have a \( Q \) of 1000 or 10000. One of the reasons why a cheap synthesizer sounds so bad is that the sound suddenly cuts off after a key is released.

### 3.3.3 Driven motion

The driven case is extremely important in science, technology, and engineering. We have an external driving force \( F = F_m \sin \omega t \), where the constant \( F_m \) indicates the maximum strength of the force in either direction. The equation of motion is now

\[
ma + bv + kx = F_m \sin \omega t
\]

[equation of motion for a driven oscillator].

After the driving force has been applied for a while, we expect that the amplitude of the oscillations will approach some constant value. This motion is known as the steady state, and it’s the most interesting thing to find out; as we’ll see later, the most general type of motion is only a minor variation on the steady-state motion. For
the steady-state motion, we’re going to look for a solution of the form

\[ x = A \sin(\omega t + \delta). \]

In contrast to the undriven case, here it’s not possible to sweep \( A \) and \( \delta \) under the rug. The amplitude of the steady-state motion, \( A \), is actually the most interesting thing to know about the steady-state motion, and it’s not true that we still have a solution no matter how we fiddle with \( A \); if we have a solution for a certain value of \( A \), then multiplying \( A \) by some constant would break the equality between the two sides of the equation of motion. It’s also no longer true that we can get rid of \( \delta \) simply by redefining when we start the clock; here \( \delta \) represents a difference in time between the start of one cycle of the driving force and the start of the corresponding cycle of the motion.

The velocity and acceleration are \( v = \omega A \cos(\omega t + \delta) \) and \( a = -\omega^2 A \sin(\omega t + \delta) \), and if we plug these into the equation of motion, [1], and simplify a little, we find

\[ (k - m\omega^2) \sin(\omega t + \delta) + \omega b \cos(\omega t + \delta) = \frac{F_m}{A} \sin \omega t. \]

The sum of any two sinusoidal functions with the same frequency is also a sinusoidal, so the whole left side adds up to a sinusoidal. By fiddling with \( A \) and \( \delta \) we can make the amplitudes and phases of the two sides of the equation match up.

**Steady state, no damping**

\( A \) and \( \delta \) are easy to find in the case where there is no damping at all. There are now no cosines in equation [2] above, only sines, so if we wish we can set \( \delta \) to zero, and we find \( A = \frac{F_m}{(k - m\omega^2)} = \frac{F_m}{m(\omega_0^2 - \omega^2)} \). This, however, makes \( A \) negative for \( \omega > \omega_0 \). The variable \( \delta \) was designed to represent this kind of phase relationship, so we prefer to keep \( A \) positive and set \( \delta = \pi \) for \( \omega > \omega_0 \). Our results are then

\[ A = \frac{F_m}{m |\omega^2 - \omega_0^2|} \]

and

\[ \delta = \begin{cases} 0, & \omega < \omega_0 \\ \pi, & \omega > \omega_0 \end{cases}. \]

The most important feature of the result is that there is a resonance: the amplitude becomes greater and greater, and approaches infinity, as \( \omega \) approaches the resonant frequency \( \omega_0 \). This is the physical behavior we anticipated on page 175 in the example of pushing a child on a swing. If the driving frequency matches the frequency of the free vibrations, then the driving force will always be in the h / Dependence of the amplitude and phase angle on the driving frequency, for an undamped oscillator. The amplitudes were calculated with \( F_m, m, \) and \( \omega_0 \), all set to 1.
right direction to add energy to the swing. At a driving frequency very different from the resonant frequency, we might get lucky and push at the right time during one cycle, but our next push would come at some random point in the next cycle, possibly having the effect of slowing the swing down rather than speeding it up.

The interpretation of the infinite amplitude at \( \omega = \omega_o \) is that there really isn’t any steady state if we drive the system exactly at resonance — the amplitude will just keep on increasing indefinitely. In real life, the amplitude can’t be infinite both because there is always some damping and because there will always be some difference, however small, between \( \omega \) and \( \omega_o \). Even though the infinity is unphysical, it has entered into the popular consciousness, starting with the eccentric Serbian-American inventor and physicist Nikola Tesla. Around 1912, the tabloid newspaper The World Today credulously reported a story which Tesla probably fabricated — or wildly exaggerated — for the sake of publicity. Supposedly he created a steam-powered device “no larger than an alarm clock,” containing a piston that could be made to vibrate at a tunable and precisely controlled frequency. “He put his little vibrator in his coat-pocket and went out to hunt a half-erected steel building. Down in the Wall Street district, he found one — ten stories of steel framework without a brick or a stone laid around it. He clamped the vibrator to one of the beams, and fussed with the adjustment [presumably hunting for the building’s resonant frequency] until he got it. Tesla said finally the structure began to creak and weave and the steel-workers came to the ground panic-stricken, believing that there had been an earthquake. Police were called out. Tesla put the vibrator in his pocket and went away. Ten minutes more and he could have laid the building in the street. And, with the same vibrator he could have dropped the Brooklyn Bridge into the East River in less than an hour.”

The phase angle \( \delta \) also exhibits surprising behavior. As the frequency is tuned upward past resonance, the phase abruptly shifts so that the phase of the response is opposite to that of the driving force. There is a simple interpretation for this. The system’s mechanical energy can only change due to work done by the driving force, since there is no damping to convert mechanical energy to heat. In the steady state, then, the power transmitted by the driving force over a full cycle of motion must average out to zero. In general, the work theorem \( \text{d}E = F \text{d}x \) can always be divided by \( \text{d}t \) on both sides to give the useful relation \( P = Fv \). If \( Fv \) is to average out to zero, then \( F \) and \( v \) must be out of phase by \( \pm \pi/2 \), and since \( v \) is ahead of \( x \) by a phase angle of \( \pi/2 \), the phase angle between \( x \) and \( F \) must be zero or \( \pi \).

Given that these are the two possible phases, why is there a difference in behavior between \( \omega < \omega_o \) and \( \omega > \omega_o \)? At the low-frequency limit, consider \( \omega = 0 \), i.e., a constant force. A constant
force will simply displace the oscillator to one side, reaching an equilibrium that is offset from the usual one. The force and the response are in phase, e.g., if the force is to the right, the equilibrium will be offset to the right. This is the situation depicted in the amplitude graph of figure h at $\omega = 0$. The response, which is not zero, is simply this static displacement of the oscillator to one side.

At high frequencies, on the other hand, imagine shaking the poor child on the swing back and forth with a force that oscillates at 10 Hz. This is so fast that there is essentially no time for the force $F = -kx$ from gravity and the chain to act from one cycle to the next. The problem becomes equivalent to the oscillation of a free object. If the driving force varies like $\sin(\omega t)$, with $\delta = 0$, then the acceleration is also proportional to the sine. Integrating, we find that the velocity goes like minus a cosine, and a second integration gives a position that varies as minus the sine — opposite in phase to the driving force. Intuitively, this mathematical result corresponds to the fact that at the moment when the object has reached its maximum displacement to the right, that is the time when the greatest force is being applied to the left, in order to turn it around and bring it back toward the center.

\[ A = \frac{F_m}{(m|\omega^2 - \omega_0^2|)} \]

This is simply because a given force will produce less acceleration when applied to a more massive object. An application is shown in figure 45.

In a stringed instrument, the strings themselves don’t have enough surface area to excite sound waves very efficiently. In instruments of the violin family, as the strings vibrate from left to right, they cause the bridge (the piece of wood they pass over) to wiggle clockwise and counterclockwise, and this motion is transmitted to the top panel of the instrument, which vibrates and creates sound waves in the air.

A string player who wants to practice at night without bothering the neighbors can add some mass to the bridge. Adding mass to the bridge causes the amplitude of the vibrations to be smaller, and the sound to be much softer. A similar effect is seen when an electric guitar is used without an amp. The body of an electric guitar is so much more massive than the body of an acoustic guitar that the amplitude of its vibrations is very small.

\[ A = \frac{F_m}{(m|\omega^2 - \omega_0^2|)} \]
Steady state, with damping

The extension of the analysis to the damped case involves some lengthy algebra, which I’ve outlined on page 1027 in appendix 2. The results are shown in figure j. It’s not surprising that the steady state response is weaker when there is more damping, since the steady state is reached when the power extracted by damping matches the power input by the driving force. The maximum amplitude, at the peak of the resonance curve, is approximately proportional to $Q$.

**self-check 1**

From the final result of the analysis on page 1027, substitute $\omega = \omega_o$, and satisfy yourself that the result is proportional to $Q$. Why is $A_{res} \propto Q$ only an approximation?

What is surprising is that the amplitude is strongly affected by damping close to resonance, but only weakly affected far from it. In other words, the shape of the resonance curve is broader with more damping, and even if we were to scale up a high-damping curve so that its maximum was the same as that of a low-damping curve, it would still have a different shape. The standard way of describing the shape numerically is to give the quantity $\Delta \omega$, called the *full width at half-maximum*, or FWHM, which is defined in figure k. Note that the $y$ axis is energy, which is proportional to the square of the amplitude. Our previous observations amount to a statement that $\Delta \omega$ is greater when the damping is stronger, i.e., when the $Q$ is lower. It’s not hard to show from the equations on page 1027 that for large $Q$, the FWHM is given approximately by

$$\Delta \omega \approx \omega_o/Q.$$  

Another thing we notice in figure j is that for small values of $Q$ the frequency $\omega_{res}$ of the maximum $A$ is less than $\omega_o$.\(^{11}\) At even

\(^{11}\) The relationship is $\omega_{max} A/\omega_o = \sqrt{1 - 1/2Q^2}$, which is similar in form to the equation for the frequency of the free vibration, $\omega_f/\omega_o = \sqrt{1 - 1/4Q^2}$. A subtle point here is that although the maximum of $A$ and the maximum of $A^2$ must occur at the same frequency, the maximum energy does not occur, as we might expect, at the same frequency as the maximum of $A^2$. This is because
lower values of $Q$, like $Q = 1$, the $A - \omega$ curve doesn’t even have a maximum near $\omega > 0$.

An opera singer breaking a wineglass example 46
In order to break a wineglass by singing, an opera singer must first tap the glass to find its natural frequency of vibration, and then sing the same note back, so that her driving force will produce a response with the greatest possible amplitude. If she’s shopping for the right glass to use for this display of her prowess, she should look for one that has the greatest possible $Q$, since the resonance curve has a higher maximum for higher values of $Q$.

The interaction energy is proportional to $A^2$ regardless of frequency, but the kinetic energy is proportional to $A^2\omega^2$. The maximum energy actually occurs are precisely $\omega_0$. 

Section 3.3 Resonance 185
The Nimitz Freeway Collapse example 47

Figure 1 shows a section of the Nimitz Freeway in Oakland, CA, that collapsed during an earthquake in 1989. An earthquake consists of many low-frequency vibrations that occur simultaneously, which is why it sounds like a rumble of indeterminate pitch rather than a low hum. The frequencies that we can hear are not even the strongest ones; most of the energy is in the form of vibrations in the range of frequencies from about 1 Hz to 10 Hz.

All the structures we build are resting on geological layers of dirt, mud, sand, or rock. When an earthquake wave comes along, the topmost layer acts like a system with a certain natural frequency of vibration, sort of like a cube of jello on a plate being shaken from side to side. The resonant frequency of the layer depends on how stiff it is and also on how deep it is. The ill-fated section of the Nimitz freeway was built on a layer of mud, and analysis by geologist Susan E. Hough of the U.S. Geological Survey shows that the mud layer’s resonance was centered on about 2.5 Hz, and had a width covering a range from about 1 Hz to 4 Hz.

When the earthquake wave came along with its mixture of frequencies, the mud responded strongly to those that were close to its own natural 2.5 Hz frequency. Unfortunately, an engineering analysis after the quake showed that the overpass itself had a resonant frequency of 2.5 Hz as well! The mud responded strongly to the earthquake waves with frequencies close to 2.5 Hz, and the bridge responded strongly to the 2.5 Hz vibrations of the mud, causing sections of it to collapse.

Physical reason for the relationship between Q and the FWHM

What is the reason for this surprising relationship between the damping and the width of the resonance? Fundamentally, it has to do with the fact that friction causes a system to lose its “memory” of its previous state. If the Pioneer 10 space probe, coasting through the frictionless vacuum of interplanetary space, is detected by aliens a million years from now, they will be able to trace its trajectory backwards and infer that it came from our solar system. On the other hand, imagine that I shove a book along a tabletop, it comes to rest, and then someone else walks into the room. There will be no clue as to which direction the book was moving before it stopped — friction has erased its memory of its motion. Now consider the playground swing driven at twice its natural frequency, figure m, where the undamped case is repeated from figure b on page 175. In the undamped case, the first push starts the swing moving with momentum $p$, but when the second push comes, if there is no friction at all, it now has a momentum of exactly $-p$, and the momentum transfer from the second push is exactly enough to stop it dead. With moderate damping, however, the momentum on the rebound is not quite $-p$, and the second push’s effect isn’t quite as disas-
tous. With very strong damping, the swing comes essentially to rest long before the second push. It has lost all its memory, and the second push puts energy into the system rather than taking it out. Although the detailed mathematical results with this kind of impulsive driving force are different, the general results are the same as for sinusoidal driving: the less damping there is, the greater the penalty you pay for driving the system off of resonance.

1High-Q speakers example 48
Most good audio speakers have \( Q \approx 1 \), but the resonance curve for a higher-\( Q \) oscillator always lies above the corresponding curve for one with a lower \( Q \), so people who want their car stereos to be able to rattle the windows of the neighboring cars will often choose speakers that have a high \( Q \). Of course they could just use speakers with stronger driving magnets to increase \( F_m \), but the speakers might be more expensive, and a high-\( Q \) speaker also has less friction, so it wastes less energy as heat.

One problem with this is that whereas the resonance curve of a low-\( Q \) speaker (its “response curve” or “frequency response” in audiophile lingo) is fairly flat, a higher-\( Q \) speaker tends to emphasize the frequencies that are close to its natural resonance. In audio, a flat response curve gives more realistic reproduction of sound, so a higher quality factor, \( Q \), really corresponds to a lower-quality speaker.

Another problem with high-\( Q \) speakers is discussed in example 51 on page 189.

1Changing the pitch of a wind instrument example 49
▷ A saxophone player normally selects which note to play by choosing a certain fingering, which gives the saxophone a certain resonant frequency. The musician can also, however, change the pitch significantly by altering the tightness of her lips. This corresponds to driving the horn slightly off of resonance. If the pitch can be altered by about 5% up or down (about one musical half-step) without too much effort, roughly what is the \( Q \) of a saxophone?

▷ Five percent is the width on one side of the resonance, so the full width is about 10%, \( \Delta f / f_o \approx 0.1 \). The equation \( \Delta \omega = \omega_o / Q \) is defined in terms of angular frequency, \( \omega = 2\pi f \), and we’ve been given our data in terms of ordinary frequency, \( f \). The factors of \( 2\pi \)

12For example, the graphs calculated for sinusoidal driving have resonances that are somewhat below the natural frequency, getting lower with increasing damping, until for \( Q \leq 1 \) the maximum response occurs at \( \omega = 0 \). In figure m, however, we can see that impulsive driving at \( \omega = 2\omega_o \) produces a steady state with more energy than at \( \omega = \omega_o \).
end up canceling out, however:

\[ Q = \frac{\omega_0}{\Delta \omega} \]
\[ = \frac{2\pi f_0}{2\pi \Delta f} \]
\[ = \frac{f_0}{\Delta f} \]
\[ \approx 10 \]

In other words, once the musician stops blowing, the horn will continue sounding for about 10 cycles before its energy falls off by a factor of 535. (Blues and jazz saxophone players will typically choose a mouthpiece that gives a low \( Q \), so that they can produce the bluesy pitch-slides typical of their style. “Legit,” i.e., classically oriented players, use a higher-\( Q \) setup because their style only calls for enough pitch variation to produce a vibrato, and the higher \( Q \) makes it easier to play in tune.)

### Q of a radio receiver example 50

A radio receiver used in the FM band needs to be tuned in to within about 0.1 MHz for signals at about 100 MHz. What is its \( Q \)?

As in the last example, we’re given data in terms of \( f_s \), not \( \omega_s \), but the factors of \( 2\pi \) cancel. The resulting \( Q \) is about 1000, which is extremely high compared to the \( Q \) values of most mechanical systems.

### Transients

What about the motion before the steady state is achieved? When we computed the undriven motion numerically on page 176, the program had to initialize the position and velocity. By changing these two variables, we could have gotten any of an infinite number of simulations. The same is true when we have an equation of motion with a driving term, \( ma + bv + kx = F_m \sin \omega t \) (p. 180, equation [1]). The steady-state solutions, however, have no adjustable parameters at all — \( A \) and \( \delta \) are uniquely determined by the parameters of the driving force and the oscillator itself. If the oscillator isn’t initially in the steady state, then it will not have the steady-state motion at first. What kind of motion will it have?

The answer comes from realizing that if we start with the solution to the driven equation of motion, and then add to it any solution to the free equation of motion, the result,

\[ x = A \sin(\omega t + \delta) + A' e^{-\omega t} \sin(\omega' t + \delta'), \]

If you’ve learned about differential equations, you’ll know that any second-order differential equation requires the specification of two boundary conditions in order to specify solution uniquely.
is also a solution of the driven equation. Here, as before, $\omega_f$ is the frequency of the free oscillations ($\omega_f \approx \omega_0$ for small $Q$), $\omega$ is the frequency of the driving force, $A$ and $\delta$ are related as usual to the parameters of the driving force, and $A'$ and $\delta'$ can have any values at all. Given the initial position and velocity, we can always choose $A'$ and $\delta'$ to reproduce them, but this is not something one often has to do in real life. What’s more important is to realize that the second term dies out exponentially over time, decaying at the same rate at which a free vibration would. For this reason, the $A'$ term is called a transient. A high-$Q$ oscillator’s transients take a long time to die out, while a low-$Q$ oscillator always settles down to its steady state very quickly.

**Boomy bass**  
In example 48 on page 187, I’ve already discussed one of the drawbacks of a high-$Q$ speaker, which is an uneven response curve. Another problem is that in a high-$Q$ speaker, transients take a long time to die out. The bleeding-eardrums crowd tend to focus mostly on making their bass loud, so it’s usually their woofers that have high $Q$s. The result is that bass notes, “ring” after the onset of the note, a phenomenon referred to as “boomy bass.”

**Overdamped motion**

The treatment of free, damped motion on page 178 skipped over a subtle point: in the equation $\omega_f = \sqrt{k/m - b^2/4m^2} = \omega_0\sqrt{1 - 1/4Q^2}$, $Q < 1/2$ results in an answer that is the square root of a negative number. For example, suppose we had $k = 0$, which corresponds to a neutral equilibrium. A physical example would be a mass sitting in a tub of syrup. If we set it in motion, it won’t oscillate — it will simply slow to a stop. This system has $Q = 0$. The equation of motion in this case is $ma + bv = 0$, or, more suggestively,

$$m\frac{dv}{dt} + bv = 0.$$

One can easily verify that this has the solution $v = (\text{constant})e^{-bt/m}$, and integrating, we find $x = (\text{constant})e^{-bt/m} + (\text{constant})$. In other words, the reason $\omega_f$ comes out to be mathematical nonsense is that we were incorrect in assuming a solution that oscillated at a frequency $\omega_f$. The actual motion is not oscillatory at all.

In general, systems with $Q < 1/2$, called overdamped systems, do not display oscillatory motion. Most cars’ shock absorbers are designed with $Q \approx 1/2$, since it’s undesirable for the car to undulate up and down for a while after you go over a bump. (Shocks with extremely low values of $Q$ are not good either, because such a system takes a very long time to come back to equilibrium.) It’s not par-

---

14 Actually, if you know about complex numbers and Euler’s theorem, it’s not quite so nonsensical.
particularly important for our purposes, but for completeness I’ll note, as you can easily verify, that the general solution to the equation of motion for $0 < Q < 1/2$ is of the form $x = Ae^{-ct} + Be^{-dt}$, while $Q = 1/2$, called the critically damped case, gives $x = (A + Bt)e^{-ct}$.
3.4 Motion in three dimensions

3.4.1 The Cartesian perspective

When my friends and I were bored in high school, we used to play a paper-and-pencil game which, although we never knew it, was Very Educational — in fact, it pretty much embodies the entire worldview of classical physics. To play the game, you draw a racetrack on graph paper, and try to get your car around the track before anyone else. The default is for your car to continue at constant speed in a straight line, so if it moved three squares to the right and one square up on your last turn, it will do the same this turn. You can also control the car’s motion by changing its $\Delta x$ and $\Delta y$ by up to one unit. If it moved three squares to the right last turn, you can have it move anywhere from two to four squares to the right this turn.

French mathematician René Descartes invented analytic geometry; Cartesian $(xyz)$ coordinates are named after him. He did work in philosophy, and was particularly interested in the mind-body problem. He was a skeptic and an antiaristotelian, and, probably for fear of religious persecution, spent his adult life in the Netherlands, where he fathered a daughter with a Protestant peasant whom he could not marry. He kept his daughter’s existence secret from his enemies in France to avoid giving them ammunition, but he was crushed when she died of scarlatina at age 5. A pious Catholic, he was widely expected to be sainted. His body was buried in Sweden but then reburied several times in France, and along the way everything but a few fingerbones was stolen by peasants who expected the body parts to become holy relics.

The fundamental way of dealing with the direction of an object’s motion in physics is to use conservation of momentum, since momentum depends on direction. Up until now, we’ve only done momentum in one dimension. How does this relate to the racetrack game? In the game, the motion of a car from one turn to the next is represented by its $\Delta x$ and $\Delta y$. In one dimension, we would only need $\Delta x$, which could be related to the velocity, $\Delta x/\Delta t$, and the momentum, $m\Delta x/\Delta t$. In two dimensions, the rules of the game amount to a statement that if there is no momentum transfer, then both $m\Delta x/\Delta t$ and $m\Delta y/\Delta t$ stay the same. In other words, there are two flavors of momentum, and they are separately conserved. All of this so far has been done with an artificial division of time into “turns,” but we can fix that by redefining everything in terms of derivatives, and for motion in three dimensions rather than two, we augment $x$ and $y$ with $z$:

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$

and

$$p_x = mv_x, \quad p_y = mv_y, \quad p_z = mv_z$$
Bullets are dropped and shot at the same time.

We call these the $x$, $y$, and $z$ components of the velocity and the momentum.

There is both experimental and theoretical evidence that the $x$, $y$, and $z$ momentum components are separately conserved, and that a momentum transfer (force) along one axis has no effect on the momentum components along the other two axes. On page 89, for example, I argued that it was impossible for an air hockey puck to make a 180-degree turn spontaneously, because then in the frame moving along with the puck, it would have begun moving after starting from rest. Now that we’re working in two dimensions, we might wonder whether the puck could spontaneously make a 90-degree turn, but exactly the same line of reasoning shows that this would be impossible as well, which proves that the puck can’t trade $x$-momentum for $y$-momentum. A more general proof of separate conservation will be given on page 218, after some of the appropriate mathematical techniques have been introduced.

As an example of the experimental evidence for separate conservation of the momentum components, figure c shows correct and incorrect predictions of what happens if you shoot a rifle and arrange for a second bullet to be dropped from the same height at exactly the same moment when the first one left the barrel. Nearly everyone expects that the dropped bullet will reach the dirt first, and Aristotle would have agreed, since he believed that the bullet had to lose its horizontal motion before it could start moving vertically. In reality, we find that the vertical momentum transfer between the earth and the bullet is completely unrelated to the horizontal momentum. The bullet ends up with $p_y < 0$, while the planet picks up an upward momentum $p_y > 0$, and the total momentum in the $y$ direction remains zero. Both bullets hit the ground at the same time. This is much simpler than the Aristotelian version!

The Pelton waterwheel example 52

There is a general class of machines that either do work on a
gas or liquid, like a boat’s propeller, or have work done on them by a gas or liquid, like the turbine in a hydroelectric power plant. Figure d shows two types of surfaces that could be attached to the circumference of an old-fashioned waterwheel. Compare the force exerted by the water in the two cases.

Let the $x$ axis point to the right, and the $y$ axis up. In both cases, the stream of water rushes down onto the surface with momentum $p_{y,i} = -p_0$, where the subscript $i$ stands for “initial,” i.e., before the collision.

In the case of surface 1, the streams of water leaving the surface have no momentum in the $y$ direction, and their momenta in the $x$ direction cancel. The final momentum of the water is zero along both axes, so its entire momentum, $-p_0$, has been transferred to the waterwheel.

When the water leaves surface 2, however, its momentum isn’t zero. If we assume there is no friction, it’s $p_{y,f} = +p_0$, with the positive sign indicating upward momentum. The change in the water’s momentum is $p_{y,f} - p_{y,i} = 2p_0$, and the momentum transferred to the waterwheel is $-2p_0$.

Force is defined as the rate of transfer of momentum, so surface 2 experiences double the force. A waterwheel constructed in this way is known as a Pelton waterwheel.

We think of the planets and asteroids as inhabiting their orbits permanently, but it is possible for an orbit to change over periods of millions or billions of years, due to a variety of effects. For asteroids with diameters of a few meters or less, an important mechanism is the Yarkovsky effect, which is easiest to understand if we consider an asteroid spinning about an axis that is exactly perpendicular to its orbital plane.

The illuminated side of the asteroid is relatively hot, and radiates more infrared light than the dark (night) side. Light has momentum, and a total force away from the sun is produced by combined effect of the sunlight hitting the asteroid and the imbalance between the momentum radiated away on the two sides. This force, however, doesn’t cause the asteroid’s orbit to change over time, since it simply cancels a tiny fraction of the sun’s gravitational attraction. The result is merely a tiny, undetectable violation of Kepler’s law of periods.

Consider the sideways momentum transfers, however. In figure e, the part of the asteroid on the right has been illuminated for half a spin-period (half a “day”) by the sun, and is hot. It radiates more light than the morning side on the left. This imbalance produces a total force in the $x$ direction which points to the left. If the asteroid’s orbital motion is to the left, then this is a force in the same
direction as the motion, which will do positive work, increasing the asteroid's energy and boosting it into an orbit with a greater radius. On the other hand, if the asteroid's spin and orbital motion are in opposite directions, the Yarkovsky push brings the asteroid spiraling in closer to the sun.

Calculations show that it takes on the order of $10^7$ to $10^8$ years for the Yarkovsky effect to move an asteroid out of the asteroid belt and into the vicinity of earth's orbit, and this is about the same as the typical age of a meteorite as estimated by its exposure to cosmic rays. The Yarkovsky effect doesn’t remove all the asteroids from the asteroid belt, because many of them have orbits that are stabilized by gravitational interactions with Jupiter. However, when collisions occur, the fragments can end up in orbits which are not stabilized in this way, and they may then end up reaching the earth due to the Yarkovsky effect. The cosmic-ray technique is really telling us how long it has been since the fragment was broken out of its parent.

**Discussion Questions**

**A** The following is an incorrect explanation of a fact about target shooting:

“Shooting a high-powered rifle with a high muzzle velocity is different from shooting a less powerful gun. With a less powerful gun, you have to aim quite a bit above your target, but with a more powerful one you don’t have to aim so high because the bullet doesn’t drop as fast.”

What is the correct explanation?

**B** You have thrown a rock, and it is flying through the air in an arc. If the earth’s gravitational force on it is always straight down, why doesn’t it just go straight down once it leaves your hand?

**C** Consider the example of the bullet that is dropped at the same moment another bullet is fired from a gun. What would the motion of the two bullets look like to a jet pilot flying alongside in the same direction as the shot bullet and at the same horizontal speed?
3.4.2 Rotational invariance

The Cartesian approach requires that we choose \( x, y, \) and \( z \) axes. How do we choose them correctly? The answer is that it had better not matter which directions the axes point (provided they’re perpendicular to each other), or where we put the origin, because if it did matter, it would mean that space was asymmetric. If there was a certain point in the universe that was the right place to put the origin, where would it be? The top of Mount Olympus? The United Nations headquarters? We find that experiments come out the same no matter where we do them, and regardless of which way the laboratory is oriented, which indicates that no location in space or direction in space is special in any way.\(^{15}\)

This is closely related to the idea of Galilean relativity stated on page 62, from which we already know that the absolute motion of a frame of reference is irrelevant and undetectable. Observers using frames of reference that are in motion relative to each other will not even agree on the permanent identity of a particular point in space, so it’s not possible for the laws of physics to depend on where you are in space. For instance, if gravitational energies were proportional to \( m_1m_2 \) in one location but to \( (m_1m_2)^{1.00001} \) in another, then it would be possible to determine when you were in a state of absolute motion, because the behavior of gravitational interactions would change as you moved from one region to the other.

Because of this close relationship, we restate the principle of Galilean relativity in a more general form. This extended principle of Galilean relativity states that the laws of physics are no different in one time and place than in another, and that they also don’t depend on your orientation or your motion, provided that your motion is in a straight line and at constant speed.

The irrelevance of time and place could have been stated in chapter 1, but since this section is the first one in which we’re dealing with three-dimensional physics in full generality, the irrelevance of orientation is what we really care about right now. This property of the laws of physics is called rotational invariance. The word “invariance” means a lack of change, i.e., the laws of physics don’t change when we reorient our frame of reference.

\(^{15}\) Of course, you could tell in a sealed laboratory which way was down, but that’s because there happens to be a big planet nearby, and the planet’s gravitational field reaches into the lab, not because space itself has a special down direction. Similarly, if your experiment was sensitive to magnetic fields, it might matter which way the building was oriented, but that’s because the earth makes a magnetic field, not because space itself comes equipped with a north direction.
Two balls roll down a cone and onto a plane.

Pythagorean theorem equals

\[ \frac{1}{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}} \]

Rotating a line segment doesn’t change its length, so this expression comes out the same regardless of which way we orient our coordinate axes. Even though \( \Delta x \), \( \Delta y \), and \( \Delta z \) are different in differently oriented coordinate systems, \( r \) is the same.

Kinetic energy example 55

Kinetic energy equals \((1/2)mv^2\), but what does that mean in three dimensions, where we have \( v_x \), \( v_y \), and \( v_z \)? If you were tempted to add the components and calculate \( K = (1/2)m(v_x + v_y + v_z)^2 \), figure g should convince you otherwise. Using that method, we’d have to assign a kinetic energy of zero to ball number 1, since its negative \( v_y \) would exactly cancel its positive \( v_x \), whereas ball number 2’s kinetic energy wouldn’t be zero. This would violate rotational invariance, since the balls would behave differently.

The only possible way to generalize kinetic energy to three dimensions, without violating rotational invariance, is to use an expression that resembles the Pythagorean theorem,

\[ v = \sqrt{v_x^2 + v_y^2 + v_z^2}, \]

which results in

\[ K = \frac{1}{2}m \left( v_x^2 + v_y^2 + v_z^2 \right). \]

Since the velocity components are squared, the positive and negative signs don’t matter, and the two balls in the example behave the same way.
3.4.3 Vectors

Remember the title of this book? It would have been possible to obtain the result of example 55 by applying the Pythagorean theorem to \( dx \), \( dy \), and \( dz \), and then dividing by \( dt \), but the rotational invariance approach is simpler, and is useful in a much broader context. Even with a quantity you presently know nothing about, say the magnetic field, you can infer that if the components of the magnetic field are \( B_x \), \( B_y \), and \( B_z \), then the physically useful way to talk about the strength of the magnetic field is to define it as \( \sqrt{B_x^2 + B_y^2 + B_z^2} \). Nature knows your brain cells are precious, and doesn’t want you to have to waste them by memorizing mathematical rules that are different for magnetic fields than for velocities.

When mathematicians see that the same set of techniques is useful in many different contexts, that’s when they start making definitions that allow them to stop reinventing the wheel. The ancient Greeks, for example, had no general concept of fractions. They couldn’t say that a circle’s radius divided by its diameter was equal to the number 1/2. They had to say that the radius and the diameter were in the ratio of one to two. With this limited number concept, they couldn’t have said that water was dripping out of a tank at a rate of 3/4 of a barrel per day; instead, they would have had to say that over four days, three barrels worth of water would be lost. Once enough of these situations came up, some clever mathematician finally realized that it would make sense to define something called a fraction, and that one could think of these fraction thingies as numbers that lay in the gaps between the traditionally recognized numbers like zero, one, and two. Later generations of mathematicians introduced further subversive generalizations of the number concepts, inventing mathematical creatures like negative numbers, and the square root of two, which can’t be expressed as a fraction.

In this spirit, we define a vector as any quantity that has both an amount and a direction in space. In contradistinction, a scalar has an amount, but no direction. Time and temperature are scalars. Velocity, acceleration, momentum, and force are vectors. In one dimension, there are only two possible directions, and we can use positive and negative numbers to indicate the two directions. In more than one dimension, there are infinitely many possible directions, so we can’t use the two symbols + and − to indicate the direction of a vector. Instead, we can specify the three components of the vector, each of which can be either negative or positive. We represent vector quantities in handwriting by writing an arrow above them, so for example the momentum vector looks like this, \( \vec{p} \), but the arrow looks ugly in print, so in books vectors are usually shown in bold-face type: \( \mathbf{p} \). A straightforward way of thinking about vectors is that a vector equation really represents three different equations. For instance, conservation of momentum could be written in terms...
of the three components,

\[ \Delta p_x = 0 \]
\[ \Delta p_y = 0 \]
\[ \Delta p_z = 0, \]

or as a single vector equation,\(^\text{16}\)

\[ \Delta \mathbf{p} = 0. \]

The following table summarizes some vector operations.

<table>
<thead>
<tr>
<th>operation</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>\text{vector}</td>
</tr>
<tr>
<td>( \text{vector} + \text{vector}</td>
<td>)</td>
</tr>
<tr>
<td>( \text{vector} - \text{vector}</td>
<td>)</td>
</tr>
<tr>
<td>( \text{vector} \cdot \text{scalar}</td>
<td>)</td>
</tr>
<tr>
<td>( \text{vector} / \text{scalar}</td>
<td>)</td>
</tr>
</tbody>
</table>

The first of these is called the \textit{magnitude} of the vector; in one dimension, where a vector only has one component, it amounts to taking the absolute value, hence the similar notation.

\textit{self-check J}

Translate the equations \( F_x = ma_x, \) \( F_y = ma_y, \) and \( F_z = ma_z \) into a single equation in vector notation. \( \triangleright \) Answer, p. 1060

\textit{An explosion example 56}

\( \triangleright \) Astronomers observe the planet Mars as the Martians fight a nuclear war. The Martian bombs are so powerful that they rip the planet into three separate pieces of liquefied rock, all having the same mass. If one fragment flies off with velocity components \( v_{1x} = 0, \) \( v_{1y} = 1.0 \times 10^4 \text{ km/hr}, \) and the second with \( v_{2x} = 1.0 \times 10^4 \text{ km/hr}, v_{2y} = 0, \) what is the magnitude of the third one’s velocity?

\( \triangleright \) We work the problem in the center of mass frame, in which the planet initially had zero momentum. After the explosion, the vector sum of the momenta must still be zero. Vector addition can be done by adding components, so

\[ mv_{1x} + mv_{2x} + mv_{3x} = 0 \]

and

\[ mv_{1y} + mv_{2y} + mv_{3y} = 0, \]

where we have used the same symbol \( m \) for all the terms, because the fragments all have the same mass. The masses can

\(^{16}\)The zero here is really a zero \textit{vector}, i.e., a vector whose components are all zero, so we should really represent it with a boldface \( \mathbf{0} \). There’s usually not much danger of confusion, however, so most books, including this one, don’t use boldface for the zero vector.
be eliminated by dividing each equation by $m$, and we find

\[
\begin{align*}
v_{3x} &= -1.0 \times 10^4 \text{ km/hr} \\
v_{3y} &= -1.0 \times 10^4 \text{ km/hr},
\end{align*}
\]

which gives a magnitude of

\[
|v_3| = \sqrt{v_{3x}^2 + v_{3y}^2} = 1.4 \times 10^4 \text{ km/hr}.
\]

A toppling box example 57

If you place a box on a frictionless surface, it will fall over with a very complicated motion that is hard to predict in detail. We know, however, that its center of mass's motion is related to its momentum, and the rate at which momentum is transferred is the force. Moreover, we know that these relationships apply separately to each component. Let $x$ and $y$ be horizontal, and $z$ vertical. There are two forces on the box, an upward force from the table and a downward gravitational force. Since both of these are along the $z$ axis, $p_z$ is the only component of the box's momentum that can change. We conclude that the center of mass travels vertically. This is true even if the box bounces and tumbles. [Based on an example by Kleppner and Kolenkow.]

Geometric representation of vectors

A vector in two dimensions can be easily visualized by drawing an arrow whose length represents its magnitude and whose direction represents its direction. The $x$ component of a vector can then be visualized, $j$, as the length of the shadow it would cast in a beam of light projected onto the $x$ axis, and similarly for the $y$ component. Shadows with arrowheads pointing back against the direction of the positive axis correspond to negative components.

In this type of diagram, the negative of a vector is the vector with the same magnitude but in the opposite direction. Multiplying a vector by a scalar is represented by lengthening the arrow by that factor, and similarly for division.

**self-check K**

Given vector $\mathbf{Q}$ represented by an arrow below, draw arrows representing the vectors $1.5\mathbf{Q}$ and $-\mathbf{Q}$.
The way I've defined the various vector operations above aren't as arbitrary as they seem. There are many different vector operations that we could define, but only some of the possible definitions are mathematically useful. Consider the operation of multiplying two vectors component by component to produce a third vector:

\[
\begin{align*}
R_x &= P_x Q_x \\
R_y &= P_y Q_y \\
R_z &= P_z Q_z
\end{align*}
\]

As a simple example, we choose vectors \( \mathbf{P} \) and \( \mathbf{Q} \) to have length 1, and make them perpendicular to each other, as shown in figure k/1. If we compute the result of our new vector operation using the coordinate system shown in k/2, we find:

\[
\begin{align*}
R_x &= 0 \\
R_y &= 0 \\
R_z &= 0
\end{align*}
\]

The \( x \) component is zero because \( P_x = 0 \), the \( y \) component is zero because \( Q_y = 0 \), and the \( z \) component is of course zero because both vectors are in the \( x-y \) plane. However, if we carry out the same operations in coordinate system k/3, rotated 45 degrees with respect to the previous one, we find

\[
\begin{align*}
R_x &= -1/2 \\
R_y &= 1/2 \\
R_z &= 0
\end{align*}
\]

The operation's result depends on what coordinate system we use, and since the two versions of \( \mathbf{R} \) have different lengths (one being zero and the other nonzero), they don’t just represent the same answer expressed in two different coordinate systems. Such an operation will never be useful in physics, because experiments show physics works the same regardless of which way we orient the laboratory building! The useful vector operations, such as addition and scalar multiplication, are rotationally invariant, i.e., come out the same regardless of the orientation of the coordinate system.

All the vector techniques can be applied to any kind of vector, but the graphical representation of vectors as arrows is particularly natural for vectors that represent lengths and distances. We define a vector called \( \mathbf{r} \) whose components are the coordinates of a particular point in space, \( x, y, \) and \( z \). The \( \Delta \mathbf{r} \) vector, whose components are \( \Delta x, \Delta y, \) and \( \Delta z \), can then be used to represent motion that starts at