Determining \( G \) 

The constant \( G \) is not easy to determine, and Newton went to his grave without knowing an accurate value for it. If we knew the mass of the earth, then we could easily determine \( G \) from experiments with terrestrial gravity, but the only way to determine the mass of the earth accurately in units of kilograms is by finding \( G \) and reasoning the other way around! (If you estimate the average density of the earth, you can make at least a rough estimate of \( G \).) Figures g and h show how \( G \) was first measured by Henry Cavendish in the nineteenth century. The rotating arm is released from rest, and the kinetic energy of the two moving balls is measured when they pass position C. Conservation of energy gives

\[
-2 \frac{GMm}{r_{BA}} - 2 \frac{GMm}{r_{BD}} = -2 \frac{GMm}{r_{CA}} - 2 \frac{GMm}{r_{CD}} + 2K,
\]

where \( M \) is the mass of one of the large balls, \( m \) is the mass of one of the small ones, and the factors of two, which will cancel, occur because every energy is mirrored on the opposite side of the apparatus. (As discussed on page 102, it turns out that we get the right result by measuring all the distances from the center of one sphere to the center of the other.) This can easily be solved for \( G \). The best modern value of \( G \), from later versions of the same experiment, is \( 6.67 \times 10^{-11} \text{ J} \cdot \text{m/kg}^2 \).

Escape velocity

The Pioneer 10 space probe was launched in 1972, and continued sending back signals for 30 years. In the year 2001, not long before contact with the probe was lost, it was about \( 1.2 \times 10^{13} \) m from the sun, and was moving almost directly away from the sun at a velocity of \( 1.21 \times 10^4 \) m. The mass of the sun is \( 1.99 \times 10^{30} \) kg. Will Pioneer 10 escape permanently, or will it fall back into the solar system?

We want to know whether there will be a point where the probe will turn around. If so, then it will have zero kinetic energy at the turnaround point:

\[
K_i + U_i = U_f
\]

\[
\frac{1}{2} mv^2 - \frac{GMm}{r_i} = - \frac{GMm}{r_f},
\]

where \( M \) is the mass of the sun, \( m \) is the (irrelevant) mass of the probe, and \( r_i \) is the distance from the sun of the hypothetical turnaround point. Plugging in numbers on the left, we get a positive result. There can therefore be no solution, since the right side is negative. There won't be any turnaround point, and Pioneer 10 is never coming back.
The minimum velocity required for this to happen is called escape velocity. For speeds above escape velocity, the orbits are open-ended hyperbolas, rather than repeating elliptical orbits. In figure i, Pioneer’s hyperbolic trajectory becomes almost indistinguishable from a line at large distances from the sun. The motion slows perceptibly in the first few years after 1974, but later the speed becomes nearly constant, as shown by the nearly constant spacing of the dots.

The gravitational field

We got the energy equation $U = -\frac{Gm_1 m_2}{r}$ by integrating $g \propto \frac{1}{r^2}$ and then inserting a constant of proportionality to make the proportionality into an equation. The opposite of an integral is a derivative, so we can now go backwards and insert a constant of proportionality in $g \propto \frac{1}{r^2}$ that will be consistent with the energy equation:

$$dU = m_2 g_1 dr$$
$$g_1 = \frac{1}{m_2} \frac{dU}{dr}$$
$$= \frac{1}{m_2} \frac{d}{dr} \left( -\frac{Gm_1 m_2}{r} \right)$$
$$= -Gm_1 \frac{d}{dr} \left( \frac{1}{r} \right)$$
$$= \frac{Gm_1}{r^2}$$

This kind of inverse-square law occurs all the time in nature. For instance, if you go twice as far away from a lightbulb, you receive 1/4 as much light from it, because as the light spreads out, it is like an expanding sphere, and a sphere with twice the radius has four times the surface area. It’s like spreading the same amount of peanut butter on four pieces of bread instead of one — we have to spread it thinner.

Discussion Questions

A bowling ball interacts gravitationally with the earth. Would it make sense for the gravitational energy to be inversely proportional to the distance between their surfaces rather than their centers?

2.3.5 The shell theorem

Newton’s great insight was that gravity near the earth’s surface was the same kind of interaction as the one that kept the planets from flying away from the sun. He told his niece that the idea came to him when he saw an apple fall from a tree, which made him wonder whether the earth might be affecting the apple and the moon in the same way. Up until now, we’ve generally been dealing with gravitational interactions between objects that are small compared to the distances between them, but that assumption doesn’t apply to
the apple. A kilogram of dirt a few feet under his garden in England would interact much more strongly with the apple than a kilogram of molten rock deep under Australia, thousands of miles away. Also, we know that the earth has some parts that are more dense, and some parts that are less dense. The solid crust, on which we live, is considerably less dense than the molten rock on which it floats. By all rights, the computation of the total gravitational energy of the apple should be a horrendous mess. Surprisingly, it turns out to be fairly simple in the end. First, we note that although the earth doesn’t have the same density throughout, it does have spherical symmetry: if we imagine dividing it up into thin concentric shells, the density of each shell is uniform.

Second, it turns out that a uniform spherical shell interacts with external masses as if all its mass were concentrated at its center.

The shell theorem: The gravitational energy of a uniform spherical shell of mass $M$ interacting with a pointlike mass $m$ outside it equals $-\frac{GMm}{s}$, where $s$ is the center-to-center distance. If mass $m$ is inside the shell, then the energy is constant, i.e., the shell’s interior gravitational field is zero.

Proof: Let $b$ be the radius of the shell, $h$ its thickness, and $\rho$ its density. Its volume is then $V = (\text{area})(\text{thickness}) = 4\pi b^2 h$, and its mass is $M = \rho V = 4\pi \rho b^2 h$. The strategy is to divide the shell up into rings as shown in figure j, with each ring extending from $\theta$ to $\theta + d\theta$. Since the ring is infinitesimally skinny, its entire mass lies at the same distance, $r$, from mass $m$. The width of such a ring is found by the definition of radian measure to be $w = b d\theta$, and its mass is $dM = (\rho)(\text{circumference})(\text{thickness})(\text{width}) = (\rho)(2\pi b \sin \theta)(h)(b d\theta) = 2\pi \rho b^2 h \sin \theta d\theta$. The gravitational energy of the ring interacting with mass $m$ is therefore 

$$dU = -\frac{Gm dM}{r} = -2\pi G\rho b^2 hm \sin \theta \frac{d\theta}{r}.$$ 

Integrating both sides, we find the total gravitational energy of the shell:

$$U = -2\pi G\rho b^2 hm \int_0^\pi \frac{\sin \theta \ d\theta}{r}.$$ 

The integral has a mixture of the variables $r$ and $\theta$, which are related by the law of cosines,

$$r^2 = b^2 + s^2 - 2bs \cos \theta,$$

and to evaluate the integral, we need to get everything in terms of either $r$ and $dr$ or $\theta$ and $d\theta$. The relationship between the differentials is found by differentiating the law of cosines,

$$2r \ dr = 2bs \sin \theta \ d\theta.$$
The gravitational energy of a mass \( m \) at a distance \( s \) from the center of a hollow spherical shell of mass.

Since \( \sin \theta \, d\theta \) occurs in the integral, the easiest path is to substitute for it, and get everything in terms of \( r \) and \( dr \):

\[
U = -\frac{2\pi G\rho bhm}{s} \int_{s-b}^{s+b} dr
= -\frac{4\pi G\rho b^2 hm}{s}
= -\frac{GMm}{s}
\]

This was all under the assumption that mass \( m \) was on the outside of the shell. To complete the proof, we consider the case where it’s inside. In this case, the only change is that the limits of integration are different:

\[
U = -\frac{2\pi G\rho bhm}{s} \int_{b-s}^{b+s} dr
= -\frac{4\pi G\rho bhm}{b}
= -\frac{GMm}{b}
\]

The two results are equal at the surface of the sphere, \( s = b \), so the constant-energy part joins continuously onto the \( 1/s \) part, and the effect is to chop off the steepest part of the graph that we would have had if the whole mass \( M \) had been concentrated at its center. Dropping a mass \( m \) from A to B in figure k releases the same amount of energy as if mass \( M \) had been concentrated at its center, but there is no release of gravitational energy at all when moving between two interior points like C and D. In other words, the internal gravitational field is zero. Moving from C to D brings mass \( m \) farther away from the nearby side of the shell, but closer to the far side, and the cancellation between these two effects turns out to be perfect. Although the gravitational field has to be zero at the center due to symmetry, it’s much more surprising that it cancels out perfectly in the whole interior region; this is a special mathematical characteristic of a \( 1/r \) interaction like gravity.

\[\text{Newton’s apple}\]

Over a period of 27.3 days, the moon travels the circumference of its orbit, so using data from Appendix 5, we can calculate its speed, and solve the circular orbit condition to determine the strength of the earth’s gravitational field at the moon’s distance from the earth, \( g = \frac{v^2}{r} = 2.72 \times 10^{-3} \text{ m/s}^2 \), which is 3600 times smaller than the gravitational field at the earth’s surface. The center-to-center distance from the moon to the earth is 60 times greater than the radius of the earth. The earth is, to a very good approximation, a sphere made up of concentric shells, each with uniform density, so the shell theorem tells us that its external gravitational field is the same as if all its mass was concentrated...
at its center. We already know that a gravitational energy that varies as $-1/r$ is equivalent to a gravitational field proportional to $1/r^2$, so it makes sense that a distance that is greater by a factor of 60 corresponds to a gravitational field that is $60 \times 60 = 3600$ times weaker. Note that the calculation didn't require knowledge of the earth's mass or the gravitational constant, which Newton didn't know.

In 1665, shortly after Newton graduated from Cambridge, the Great Plague forced the college to close for two years, and Newton returned to the family farm and worked intensely on scientific problems. During this productive period, he carried out this calculation, but it came out wrong, causing him to doubt his new theory of gravity. The problem was that during the plague years, he was unable to use the university's library, so he had to use a figure for the radius of the moon's orbit that he had memorized, and he forgot that the memorized value was in units of nautical miles rather than statute miles. Once he realized his mistake, he found that the calculation came out just right, and became confident that his theory was right after all.  

\[ \text{Weighing the earth example 18} \]

Once Cavendish had found $G = 6.67 \times 10^{-11} \text{J} \cdot \text{m/kg}^2$ (p. 101, example 15), it became possible to determine the mass of the earth. By the shell theorem, the gravitational energy of a mass $m$ at a distance $r$ from the center of the earth is $U = -G M m/r$, where $M$ is the mass of the earth. The gravitational field is related to this by $mg \, dr = dU$, or $g = (1/m) dU/\, dr = GM/r^2$. Solving for $M$, we have

\[
M = \frac{gr^2}{G} = \frac{(9.8 \text{ m/s}^2)(6.4 \times 10^6 \text{ m})^2}{6.67 \times 10^{-11} \text{ J} \cdot \text{m/kg}^2} = 6.0 \times 10^{24} \frac{\text{m}^2 \cdot \text{kg}}{\text{J} \cdot \text{s}^2} = 6.0 \times 10^{24} \text{ kg}
\]

\[ \text{Gravity inside the earth example 19} \]

The earth is somewhat more dense at greater depths, but as an approximation let's assume it has a constant density throughout. How does its internal gravitational field vary with the distance $r$ from the center?

Let's write $b$ for the radius of the earth. The shell theorem tell us that at a given location $r$, we only need to consider the mass $M_{\leq r}$.

\[ ^9\text{Some historians are suspicious that the story of the apple and the mistake in conversions may have been fabricated by Newton later in life. The conversion incident may have been a way of explaining his long delay in publishing his work, which led to a conflict with Leibniz over priority in the invention of calculus.} \]
that is deeper than \( r \). Under the assumption of constant density, this mass is related to the total mass of the earth by

\[
\frac{M_{<r}}{M} = \frac{r^3}{b^3},
\]

and by the same reasoning as in example 18,

\[
g = \frac{G M_{<r}}{r^2},
\]

so

\[
g = \frac{G M r}{b^3}.
\]

In other words, the gravitational field interpolates linearly between zero at \( r = 0 \) and its ordinary surface value at \( r = b \).

The following example applies the numerical techniques of section 2.2.

From the earth to the moon example 20

The Apollo 11 mission landed the first humans on the moon in 1969. In this example, we’ll estimate the time it took to get to the moon, and compare our estimate with the actual time, which was 73.0708 hours from the engine burn that took the ship out of earth orbit to the engine burn that inserted it into lunar orbit. During this time, the ship was coasting with the engines off, except for a small course-correction burn, which we neglect. More importantly, we do the calculation for a straight-line trajectory rather than the real S-shaped one, so the result can only be expected to agree roughly with what really happened. The following data come from the original press kit, which NASA has scanned and posted on the Web:

- initial altitude: \( 3.363 \times 10^5 \) m
- initial velocity: \( 1.083 \times 10^4 \) m/s

The endpoint of the the straight-line trajectory is a free-fall impact on the lunar surface, which is also unrealistic (luckily for the astronauts).

The ship’s energy is

\[
E = -\frac{G M_e m}{r} - \frac{G M_m m}{r_m - r} + \frac{1}{2} m v^2,
\]

but since everything is proportional to the mass of the ship, \( m \), we can divide it out

\[
\frac{E}{m} = -\frac{G M_e}{r} - \frac{G M_m}{r_m - r} + \frac{1}{2} v^2,
\]

and the energy variables in the program with names like \( e \), \( k \), and \( u \) are actually energies per unit mass. The program is a straightforward modification of the function \texttt{time3} on page 93.
import math
def tmoon(vi,ri,rf,n):
    bigg=6.67e-11  # gravitational constant
    me=5.97e24     # mass of earth
    mm=7.35e22     # mass of moon
    rm=3.84e8      # earth-moon distance
    r=ri
    v=vi
    dr = (rf-ri)/n
    e=-bigg*me/ri-bigg*mm/(rm-ri)+.5*vi**2
    t=0
    for i in range(n):
        u_old = -bigg*me/r-bigg*mm/(rm-r)
        k_old = e - u_old
        v_old = math.sqrt(2.*k_old)
        r = r+dr
        u = -bigg*me/r-bigg*mm/(rm-r)
        k = e - u
        v = math.sqrt(2.*k)
        v_avg = .5*(v_old+v)
        dt=dr/v_avg
        t=t+dt
    return t

>>> re=6.378e6  # radius of earth
>>> rm=1.74e6    # radius of moon
>>> ri=re+3.363e5  # re+initial altitude
>>> rf=3.8e8-rm # earth-moon distance minus rm
>>> vi=1.083e4  # initial velocity
>>> print(tmoon(vi,ri,rf,1000)/3600.) # convert seconds to hours
59.6540744197656

This is pretty decent agreement, considering the wildly inaccurate trajectory assumed. It’s interesting to see how much the duration of the trip changes if we increase the initial velocity by only ten percent:

>>> vi=1.2e4
>>> print(tmoon(vi,ri,rf,1000)/3600.)
18.1775263611167

The most important reason for using the lower speed was that if something had gone wrong, the ship would have been able to whip around the moon and take a “free return” trajectory back to the earth, without having to do any further burns. At a higher speed, the ship would have had so much kinetic energy that in the absence of any further engine burns, it would have escaped from the earth-moon system. The Apollo 13 mission had to take a free return trajectory after an explosion crippled the spacecraft.
2.3.6 Evidence for repulsive gravity

Until recently, physicists thought they understood gravity fairly well. Einstein had modified Newton’s theory, but certain characteristics of gravitational forces were firmly established. For one thing, they were always attractive. If gravity always attracts, then it is logical to ask why the universe doesn’t collapse. Newton had answered this question by saying that if the universe was infinite in all directions, then it would have no geometric center toward which it would collapse; the forces on any particular star or planet exerted by distant parts of the universe would tend to cancel out by symmetry. More careful calculations, however, show that Newton’s universe would have a tendency to collapse on smaller scales: any part of the universe that happened to be slightly more dense than average would contract further, and this contraction would result in stronger gravitational forces, which would cause even more rapid contraction, and so on.

When Einstein overhauled gravity, the same problem reared its ugly head. Like Newton, Einstein was predisposed to believe in a universe that was static, so he added a special repulsive term to his equations, intended to prevent a collapse. This term was not associated with any interaction of mass with mass, but represented merely an overall tendency for space itself to expand unless restrained by the matter that inhabited it. It turns out that Einstein’s solution, like Newton’s, is unstable. Furthermore, it was soon discovered observationally that the universe was expanding, and this was interpreted by creating the Big Bang model, in which the universe’s current expansion is the aftermath of a fantastically hot explosion. An expanding universe, unlike a static one, was capable of being explained with Einstein’s equations, without any repulsion term. The universe’s expansion would simply slow down over time due to the attractive gravitational forces. After these developments, Einstein said woefully that adding the repulsive term, known as the cosmological constant, had been the greatest blunder of his life.

m The WMAP probe’s map of the cosmic microwave background is like a “baby picture” of the universe.

10 Subsection 6.1.5 presents some evidence for the Big Bang theory.
This was the state of things until 1999, when evidence began to turn up that the universe’s expansion has been speeding up rather than slowing down! The first evidence came from using a telescope as a sort of time machine: light from a distant galaxy may have taken billions of years to reach us, so we are seeing it as it was far in the past. Looking back in time, astronomers saw the universe expanding at speeds that were lower, rather than higher. At first they were mortified, since this was exactly the opposite of what had been expected. The statistical quality of the data was also not good enough to constitute ironclad proof, and there were worries about systematic errors. The case for an accelerating expansion has however been supported by high-precision mapping of the dim, sky-wide afterglow of the Big Bang, known as the cosmic microwave background.

So now Einstein’s “greatest blunder” has been resurrected. Since we don’t actually know whether or not this self-repulsion of space has a constant strength, the term “cosmological constant” has lost currency. Nowadays physicists usually refer to the phenomenon as “dark energy.” Picking an impressive-sounding name for it should not obscure the fact that we know absolutely nothing about the nature of the effect or why it exists.

2.4 Atomic Phenomena

Variety is the spice of life, not of science. So far this chapter has focused on heat energy, kinetic energy, and gravitational energy, but it might seem that in addition to these there is a bewildering array of other forms of energy. Gasoline, chocolate bars, batteries, melting water — in each case there seems to be a whole new type of energy. The physicist’s psyche rebels against the prospect of a long laundry list of types of energy, each of which would require its own equations, concepts, notation, and terminology. The point at which we’ve arrived in the study of energy is analogous to the period in the 1960’s when a half a dozen new subatomic particles were being discovered every year in particle accelerators. It was an embarrassment. Physicists began to speak of the “particle zoo,” and it seemed that the subatomic world was distressingly complex. The particle zoo was simplified by the realization that most of the new particles being whipped up were simply clusters of a previously unsuspected set of fundamental particles (which were whimsically dubbed quarks, a made-up word from a line of poetry by James Joyce, “Three quarks for Master Mark.”) The energy zoo can also be simplified, and it’s the purpose of this section to demonstrate the hidden similarities between forms of energy as seemingly different as heat and motion.
2.4.1 **Heat is kinetic energy.**

What is heat really? Is it an invisible fluid that your bare feet soak up from a hot sidewalk? Can one ever remove all the heat from an object? Is there a maximum to the temperature scale?

The theory of heat as a fluid seemed to explain why colder objects absorbed heat from hotter ones, but once it became clear that heat was a form of energy, it began to seem unlikely that a material substance could transform itself into and out of all those other forms of energy like motion or light. For instance, a compost pile gets hot, and we describe this as a case where, through the action of bacteria, chemical energy stored in the plant cuttings is transformed into heat energy. The heating occurs even if there is no nearby warmer object that could have been leaking “heat fluid” into the pile.

An alternative interpretation of heat was suggested by the theory that matter is made of atoms. Since gases are thousands of times less dense than solids or liquids, the atoms (or clusters of atoms called molecules) in a gas must be far apart. In that case, what is keeping all the air molecules from settling into a thin film on the floor of the room in which you are reading this book? The simplest explanation is that they are moving very rapidly, continually ricocheting off of the floor, walls, and ceiling. Though bizarre, the cloud-of-bullets image of a gas did give a natural explanation for the surprising ability of something as tenuous as a gas to exert huge forces.

The experiment shown in figure a, for instance, can be explained as follows. The high temperature of the steam is interpreted as a high average speed of random motions of its molecules. Before the lid was put on the can, the rapidly moving steam molecules pushed their way out of the can, forcing the slower air molecules out of the way. As the steam inside the can thinned out, a stable situation was soon achieved, in which the force from the less dense steam molecules moving at high speed balanced against the force from the more dense but slower air molecules outside. The cap was put on, and after a while the steam inside the can began to cool off. The force from the cooler, thin steam no longer matched the force from the cool, dense air outside, and the imbalance of forces crushed the can.

This type of observation leads naturally to the conclusion that hotter matter differs from colder in that its atoms’ random motion is more rapid. In a liquid, the motion could be visualized as people in a milling crowd shoving past each other more quickly. In a solid, where the atoms are packed together, the motion is a random vibration of each atom as it knocks against its neighbors.

We thus achieve a great simplification in the theory of heat. Heat is simply a form of kinetic energy, the total kinetic energy of random motion of all the atoms in an object. With this new understanding,
it becomes possible to answer at one stroke the questions posed at the beginning of the section. Yes, it is at least theoretically possible to remove all the heat from an object. The coldest possible temperature, known as absolute zero, is that at which all the atoms have zero velocity, so that their kinetic energies, \( K = (1/2)mv^2 \), are all zero. No, there is no maximum amount of heat that a certain quantity of matter can have, and no maximum to the temperature scale, since arbitrarily large values of \( v \) can create arbitrarily large amounts of kinetic energy per atom.

The kinetic theory of heat also provides a simple explanation of the true nature of temperature. Temperature is a measure of the amount of energy per molecule, whereas heat is the total amount of energy possessed by all the molecules in an object.

There is an entire branch of physics, called thermodynamics, that deals with heat and temperature and forms the basis for technologies such as refrigeration. Thermodynamics is discussed in more detail in chapter 5, and I’ve provided here only a brief overview of the thermodynamic concepts that relate directly to energy.

2.4.2 All energy comes from particles moving or interacting.

If I stretch the spring in figure c and then release it, it snaps taut again. The creation of some kinetic energy shows that there must have been some other form of energy that was destroyed. What was it?

We could just invent a new type of energy called “spring energy,” study its behavior, and call it quits, but that would be ugly. Are we going to have to invent a new forms of energy like this, over and over? No: the title of this book doesn’t lie, and physics really is fundamentally simple. As shown in figure d, when we bend or stretch an object, we’re really changing the distances between the atoms, resulting in a change in electrical energy. Electrical energy isn’t really our topic right now — that’s what most of the second half of this book is about — but conceptually it’s very similar to gravitational energy. Like gravitational energy, it depends on \( 1/r \), although there are some interesting new phenomena, such as the existence of both attraction and repulsion, which doesn’t occur with gravity because gravitational mass can’t be negative. The real point is that all the apparently dissimilar forms of energy in figure d turn out to be due to electrical interactions among atoms. Even if we wish to include nuclear reactions (figure e) in the picture, there still turn out to be only four fundamental types of energy:

- kinetic energy (including heat)
- gravitational energy
- electrical and magnetic energy
- nuclear energy

Astute students have often asked me how light fits into this picture.
This figure looks similar to the previous ones, but the scale is a million times smaller. The little balls are the neutrons and protons that make up the tiny nucleus at the center of a uranium atom. When the nucleus splits (fissions), the source of the kinetic energy is partly electrical and partly nuclear.

1. **Temperature during boiling**

   If you stick a thermometer in a pan of water, and watch the temperature as you bring the water to a boil, you'll notice an interesting fact. The temperature goes up until the boiling point is reached, but then stays at 100°C during the whole time the water is being boiled off. The temperature of the steam is also 100°C. Why does the temperature "stick" like this? What's happening to all the energy that the stove's burner is putting into the pan?

   As shown in figure d, boiling requires an increase in electrical energy, because the atoms coming out as gas are moving away from the other atoms, which attract them electrically. It is only this electrical energy that is increasing, not the atoms' kinetic energy, which is what the thermometer can measure.

2. **Diffusion**

   A drop of food coloring in a cup of water will gradually spread out, even if you don't do any mixing with a spoon. This is called diffusion. Why would this happen, and what effect would temperature have? What would happen with solids or gases?

   Figure b shows that the atoms in a liquid mingle because of their random thermal motion. Diffusion is slow (typically on the order of a centimeter a minute), despite the high speeds of the atoms (typically hundreds of miles per hour). This is due to the randomness of the motion: a particular atom will take a long time to travel any significant distance, because it doesn't travel in a straight line.

   Based on this picture, we expect that the speed of diffusion should increase as a function of temperature, and experiments show that this is true.

   Diffusion also occurs in gases, which is why you can smell things even when the air is still. The speeds are much faster, because the typical distance between collisions is much longer than in a liquid.

   We can see from figure b that diffusion won't occur in solids, because each atom vibrates around an equilibrium position.
Discussion Questions

A I’m not making this up. XS Energy Drink has ads that read like this: All the “Energy” ... Without the Sugar! Only 8 Calories!” Comment on this.

2.5 Oscillations

Let’s revisit the example of the stretched spring from the previous section. We know that its energy is a form of electrical energy of interacting atoms, which is nice conceptually but doesn’t help us to solve problems, since we don’t know how the energy, $U$, depends on the length of the spring. All we know is that there’s an equilibrium (figure a/1), which is a local minimum of the function $U$. An extremely important problem which arises in this connection is how to calculate oscillatory motion around an equilibrium, as in a/4-13. Even if we did special experiments to find out how the spring’s energy worked, it might seem like we’d have to go through just as much work to deal with any other kind of oscillation, such as a sapling swinging back and forth in the breeze.

Surprisingly, it’s possible to analyze this type of oscillation in a very general and elegant manner, as long as the analysis is limited to small oscillations. We’ll talk about the mass on the spring for concreteness, but there will be nothing in the discussion at all that is restricted to that particular physical system. First, let’s choose a coordinate system in which $x = 0$ corresponds to the position of the mass where the spring is in equilibrium, and since interaction energies like $U$ are only well defined up to an additive constant, we’ll simply define it to be zero at equilibrium:

$$U(0) = 0$$

Since $x = 0$ is an equilibrium, $U(x)$ must have a local minimum there, and a differentiable function (which we assume $U$ is) has a zero derivative at a local minimum:

$$rac{dU}{dx}(0) = 0$$

There are still infinitely many functions that could satisfy these criteria, including the three shown in figure b, which are $x^2/2$, $x^2/(1+x^2)$, and $(e^{3x} + e^{-3x} - 2)/18$. Note, however, how all three functions are virtually identical right near the minimum. That’s because they all have the same curvature. More specifically, each function has its second derivative equal to 1 at $x = 0$, and the second derivative is a measure of curvature. We write $k$ for the second derivative of the energy at an equilibrium point,

$$rac{d^2U}{dx^2}(0) = k$$
Physically, $k$ is a measure of stiffness. For example, the heavy-duty springs in a car’s shock absorbers would have a high value of $k$. It is often referred to as the spring constant, but we’re only using a spring as an example here. As shown in figure b, any two functions that have $U(0) = 0$, $dU/\,dx = 0$, and $d^2U/\,dx^2 = k$, with the same value of $k$, are virtually indistinguishable for small values of $x$, so if we want to analyze small oscillations, it doesn’t even matter which function we assume. For simplicity, we’ll just use $U(x) = (1/2)kx^2$ from now on.

Now we’re ready to analyze the mass-on-a-spring system, while keeping in mind that it’s really only a representative example of a whole class of similar oscillating systems. We expect that the motion is going to repeat itself over and over again, and since we’re not going to include frictional heating in our model, that repetition should go on forever without dying out. The most interesting thing to know about the motion would be the period, $T$, which is the amount of time required for one complete cycle of the motion. We might expect that the period would depend on the spring constant, $k$, the mass, $m$, and the amplitude, $A$, defined in figure c.\footnote{Many kinds of oscillations are possible, so there is no standard definition of the amplitude. For a pendulum, the natural definition would be in terms of an angle. For a radio transmitter, we’d use some kind of electrical units.}

In examples like the brachistochrone and the Apollo 11 mission, it was generally necessary to use numerical techniques to determine the amount of time required for a certain motion. Once again, let’s dust off the time3 function from page 93 and modify it for our purposes. For flexibility, we’ll define the function $U(x)$ as a separate Python function. We really want to calculate the time required for the mass to come back to its starting point, but that would be awkward to set up, since our function works by dividing up the distance to be traveled into tiny segments. By symmetry, the time required to go from one end to the other equals the time required to come back to the start, so we’ll just calculate the time for half a cycle and then double it when we return the result at the end of the function. The test at lines 16-19 is necessary because otherwise at the very end of the motion we can end up trying to take the square root of a negative number due to rounding errors.
import math
def u(k,x):
    return 0.5*k*x**2
def osc(m,k,a,n):
    x=a
    v=0
dx = -2.*a/n
t=0
e = u(k,x)+0.5*m*v**2
for i in range(n):
    x_old = x
    v_old = v
    x = x+dx
    kinetic = e-u(k,x)
    if kinetic<0. :
        v=0.
    else :
        v = -math.sqrt(2.*kinetic/m)
        v_avg = (v+v_old)/2.
        dt=dx/v_avg
        t=t+dt
    return 2.*t

>>> print(osc(1.,1.,1.,100000))
warning, K=-1.43707268307e-12 <0
6.2831854132667919

The first thing to notice is that with this particular set of inputs (m=1 kg, k = 1 J/m$^2$, and A = 1 m), the program has done an excellent job of computing $2\pi = 6.2831853...$. This is Mother Nature giving us a strong hint that the problem has an algebraic solution, not just a numerical one. The next interesting thing happens when we change the amplitude from 1 m to 2 m:

>>> print(osc(1.,1.,2.,100000))
warning, K=-5.7482907323e-12 <0
6.2831854132667919

Even though the mass had to travel double the distance in each direction, the period is the same to within the numerical accuracy of the calculation!

With these hints, it seems like we should start looking for an algebraic solution. For guidance, here’s a graph of $x$ as a function of $t$, as calculated by the osc function with n=10.
Example 23. The rod pivots on the hinge at the bottom. This looks like a cosine function, so let’s see if \( x = A \cos(\omega t + \delta) \) is a solution to the conservation of energy equation — it’s not uncommon to try to “reverse-engineer” the cryptic results of a numerical calculation like this. The symbol \( \omega = 2\pi/T \) (Greek omega), called angular frequency, is a standard symbol for the number of radians per second of oscillation. Except for the factor of \( 2\pi \), it is identical to the ordinary frequency \( f = 1/T \), which has units of \( \text{s}^{-1} \) or Hz (Hertz). The phase angle \( \delta \) is to allow for the possibility that \( t = 0 \) doesn’t coincide with the beginning of the motion. The energy is

\[
E = K + U
= \frac{1}{2}mv^2 + \frac{1}{2}kx^2
= \frac{1}{2}m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2}kx^2
= \frac{1}{2}m \left[ -A\omega \sin(\omega t + \delta) \right]^2 + \frac{1}{2}k [A \cos(\omega t + \delta)]^2
= \frac{1}{2}A^2 \left[ m\omega^2 \sin^2(\omega t + \delta) + k \cos^2(\omega t + \delta) \right]
\]

According to conservation of energy, this has to be a constant. Using the identity \( \sin^2 + \cos^2 = 1 \), we can see that it will be a constant if we have \( m\omega^2 = k \), or \( \omega = \sqrt{k/m} \), i.e., \( T = 2\pi\sqrt{m/k} \). Note that the period is independent of amplitude.

---

**A spring and a lever example 23**

- What is the period of small oscillations of the system shown in the figure? Neglect the mass of the lever and the spring. Assume that the spring is so stiff that gravity is not an important effect. The spring is relaxed when the lever is vertical.

- This is a little tricky, because the spring constant \( k \), although it is relevant, is **not** the \( k \) we should be putting into the equation \( T = 2\pi\sqrt{m/k} \). The \( k \) that goes in there has to be the second derivative of \( U \) with respect to the position, \( x \), of the mass that’s moving. The energy \( U \) stored in the spring depends on how far the tip of the lever is from the center. This distance equals \((L/b)x\),
so the energy in the spring is

\[ U = \frac{1}{2} k \left( \frac{L}{b} x \right)^2 \]

\[ = \frac{k L^2}{2 b^2} x^2, \]

and the \( k \) we have to put in \( T = 2\pi \sqrt{m/k} \) is

\[ \frac{d^2 U}{dx^2} = \frac{k L^2}{b^2}. \]

The result is

\[ T = 2\pi \sqrt{\frac{m b^2}{k L^2}} \]

\[ = \frac{2\pi b}{L} \sqrt{\frac{m}{k}}. \]

The leverage of the lever makes it as if the spring was stronger, and decreases the period of the oscillations by a factor of \( b/L \).

Water in a U-shaped tube

\( \triangle \) What is the period of oscillation of the water in figure e?

\( \triangle \) In example 13 on p. 89, we found \( U(y) = \rho g A y^2 \), so the “spring constant,” which really isn’t a spring constant here at all, is

\[ k = \frac{d^2 U}{dy^2} \]

\[ = 2\rho g A. \]

This is an interesting example, because \( k \) can be calculated without any approximations, but the kinetic energy requires an approximation, because we don’t know the details of the pattern of flow of the water. It could be very complicated. There will be a tendency for the water near the walls to flow more slowly due to friction, and there may also be swirling, turbulent motion. However, if we make the approximation that all the water moves with the same velocity as the surface, \( dy/dt \), then the mass-on-a-spring analysis applies. Letting \( L \) be the total length of the filled part of the tube, the mass is \( \rho LA \), and we have

\[ T = 2\pi \sqrt{m/k} \]

\[ = 2\pi \sqrt{\frac{\rho LA}{2\rho g A}} \]

\[ = 2\pi \sqrt{\frac{L}{2g}}. \]

This chapter is summarized on page 951. Notation and terminology are tabulated on pages 945-946.
Problems

The symbols √, ■, etc. are explained on page 125.

1  Experiments show that the power consumed by a boat’s engine is approximately proportional to the third power of its speed. (We assume that it is moving at constant speed.)

(a) When a boat is cruising at constant speed, what type of energy transformation do you think is being performed?

(b) If you upgrade to a motor with double the power, by what factor is your boat’s maximum cruising speed increased?

▷ Solution, p. 936

2  Object A has a kinetic energy of 13.4 J. Object B has a mass that is greater by a factor of 3.77, but is moving more slowly by a factor of 2.34. What is object B’s kinetic energy?

▷ Solution, p. 936

3  My 1.25 kW microwave oven takes 126 seconds to bring 250 g of water from room temperature to a boil. What percentage of the power is being wasted? Where might the rest of the energy be going?

▷ Solution, p. 936

4  The multiflash photograph shows a collision between two pool balls. The ball that was initially at rest shows up as a dark image in its initial position, because its image was exposed several times before it was struck and began moving. By making measurements on the figure, determine numerically whether or not energy appears to have been conserved in the collision. What systematic effects would limit the accuracy of your test? [From an example in PSSC Physics.]
5 A grasshopper with a mass of 110 mg falls from rest from a height of 310 cm. On the way down, it dissipates 1.1 mJ of heat due to air resistance. At what speed, in m/s, does it hit the ground?

Solution, p. 937

6 A ball rolls up a ramp, turns around, and comes back down. When does it have the greatest gravitational energy? The greatest kinetic energy? [Based on a problem by Serway and Faughn.]

7 (a) You release a magnet on a tabletop near a big piece of iron, and the magnet leaps across the table to the iron. Does the magnetic energy increase, or decrease? Explain. (b) Suppose instead that you have two repelling magnets. You give them an initial push towards each other, so they decelerate while approaching each other. Does the magnetic energy increase, or decrease? Explain.

8 Estimate the kinetic energy of an Olympic sprinter.

9 You are driving your car, and you hit a brick wall head on, at full speed. The car has a mass of 1500 kg. The kinetic energy released is a measure of how much destruction will be done to the car and to your body. Calculate the energy released if you are traveling at (a) 40 mi/hr, and again (b) if you're going 80 mi/hr. What is counterintuitive about this, and what implication does this have for driving at high speeds?

10 A closed system can be a bad thing — for an astronaut sealed inside a space suit, getting rid of body heat can be difficult. Suppose a 60-kg astronaut is performing vigorous physical activity, expending 200 W of power. If none of the heat can escape from her space suit, how long will it take before her body temperature rises by 6°C (11°F), an amount sufficient to kill her? Assume that the amount of heat required to raise her body temperature by 1°C is the same as it would be for an equal mass of water. Express your answer in units of minutes.

11 The following table gives the amount of energy required in order to heat, melt, or boil a gram of water.

<table>
<thead>
<tr>
<th>Heat/Melt/Boil</th>
<th>Energy (J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 g of ice by 1°C</td>
<td>2.05 J</td>
</tr>
<tr>
<td>1 g of ice</td>
<td>333 J</td>
</tr>
<tr>
<td>1 g of liquid by 1°C</td>
<td>4.19 J</td>
</tr>
<tr>
<td>1 g of water</td>
<td>2500 J</td>
</tr>
<tr>
<td>1 g of steam by 1°C</td>
<td>2.01 J</td>
</tr>
</tbody>
</table>

(a) How much energy is required in order to convert 1.00 g of ice at -20 °C into steam at 137 °C?

(b) What is the minimum amount of hot water that could melt 1.00 g of ice?

12 Anya climbs to the top of a tree, while Ivan climbs half-way to the top. They both drop pennies to the ground. Compare the kinetic energies and velocities of the pennies on impact, using ratios.
13 Anya and Ivan lean over a balcony side by side. Anya throws a penny downward with an initial speed of 5 m/s. Ivan throws a penny upward with the same speed. Both pennies end up on the ground below. Compare their kinetic energies and velocities on impact.

14 (a) A circular hoop of mass $m$ and radius $r$ spins like a wheel while its center remains at rest. Let $\omega$ (Greek letter omega) be the number of radians it covers per unit time, i.e., $\omega = 2\pi/T$, where the period, $T$, is the time for one revolution. Show that its kinetic energy equals $(1/2)m\omega^2r^2$.

(b) Show that the answer to part a has the right units. (Note that radians aren’t really units, since the definition of a radian is a unitless ratio of two lengths.)

(c) If such a hoop rolls with its center moving at velocity $v$, its kinetic energy equals $(1/2)mv^2$, plus the amount of kinetic energy found in part a. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have.

15 On page 83, I used the chain rule to prove that the acceleration of a free-falling object is given by $a = -g$. In this problem, you’ll use a different technique to prove the same thing. Assume that the acceleration is a constant, $a$, and then integrate to find $v$ and $y$, including appropriate constants of integration. Plug your expressions for $v$ and $y$ into the equation for the total energy, and show that $a = -g$ is the only value that results in constant energy.

16 The figure shows two unequal masses, $m_1$ and $m_2$, connected by a string running over a pulley. Find the acceleration.

17 What ratio of masses will balance the pulley system shown in the figure?

18 (a) For the apparatus shown in the figure, find the equilibrium angle $\theta$ in terms of the two masses.

(b) Interpret your result in the case of $M \gg m$ ($M$ much greater than $m$). Does it make sense physically?

(c) For what combinations of masses would your result give nonsense? Interpret this physically.

19 In the system shown in the figure, the pulleys on the left and right are fixed, but the pulley in the center can move to the left or right. The two hanging masses are identical, and the pulleys and ropes are all massless. Find the upward acceleration of the mass on the left, in terms of $g$ only.

20 Two atoms will interact via electrical forces between their protons and electrons. One fairly good approximation to the elec-
trical energy is the Lennard-Jones formula,

\[ U(r) = k \left[ \left( \frac{a}{r} \right)^{12} - 2 \left( \frac{a}{r} \right)^6 \right], \]

where \( r \) is the center-to-center distance between the atoms and \( k \) is a positive constant. Show that (a) there is an equilibrium point at \( r = a \), (b) the equilibrium is stable, and (c) the energy required to bring the atoms from their equilibrium separation to infinity is \( k \).

\[ \triangle \text{Hint, p. 922} \]

21 The International Space Station orbits at an altitude of about 360 to 400 km. What is the gravitational field of the earth at this altitude?

22 (a) A geosynchronous orbit is one in which the satellite orbits above the equator, and has an orbital period of 24 hours, so that it is always above the same point on the spinning earth. Calculate the altitude of such a satellite. 

(b) What is the gravitational field experienced by the satellite? Give your answer as a percentage in relation to the gravitational field at the earth’s surface.

\[ \triangle \text{Hint, p. 922} \]

23 Astronomers calculating orbits of planets often work in a nonmetric system of units, in which the unit of time is the year, the unit of mass is the sun’s mass, and the unit of distance is the astronomical unit (A.U.), defined as half the long axis of the earth’s orbit. In these units, find an exact expression for the gravitational constant, \( G \).

\[ \triangle \text{Hint, p. 922} \]

24 The star Lalande 21185 was found in 1996 to have two planets in roughly circular orbits, with periods of 6 and 30 years. What is the ratio of the two planets’ orbital radii?

25 A projectile is moving directly away from a planet of mass \( M \) at exactly escape velocity. (a) Find \( r \), the distance from the projectile to the center of the planet, as a function of time, \( t \), and also find \( v(t) \).

(b) Check the units of your answer.

(c) Does \( v \) show the correct behavior as \( t \) approaches infinity?

\[ \triangle \text{Hint, p. 922} \]

26 The purpose of this problem is to estimate the height of the tides. The main reason for the tides is the moon’s gravity, and we’ll neglect the effect of the sun. Also, real tides are heavily influenced by landforms that channel the flow of water, but we’ll think of the earth as if it was completely covered with oceans. Under these assumptions, the ocean surface should be a surface of constant \( U/m \). That is, a thimbleful of water, \( m \), should not be able to gain or lose any gravitational energy by moving from one point on the ocean surface to another. If only the spherical earth’s gravity was present,
then we’d have \( U/m = -GMe/r \), and a surface of constant \( U/m \) would be a surface of constant \( r \), i.e., the ocean’s surface would be spherical. Taking into account the moon’s gravity, the main effect is to shift the center of the sphere, but the sphere also becomes slightly distorted into an approximately ellipsoidal shape. (The shift of the center is not physically related to the tides, since the solid part of the earth tends to be centered within the oceans; really, this effect has to do with the motion of the whole earth through space, and the way that it wobbles due to the moon’s gravity.) Determine the amount by which the long axis of the ellipsoid exceeds the short axis.

\( \triangle \) Hint, p. 922

27 You are considering going on a space voyage to Mars, in which your route would be half an ellipse, tangent to the Earth’s orbit at one end and tangent to Mars’ orbit at the other. Your spacecraft’s engines will only be used at the beginning and end, not during the voyage. How long would the outward leg of your trip last? (The orbits of Earth and Mars are nearly circular, and Mars’s is bigger by a factor of 1.52.)

\( \sqrt{ } \) 28 When you buy a helium-filled balloon, the seller has to inflate it from a large metal cylinder of the compressed gas. The helium inside the cylinder has energy, as can be demonstrated for example by releasing a little of it into the air: you hear a hissing sound, and that sound energy must have come from somewhere. The total amount of energy in the cylinder is very large, and if the valve is inadvertently damaged or broken off, the cylinder can behave like a bomb or a rocket.

Suppose the company that puts the gas in the cylinders prepares cylinder A with half the normal amount of pure helium, and cylinder B with the normal amount. Cylinder B has twice as much energy, and yet the temperatures of both cylinders are the same. Explain, at the atomic level, what form of energy is involved, and why cylinder B has twice as much.

29 Explain in terms of conservation of energy why sweating cools your body, even though the sweat is at the same temperature as your body. Describe the forms of energy involved in this energy transformation. Why don’t you get the same cooling effect if you wipe the sweat off with a towel? Hint: The sweat is evaporating.

30 [This problem is now problem 3-73.]

31 All stars, including our sun, show variations in their light output to some degree. Some stars vary their brightness by a factor of two or even more, but our sun has remained relatively steady during the hundred years or so that accurate data have been collected. Nevertheless, it is possible that climate variations such as ice ages are related to long-term irregularities in the sun’s light output. If
the sun was to increase its light output even slightly, it could melt
even slightly, it could melt
enough Antarctic ice to flood all the world’s coastal cities. The total
sunlight that falls on Antarctica amounts to about $1 \times 10^{16}$ watts.
In the absence of natural or human-caused climate change, this heat
input to the poles is balanced by the loss of heat via winds, ocean
currents, and emission of infrared light, so that there is no net melting
or freezing of ice at the poles from year to year. Suppose that
the sun changes its light output by some small percentage, but there
is no change in the rate of heat loss by the polar caps. Estimate the
percentage by which the sun’s light output would have to increase
in order to melt enough ice to raise the level of the oceans by 10 me-
ters over a period of 10 years. (This would be enough to flood New
York, London, and many other cities.) Melting 1 kg of ice requires
$3 \times 10^3$ J.

32 The figure shows the oscillation of a microphone in response
to the author whistling the musical note “A.” The horizontal axis,
representing time, has a scale of 1.0 ms per square. Find the period
$T$, the frequency $f$, and the angular frequency $\omega$.

33 (a) A mass $m$ is hung from a spring whose spring constant is $k$. Write down an expression for the total interaction energy of
the system, $U$, and find its equilibrium position. $\triangleright$ Hint, p. 922
(b) Explain how you could use your result from part a to determine
an unknown spring constant.

34 A certain mass, when hung from a certain spring, causes
the spring to stretch by an amount $h$ compared to its equilibrium
length. If the mass is displaced vertically from this equilibrium, it
will oscillate up and down with a period $T_{osc}$. Give a numerical
comparison between $T_{osc}$ and $T_{fall}$, the time required for the mass
to fall from rest through a height $h$, when it isn’t attached to the
spring. (You will need the result of problem 33).

35 Find the period of vertical oscillations of the mass $m$. The
spring, pulley, and ropes have negligible mass. $\triangleright$ Hint, p. 922

36 The equilibrium length of each spring in the figure is $b$, so
when the mass $m$ is at the center, neither spring exerts any force
on it. When the mass is displaced to the side, the springs stretch;
their spring constants are both $k$.
(a) Find the energy, $U$, stored in the springs, as a function of $y$, the
distance of the mass up or down from the center. $\triangleright$
(b) Show that the period of small up-down oscillations is infinite.
37 Two springs with spring constants $k_1$ and $k_2$ are put together end-to-end. Let $x_1$ be the amount by which the first spring is stretched relative to its equilibrium length, and similarly for $x_2$. If the combined double spring is stretched by an amount $b$ relative to its equilibrium length, then $b = x_1 + x_2$. Find the spring constant, $K$, of the combined spring in terms of $k_1$ and $k_2$.

\[
\text{Hint, p. 923} \quad \text{Answer, p. 933} \quad \checkmark
\]

38 A mass $m$ on a spring oscillates around an equilibrium at $x = 0$. Any function $U(x)$ with an equilibrium at $x = 0$ can be approximated as $U(x) = (1/2)kx^2$, and if the energy is symmetric with respect to positive and negative values of $x$, then the next level of improvement in such an approximation would be $U(x) = (1/2)kx^2 + bx^4$. The general idea here is that any smooth function can be approximated locally by a polynomial, and if you want a better approximation, you can use a polynomial with more terms in it. When you ask your calculator to calculate a function like $\sin$ or $e^x$, it’s using a polynomial approximation with 10 or 12 terms. Physically, a spring with a positive value of $b$ gets stiffer when stretched strongly than an “ideal” spring with $b = 0$. A spring with a negative $b$ is like a person who cracks under stress — when you stretch it too much, it becomes more elastic than an ideal spring would. We should not expect any spring to give totally ideal behavior no matter how much it is stretched. For example, there has to be some point at which it breaks.

Do a numerical simulation of the oscillation of a mass on a spring whose energy has a nonvanishing $b$. Is the period still independent of amplitude? Is the amplitude-independent equation for the period still approximately valid for small enough amplitudes? Does the addition of a positive $x^4$ term tend to increase the period, or decrease it? Include a printout of your program and its output with your homework paper.

39 An idealized pendulum consists of a pointlike mass $m$ on the end of a massless, rigid rod of length $L$. Its amplitude, $\theta$, is the angle the rod makes with the vertical when the pendulum is at the end of its swing. Write a numerical simulation to determine the period of the pendulum for any combination of $m$, $L$, and $\theta$. Examine the effect of changing each variable while manipulating the others.

40 A ball falls from a height $h$. Without air resistance, the time it takes to reach the floor is $\sqrt{2h/g}$. A numerical version of this calculation was given in program time2 on page 92. Now suppose that air resistance is not negligible. For a smooth sphere of radius $r$, moving at speed $v$ through air of density $\rho$, the amount of energy $dQ$ dissipated as heat as the ball falls through a height $dy$ is given (ignoring signs) by $dQ = (\pi/4)\rho v^2 r^2 dy$. Modify the program to incorporate this effect, and find the resulting change in the fall
time in the case of a 21 g ball of radius 1.0 cm, falling from a height of 1.0 m. The density of air at sea level is about 1.2 kg/m$^3$. Turn in a printout of both your program and its output. Answer: 0.34 ms.

41 The factorial of an integer $n$, written $n!$, is defined as the product of all the positive integers less than or equal to $n$. For example, $3! = 1 \times 2 \times 3 = 6$. Write a Python program to compute the factorial of a number. Test it with a small number whose factorial you can check by hand. Then use it to compute $30!$. (Python computes integer results with unlimited precision, so you won’t get any problems with rounding or overflows.) Turn in a printout of your program and its output, including the test.

42 Estimate the kinetic energy of a buzzing fly’s wing. (You may wish to review subsection 0.2.3 on order-of-magnitude estimates.)

43 A blade of grass moves upward as it grows. Estimate its kinetic energy. (You may wish to review subsection 0.2.3 on order-of-magnitude estimates.)

Key to symbols:

- easy  - typical  - challenging  - difficult  - very difficult

✓ An answer check is available at www.lightandmatter.com.
Exercises

Exercise 2A: Reasoning with Ratios and Powers

Equipment:

- ping-pong balls and paddles
- two-meter sticks

You have probably bounced a ping pong ball straight up and down in the air. The time between hits is related to the height to which you hit the ball. If you take twice as much time between hits, how many times higher do you think you will have to hit the ball? Write down your hypothesis:

Your instructor will first beat out a tempo of 240 beats per minute (four beats per second), which you should try to match with the ping-pong ball. Measure the height to which the ball rises:

Now try it at 120 beats per minute:

Compare your hypothesis and your results with the rest of the class.

Exercise 2B: The Shell Theorem

This exercise is an approximate numerical test of the shell theorem. There are seven masses A-G, each being one kilogram. Masses A-E, each one meter from the center, form a shape like two Egyptian pyramids joined at their bases; this is a rough approximation to a six-kilogram spherical shell of mass. Mass G is five meters from the center of the main group. The class will divide into six groups and split up the work required in order to calculate the total gravitational energy of mass G.

1. Each group should write its results on the board in units of picojoules, retaining six significant figures of precision.

2. The class will add the results and compare with the result that would be obtained with the shell theorem.
Exercise 2C: Vibrations

Equipment:

- air track and carts of two different masses
- springs
- weights

Place the cart on the air track and attach springs so that it can vibrate.

1. Test whether the period of vibration depends on amplitude. Try at least one moderate amplitude, for which the springs do not go slack, at least one amplitude that is large enough so that they do go slack, and one amplitude that’s the very smallest you can possibly observe.

2. Try a cart with a different mass. Does the period change by the expected factor, based on the equation $T = 2\pi \sqrt{m/k}$?

3. In homework problem 33 on page 123, you showed that a spring’s spring constant can be determined by hanging a weight from it. Use this technique to find the spring constant of each of the two springs. The equivalent spring constant of these two springs, attached to the cart in this way, can be found by adding their spring constants.

4. Test the equation $T = 2\pi \sqrt{m/k}$ numerically.
Chapter 3
Conservation of Momentum

I think, therefore I am.

I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery.

René Descartes
3.1 Momentum In One Dimension

3.1.1 Mechanical momentum

In the martial arts movie *Crouching Tiger, Hidden Dragon*, those who had received mystical enlightenment are able to violate the laws of physics. Some of the violations are obvious, such as their ability to fly, but others are a little more subtle. The rebellious young heroine/antiheroine Jen Yu gets into an argument while sitting at a table in a restaurant. A young tough, Iron Arm Lu, comes running toward her at full speed, and she puts up one arm and effortlessly makes him bounce back, without even getting out of her seat or bracing herself against anything. She does all this between bites.

Although kinetic energy doesn’t depend on the direction of motion, we’ve already seen on page 89 how conservation of energy combined with Galilean relativity allows us to make some predictions about the direction of motion. One of the examples was a demonstration that it isn’t possible for a hockey puck to spontaneously reverse its direction of motion. In the scene from the movie, however, the woman’s assailant isn’t just gliding through space. He’s interacting with her, so the previous argument doesn’t apply here, and we need to generalize it to more than one object. We consider the case of a physical system composed of pointlike material particles, in which every particle interacts with every other particle through an energy $U(r)$ that depends only on the distance $r$ between them. This still allows for a fairly general mechanical system, by which I mean roughly a system made of matter, not light. The characters in the movie are made of protons, neutrons, and electrons, so they would constitute such a system if the interactions among all these particles were of the form $U(r)$.\(^1\) We might even be able to get away with thinking of each person as one big particle, if it’s a good approximation to say that every part of each person’s whole body moves in the same direction at the same speed.

The basic insight can be extracted from the special case where there are only two particles interacting, and they only move in one dimension, as in the example shown in figure b. Conservation of energy says

$$K_{1i} + K_{2i} + U_i = K_{1f} + K_{2f} + U_f .$$

For simplicity, let’s assume that the interactions start after the time we’re calling initial, and end before the instant we choose as final. This is true in figure b, for example. Then $U_i = U_f$, and we can subtract the interaction energies from both sides, giving

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 .$$

\(^1\)Electrical and magnetic interactions don’t quite behave like this, which is a point we’ll take up later in the book.
As in the one-particle argument on page 89, the trick is to require conservation of energy not just in one particular frame of reference, but in every frame of reference. In a frame of reference moving at velocity $u$ relative to the first one, the velocities all have $u$ added onto them:

\[
\frac{1}{2} m_1 (v_1 + u)^2 + \frac{1}{2} m_2 (v_2 + u)^2 = \frac{1}{2} m_1 (v_1 f + u)^2 + \frac{1}{2} m_2 (v_2 f + u)^2
\]

When we square a quantity like $(v_1 + u)^2$, we get the same $v_1^2$ that occurred in the original frame of reference, plus two $u$-dependent terms, $2v_1u + u^2$. Subtracting the original conservation of energy equation from the version in the new frame of reference, we have

\[
m_1 v_1 u + m_2 v_2 u = m_1 v_1 f u + m_2 v_2 f u
\]

or, dividing by $u$,

\[
m_1 v_1 i + m_2 v_2 i = m_1 v_1 f + m_2 v_2 f
\]

This is a statement that when you add up $mv$ for the whole system, that total remains constant over time. In other words, this is a conservation law. The quantity $mv$ is called *momentum*, notated $p$ for obscure historical reasons. Its units are kg · m/s.

Unlike kinetic energy, momentum depends on the direction of motion, since the velocity is not squared. In one dimension, motion in the same direction as the positive $x$ axis is represented with positive values of $v$ and $p$. Motion in the opposite direction has negative $v$ and $p$.

---

**Jen Yu meets Iron Arm Lu**

> Initially, Jen Yu is at rest, and Iron Arm Lu is charging to the left, toward her, at 5 m/s. Jen Yu’s mass is 50 kg, and Lu’s is 100 kg. After the collision, the movie shows Jen Yu still at rest, and Lu rebounding at 5 m/s to the right. Is this consistent with the laws of physics, or would it be impossible in real life?

> This is perfectly consistent with conservation of mass (50 kg + 100 kg = 50 kg + 100 kg), and also with conservation of energy, since neither person’s kinetic energy changes, and there is therefore no change in the total energy. (We don’t have to worry about interaction energies, because the two points in time we’re considering are ones at which the two people aren’t interacting.) To analyze whether the scene violates conservation of momentum, we have to pick a coordinate system. Let’s define positive as

---

2We can now see that the derivation would have been equally valid for $U_i \neq U_f$. The two observers agree on the distance between the particles, so they also agree on the interaction energies, even though they disagree on the kinetic energies.
The ion drive engine of the NASA Deep Space 1 probe, shown under construction (top) and being tested in a vacuum chamber (bottom) prior to its October 1998 launch. Intended mainly as a test vehicle for new technologies, the craft nevertheless also carried out a scientific program that included a rendezvous with a comet in 2004. (NASA)

One could argue that they’re not a closed system, since Lu might be exchanging momentum with the floor, and Jen Yu might be exchanging momentum with the seat of her chair. This is a reasonable objection, but in the following section we’ll see that there are physical reasons why, in this situation, the force of friction would be relatively weak, and would not be able to transfer that much momentum in a fraction of a second.

This example points to an intuitive interpretation of conservation of momentum, which is that interactions are always mutual. That is, Jen Yu can’t change Lu’s momentum without having her own momentum changed as well.

\[ A \] cannon example 2

A cannon of mass 1000 kg fires a 10-kg shell at a velocity of 200 m/s. At what speed does the cannon recoil?

The law of conservation of momentum tells us that

\[ p_{\text{cannon},i} + p_{\text{shell},i} = p_{\text{cannon},f} + p_{\text{shell},f} \]

Choosing a coordinate system in which the cannon points in the positive direction, the given information is

\[ p_{\text{cannon},i} = 0 \]
\[ p_{\text{shell},i} = 0 \]
\[ p_{\text{shell},f} = 2000 \text{ kg} \cdot \text{m/s} \]

We must have \( p_{\text{cannon},f} = -2000 \text{ kg} \cdot \text{m/s} \), so the recoil velocity of the cannon is 2 m/s.

\[ I \] on drive example 3

The experimental solar-powered ion drive of the Deep Space 1 space probe expels its xenon gas exhaust at a speed of 30,000 m/s, ten times faster than the exhaust velocity for a typical chemical-fuel rocket engine. Roughly how many times greater is the maximum speed this spacecraft can reach, compared with a chemical-fueled probe with the same mass of fuel (“reaction mass”) available for pushing out the back as exhaust?

Momentum equals mass multiplied by velocity. Both spacecraft are assumed to have the same amount of reaction mass, and the ion drive’s exhaust has a velocity ten times greater, so the momentum of its exhaust is ten times greater. Before the engine starts firing, neither the probe nor the exhaust has any momentum, so the total momentum of the system is zero. By conservation of momentum, the total momentum must also be zero after
all the exhaust has been expelled. If we define the positive direction as the direction the spacecraft is going, then the negative momentum of the exhaust is canceled by the positive momentum of the spacecraft. The ion drive allows a final speed that is ten times greater. (This simplified analysis ignores the fact that the reaction mass expelled later in the burn is not moving backward as fast, because of the forward speed of the already-moving spacecraft.)

3.1.2 Nonmechanical momentum

So far, it sounds as though conservation of momentum can be proved mathematically, unlike conservation of mass and energy, which are entirely based on observations. The proof, however, was only for a mechanical system, with interactions of the form $U(r)$. Conservation of momentum can be extended to other systems as well, but this generalization is based on experiments, not mathematical proof. Light is the most important example of momentum that doesn’t equal $mv$ — light doesn’t have mass at all, but it does have momentum. For example, a flashlight left on for an hour would absorb about $10^{-5}$ kg $\cdot$ m/s of momentum as it recoiled.

*Momentum is not always equal to $mv$. Halley’s comet, shown in figure d, has a very elongated elliptical orbit, like those of many other comets. About once per century, its orbit brings it close to the sun. The comet’s head, or nucleus, is composed of dirty ice, so the energy deposited by the intense sunlight gradually removes ice from the surface and turns it into water vapor. The bottom photo shows a view of the water coming off of the nucleus from the European Giotto space probe, which passed within 596 km of the comet’s head on March 13, 1986.

The sunlight does not just carry energy, however. It also carries momentum. Once the steam comes off, the momentum of the sunlight impacting on it pushes it away from the sun, forming a tail as shown in the top image. The tail always points away from the sun, so when the comet is receding from the sun, the tail is in front. By analogy with matter, for which momentum equals $mv$, you would expect that massless light would have zero momentum, but the equation $p = mv$ is not the correct one for light, and light does have momentum. (Some comets also have a second tail, which is propelled by electrical forces rather than by the momentum of sunlight.)

The reason for bringing this up is not so that you can plug numbers into formulas in these exotic situations. The point is that the conservation laws have proven so sturdy exactly because they can easily be amended to fit new circumstances. The momentum of light will be a natural consequence of the discussion of the theory of relativity in chapter 7.
3.1.3 Momentum compared to kinetic energy

Momentum and kinetic energy are both measures of the quantity of motion, and a sideshow in the Newton-Leibniz controversy over who invented calculus was an argument over whether $mv$ (i.e., momentum) or $mv^2$ (i.e., kinetic energy without the 1/2 in front) was the “true” measure of motion. The modern student can certainly be excused for wondering why we need both quantities, when their complementary nature was not evident to the greatest minds of the 1700s. The following table highlights their differences.

<table>
<thead>
<tr>
<th>Kinetic energy...</th>
<th>Momentum...</th>
</tr>
</thead>
<tbody>
<tr>
<td>doesn’t depend on direction.</td>
<td>depends on direction.</td>
</tr>
<tr>
<td>is always positive, and cannot cancel out.</td>
<td>cancels with momentum in the opposite direction.</td>
</tr>
<tr>
<td>can be traded for forms of energy that do not involve motion. Kinetic energy is not a conserved quantity by itself.</td>
<td>is always conserved in a closed system.</td>
</tr>
<tr>
<td>is quadrupled if the velocity is doubled.</td>
<td>is doubled if the velocity is doubled.</td>
</tr>
</tbody>
</table>

Here are some examples that show the different behaviors of the two quantities.

**A spinning top example 5**

A spinning top has zero total momentum, because for every moving point, there is another point on the opposite side that cancels its momentum. It does, however, have kinetic energy.

**Why a tuning fork has two prongs example 6**

A tuning fork is made with two prongs so that they can vibrate in opposite directions, canceling their momenta. In a hypothetical version with only one prong, the momentum would have to oscillate, and this momentum would have to come from somewhere, such as the hand holding the fork. The result would be that vibrations would be transmitted to the hand and rapidly die out. In a two-prong fork, the two momenta cancel, but the energies don’t.

**Momentum and kinetic energy in firing a rifle example 7**

The rifle and bullet have zero momentum and zero kinetic energy to start with. When the trigger is pulled, the bullet gains some momentum in the forward direction, but this is canceled by the rifle’s backward momentum, so the total momentum is still zero. The kinetic energies of the gun and bullet are both positive numbers, however, and do not cancel. The total kinetic energy is allowed to
increase, because kinetic energy is being traded for other forms of energy. Initially there is chemical energy in the gunpowder. This chemical energy is converted into heat, sound, and kinetic energy. The gun's “backward” kinetic energy does not refrigerate the shooter's shoulder!

1. The wobbly earth  

As the moon completes half a circle around the earth, its motion reverses direction. This does not involve any change in kinetic energy. The reversed velocity does, however, imply a reversed momentum, so conservation of momentum in the closed earth-moon system tells us that the earth must also reverse its momentum. In fact, the earth wobbles in a little “orbit” about a point below its surface on the line connecting it and the moon. The two bodies' momenta always point in opposite directions and cancel each other out.

1. The earth and moon get a divorce  

Why can't the moon suddenly decide to fly off one way and the earth the other way? It is not forbidden by conservation of momentum, because the moon's newly acquired momentum in one direction could be canceled out by the change in the momentum of the earth, supposing the earth headed off in the opposite direction at the appropriate, slower speed. The catastrophe is forbidden by conservation of energy, because their energies would have to increase greatly.

1. Momentum and kinetic energy of a glacier  

A cubic-kilometer glacier would have a mass of about $10^{12}$ kg. If it moves at a speed of $10^{-5}$ m/s, then its momentum is $10^7$ kg $\cdot$ m/s. This is the kind of heroic-scale result we expect, perhaps the equivalent of the space shuttle taking off, or all the cars in LA driving in the same direction at freeway speed. Its kinetic energy, however, is only 50 J, the equivalent of the calories contained in a poppy seed or the energy in a drop of gasoline too small to be seen without a microscope. The surprisingly small kinetic energy is because kinetic energy is proportional to the square of the velocity, and the square of a small number is an even smaller number.

Discussion Questions

A If all the air molecules in the room settled down in a thin film on the floor, would that violate conservation of momentum as well as conservation of energy?

B A refrigerator has coils in back that get hot, and heat is molecular motion. These moving molecules have both energy and momentum. Why doesn't the refrigerator need to be tied to the wall to keep it from recoiling from the momentum it loses out the back?
Collisions in one dimension

Physicists employ the term “collision” in a broader sense than in ordinary usage, applying it to any situation where objects interact for a certain period of time. A bat hitting a baseball, a cosmic ray damaging DNA, and a gun and a bullet going their separate ways are all examples of collisions in this sense. Physical contact is not even required. A comet swinging past the sun on a hyperbolic orbit is considered to undergo a collision, even though it never touches the sun. All that matters is that the comet and the sun interacted gravitationally with each other.

The reason for broadening the term “collision” in this way is that all of these situations can be attacked mathematically using the same conservation laws in similar ways. In our first example, conservation of momentum is all that is required.

\[ m_N v = m_N v' + m_C v' \]

Solving for the unknown, \( v \), we find

\[ v = \left( 1 + \frac{m_C}{m_N} \right) v' = 80 \text{ km/hr} \]

He is lying.

The above example was simple because both cars had the same velocity afterward. In many one-dimensional collisions, however, the two objects do not stick. If we wish to predict the result of such a collision, conservation of momentum does not suffice, because both velocities after the collision are unknown, so we have one equation in two unknowns.

Conservation of energy can provide a second equation, but its application is not as straightforward, because kinetic energy is only the particular form of energy that has to do with motion. In many collisions, part of the kinetic energy that was present before the collision is used to create heat or sound, or to break the objects...
The equation $A + B = C + D$ says that the change in one ball’s velocity is equal and opposite to the change in the other’s. We invent a symbol $x = C - A$ for the change in ball 1’s velocity. The second equation can then be rewritten as $A^2 + B^2 = (A + x)^2 + (B - x)^2$. Squaring out the quantities in parentheses and then simplifying, we get $0 = Ax - Bx + x^2$. The equation has the trivial solution $x = 0$, i.e., neither ball’s velocity is changed, but this is physically impossible because the balls can’t travel through each other like ghosts. Assuming $x \neq 0$, we can divide by $x$ and solve for $x = B - A$. This means that ball 1 has gained an amount of velocity exactly sufficient to match ball 2’s initial velocity, and vice-versa. The balls must have swapped velocities.

Often, as in example 12, the details of the algebra are the least interesting part of the problem, and considerable physical insight can be gained simply by counting the number of unknowns and comparing to the number of equations. Suppose a beginner at pool notices a case where her cue ball hits an initially stationary ball and stops dead. “Wow, what a good trick,” she thinks. “I bet I could

or permanently bend them. Cars, in fact, are carefully designed to crumple in a collision. Crumpling the car uses up energy, and that’s good because the goal is to get rid of all that kinetic energy in a relatively safe and controlled way. At the opposite extreme, a superball is “super” because it emerges from a collision with almost all its original kinetic energy, having only stored it briefly as interatomic electrical energy while it was being squashed by the impact.

Collisions of the superball type, in which almost no kinetic energy is converted to other forms of energy, can thus be analyzed more thoroughly, because they have $K_f = K_i$, as opposed to the less useful inequality $K_f < K_i$ for a case like a tennis ball bouncing on grass.

The masses and the factors of $1/2$ can be divided out, and we eliminate the cumbersome subscripts by replacing the symbols $v_{1i},...$ with the symbols $A, B, C, and D$:

$A + B = C + D$


A little experimentation with numbers shows that given values of $A$ and $B$, it is impossible to find $C$ and $D$ that satisfy these equations unless $C$ and $D$ equal $A$ and $B$, or $C$ and $D$ are the same as $A$ and $B$ but swapped around. A formal proof of this fact is given in the sidebar. In the special case where ball 2 is initially at rest, this tells us that ball 1 is stopped dead by the collision, and ball 2 heads off at the velocity originally possessed by ball 1. This behavior will be familiar to players of pool.
never do that again in a million years.” But she tries again, and finds that she can’t help doing it even if she doesn’t want to. Luckily she has just learned about collisions in her physics course. Once she has written down the equations for conservation of momentum and no loss of kinetic energy, she really doesn’t have to complete the algebra. She knows that she has two equations in two unknowns, so there must be a well-defined solution. Once she has seen the result of one such collision, she knows that the same thing must happen every time. The same thing would happen with colliding marbles or croquet balls. It doesn’t matter if the masses or velocities are different, because that just multiplies both equations by some constant factor.

The discovery of the neutron

This was the type of reasoning employed by James Chadwick in his 1932 discovery of the neutron. At the time, the atom was imagined to be made out of two types of fundamental particles, protons and electrons. The protons were far more massive, and clustered together in the atom’s core, or nucleus. Electrical attraction caused the electrons to orbit the nucleus in circles, in much the same way that gravity kept the planets from cruising out of the solar system. Experiments showed, for example, that twice as much energy was required to strip the last electron off of a helium atom as was needed to remove the single electron from a hydrogen atom, and this was explained by saying that helium had two protons to hydrogen’s one. The trouble was that according to this model, helium would have two electrons and two protons, giving it precisely twice the mass of a hydrogen atom with one of each. In fact, helium has about four times the mass of hydrogen.

Chadwick suspected that the helium nucleus possessed two additional particles of a new type, which did not participate in electrical interactions at all, i.e., were electrically neutral. If these particles had very nearly the same mass as protons, then the four-to-one mass ratio of helium and hydrogen could be explained. In 1930, a new type of radiation was discovered that seemed to fit this description. It was electrically neutral, and seemed to be coming from the nuclei of light elements that had been exposed to other types of radiation. At this time, however, reports of new types of particles were a dime a dozen, and most of them turned out to be either clusters made of previously known particles or else previously known particles with higher energies. Many physicists believed that the “new” particle that had attracted Chadwick’s interest was really a previously known particle called a gamma ray, which was electrically neutral. Since gamma rays have no mass, Chadwick decided to try to determine the new particle’s mass and see if it was nonzero and approximately equal to the mass of a proton.

Unfortunately a subatomic particle is not something you can
Chadwick’s subatomic pool table. A disk of the naturally occurring metal polonium provides a source of radiation capable of kicking neutrons out of the beryllium nuclei. The type of radiation emitted by the polonium is easily absorbed by a few mm of air, so the air has to be pumped out of the left-hand chamber. The neutrons, Chadwick’s mystery particles, penetrate matter far more readily, and fly out through the wall and into the chamber on the right, which is filled with nitrogen or hydrogen gas. When a neutron collides with a nitrogen or hydrogen nucleus, it kicks it out of its atom at high speed, and this recoiling nucleus then rips apart thousands of other atoms of the gas. The result is an electrical pulse that can be detected in the wire on the right. Physicists had already calibrated this type of apparatus so that they could translate the strength of the electrical pulse into the velocity of the recoiling nucleus. The whole apparatus shown in the figure would fit in the palm of your hand, in dramatic contrast to today’s giant particle accelerators.

just put on a scale and weigh. Chadwick came up with an ingenious solution. The masses of the nuclei of the various chemical elements were already known, and techniques had already been developed for measuring the speed of a rapidly moving nucleus. He therefore set out to bombard samples of selected elements with the mysterious new particles. When a direct, head-on collision occurred between a mystery particle and the nucleus of one of the target atoms, the nucleus would be knocked out of the atom, and he would measure its velocity.

Suppose, for instance, that we bombard a sample of hydrogen atoms with the mystery particles. Since the participants in the collision are fundamental particles, there is no way for kinetic energy to be converted into heat or any other form of energy, and Chadwick thus had two equations in three unknowns:

equation #1: conservation of momentum

unknown #1: mass of the mystery particle

unknown #2: initial velocity of the mystery particle
The highjumper’s body passes over the bar, but his center of mass passes under it. (Dunia Young)

unknown #3: final velocity of the mystery particle

The number of unknowns is greater than the number of equations, so there is no unique solution. But by creating collisions with nuclei of another element, nitrogen, he gained two more equations at the expense of only one more unknown:

equation #3: conservation of momentum in the new collision

equation #4: no loss of kinetic energy in the new collision

unknown #4: final velocity of the mystery particle in the new collision

He was thus able to solve for all the unknowns, including the mass of the mystery particle, which was indeed within 1% of the mass of a proton. He named the new particle the neutron, since it is electrically neutral.

Discussion Questions

A  Good pool players learn to make the cue ball spin, which can cause it not to stop dead in a head-on collision with a stationary ball. If this does not violate the laws of physics, what hidden assumption was there in the example in the text where it was proved that the cue ball must stop?

3.1.5 The center of mass

Figures i and k show two examples where a motion that appears complicated actually has a very simple feature. In both cases, there is a particular point, called the center of mass, whose motion is surprisingly simple. The highjumper flexes his body as he passes over the bar, so his motion is intrinsically very complicated, and yet his center of mass’s motion is a simple parabola, just like the parabolic arc of a pointlike particle. The wrench’s center of mass travels in a straight line as seen from above, which is what we’d expect for a pointlike particle flying through the air.

The highjumper and the wrench are both complicated systems, each consisting of zillions of subatomic particles. To understand what’s going on, let’s instead look at a nice simple system, two pool balls colliding. We assume the balls are a closed system (i.e., their
interaction with the felt surface is not important) and that their rotation is unimportant, so that we’ll be able to treat each one as a single particle. By symmetry, the only place their center of mass can be is half-way in between, at an \( x \) coordinate equal to the average of the two balls’ positions, \( x_{cm} = (x_1 + x_2)/2 \).

Figure j makes it appear that the center of mass, marked with an \( \times \), moves with constant velocity to the right, regardless of the collision, and we can easily prove this using conservation of momentum:

\[
v_{cm} = \frac{dx_{cm}}{dt} = \frac{1}{2} (v_1 + v_2) = \frac{1}{2m} (mv_1 + mv_2) = \frac{p_{total}}{m_{total}}
\]

Since momentum is conserved, the last expression is constant, which proves that \( v_{cm} \) is constant.

Rearranging this a little, we have \( p_{total} = m_{total} v_{cm} \). In other words, the total momentum of the system is the same as if all its mass was concentrated at the center of mass point.

**Sigma notation**

When there is a large, potentially unknown number of particles, we can write sums like the ones occurring above using symbols like “+...,” but that gets awkward. It’s more convenient to use the Greek uppercase sigma, \( \Sigma \), to indicate addition. For example, the sum \( 1^2 + 2^2 + 3^2 + 4^2 = 30 \) could be written as

\[
\sum_{j=1}^{n} j^2 = 30
\]

read “the sum from \( j = 1 \) to \( n \) of \( j^2 \).” The variable \( j \) is a dummy variable, just like the \( dx \) in an integral that tells you you’re integrating with respect to \( x \), but has no significance outside the integral. The \( j \) below the sigma tells you what variable is changing from one term of the sum to the next, but \( j \) has no significance outside the sum.

As an example, let’s generalize the proof of \( p_{total} = m_{total} v_{cm} \) to the case of an arbitrary number \( n \) of identical particles moving in one dimension, rather than just two particles. The center of mass is at

\[
x_{cm} = \frac{1}{n} \sum_{j=1}^{n} x_j
\]
An improperly balanced wheel has a center of mass that is not at its geometric center. When you get a new tire, the mechanic clamps little weights to the rim to balance the wheel.

This toy was intentionally designed so that the mushroom-shaped piece of metal on top would throw off the center of mass. When you wind it up, the mushroom spins, but the center of mass doesn’t want to move, so the rest of the toy tends to counter the mushroom’s motion, causing the whole thing to jump around.

Where $x_1$ is the mass of the first particle, and so on. The velocity of the center of mass is

$$v_{cm} = \frac{dx_{cm}}{dt} = \frac{1}{n} \sum_{j=1}^{n} v_j = \frac{1}{nm} \sum_{j=1}^{n} mv_j = \frac{p_{total}}{m_{total}}$$

What about a system containing objects with unequal masses, or containing more than two objects? The reasoning above can be generalized to a weighted average:

$$x_{cm} = \frac{\sum_{j=1}^{n} m_j x_j}{\sum_{j=1}^{n} m_j}$$

**The solar system’s center of mass example 13**

In the discussion of the sun’s gravitational field on page 99, I mentioned in a footnote that the sun doesn’t really stay in one place while the planets orbit around it. Actually, motion is relative, so it’s meaningless to ask whether the sun is absolutely at rest, but it is meaningful to ask whether it moves in a straight line at constant velocity. We can now see that since the solar system is a closed system, its total momentum must be constant, and $p_{total} = m_{total}v_{cm}$ then tells us that it’s the solar system’s center of mass that has constant velocity, not the sun. The sun wobbles around this point irregularly due to its interactions with the planets, Jupiter in particular.

**The earth-moon system example 14**

The earth-moon system is much simpler than the solar system because it contains only two objects. Where is the center of mass of this system? Let $x=0$ be the earth’s center, so that the moon lies at $x = 3.8 \times 10^5$ km. Then

$$x_{cm} = \frac{\sum_{j=1}^{2} m_j x_j}{\sum_{j=1}^{2} m_j} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

and letting 1 be the earth and 2 the moon, we have

$$x_{cm} = \frac{m_{earth} \times 0 + m_{moon} x_{moon}}{m_{earth} + m_{moon}} = \frac{m_{earth} x_{moon}}{m_{earth} + m_{moon}} = 4600 \text{ km}$$

or about three quarters of the way from the earth’s center to its surface.
The center of mass as an average example 15

Explain how we know that the center of mass of each object is at the location shown in figure p.

Example 15.

The center of mass is a sort of average, so the height of the centers of mass in 1 and 2 has to be midway between the two squares, because that height is the average of the heights of the two squares. Example 3 is a combination of examples 1 and 2, so we can find its center of mass by averaging the horizontal positions of their centers of mass. In example 4, each square has been skewed a little, but just as much mass has been moved up as down, so the average vertical position of the mass hasn't changed. Example 5 is clearly not all that different from example 4, the main difference being a slight clockwise rotation, so just as in example 4, the center of mass must be hanging in empty space, where there isn't actually any mass. Horizontally, the center of mass must be between the heels and toes, or else it wouldn't be possible to stand without tipping over.

Momentum and Galilean relativity example 16

The principle of Galilean relativity states that the laws of physics are supposed to be equally valid in all inertial frames of reference. If we first calculate some momenta in one frame of reference and find that momentum is conserved, and then rework the whole problem in some other frame of reference that is moving with respect to the first, the numerical values of the momenta will all be different. Even so, momentum will still be conserved. All that matters is that we work a single problem in one consistent frame of reference.

One way of proving this is to apply the equation $p_{\text{total}} = m_{\text{total}} v_{\text{cm}}$. If the velocity of one frame relative to the other is $u$, then the only effect of changing frames of reference is to change $v_{\text{cm}}$ from its original value to $v_{\text{cm}} + u$. This adds a constant onto the momentum, which has no effect on conservation of momentum.

self-check A

The figure shows a gymnast holding onto the inside of a big wheel. From inside the wheel, how could he make it roll one way or the other?

Answer, p. 926
3.1.6 The center of mass frame of reference

A particularly useful frame of reference in many cases is the frame that moves along with the center of mass, called the center of mass (c.m.) frame. In this frame, the total momentum is zero. The following examples show how the center of mass frame can be a powerful tool for simplifying our understanding of collisions.

A collision of pool balls viewed in the c.m. frame example 17
If you move your head so that your eye is always above the point halfway in between the two pool balls, as in figure r, you are viewing things in the center of mass frame. In this frame, the balls come toward the center of mass at equal speeds. By symmetry, they must therefore recoil at equal speeds along the lines on which they entered. Since the balls have essentially swapped paths in the center of mass frame, the same must also be true in any other frame. This is the same result that required laborious algebra to prove previously without the concept of the center of mass frame.

The slingshot effect example 18
It is a counterintuitive fact that a spacecraft can pick up speed by swinging around a planet, if it arrives in the opposite direction compared to the planet’s motion. Although there is no physical contact, we treat the encounter as a one-dimensional collision, and analyze it in the center of mass frame. Since Jupiter is so much more massive than the spacecraft, the center of mass is essentially fixed at Jupiter’s center, and Jupiter has zero velocity in the center of mass frame, as shown in figure 3.1.6. The c.m. frame is moving to the left compared to the sun-fixed frame used in figure 3.1.6, so the spacecraft’s initial velocity is greater in this frame than in the sun’s frame.

Things are simpler in the center of mass frame, because it is more symmetric. In the sun-fixed frame, the incoming leg of the encounter is rapid, because the two bodies are rushing toward each other, while their separation on the outbound leg is more gradual, because Jupiter is trying to catch up. In the c.m. frame, Jupiter is sitting still, and there is perfect symmetry between the incoming and outgoing legs, so by symmetry we have \( v_{1f} = -v_{1i} \).

Going back to the sun-fixed frame, the spacecraft’s final velocity is increased by the frames’ motion relative to each other. In the sun-fixed frame, the spacecraft’s velocity has increased greatly.

Einstein’s motorcycle example 19
We’ve assumed we were dealing with a system of material objects, for which the equation \( p = mv \) was true. What if our system contains only light rays, or a mixture of light and matter? As a college student, Einstein kept worrying about what a beam of light would look like if you could ride alongside it on a motorcycle. In other words, he imagined putting himself in the light beam’s
center of mass frame. Chapter 7 discusses Einstein’s resolution of this problem, but the basic point is that you can’t ride the motorcycle alongside the light beam, because material objects can’t go as fast as the speed of light. A beam of light has no center of mass frame of reference.

Discussion Questions

A  Make up a numerical example of two unequal masses moving in one dimension at constant velocity, and verify the equation \( p_{\text{total}} = m_{\text{total}} v_{\text{cm}} \) over a time interval of one second.

B  A more massive tennis racquet or baseball bat makes the ball fly off faster. Explain why this is true, using the center of mass frame. For simplicity, assume that the racquet or bat is simply sitting still before the collision, and that the hitter’s hands do not make any force large enough to have a significant effect over the short duration of the impact.

3.2 Force In One Dimension

3.2.1 Momentum transfer

For every conserved quantity, we can define an associated rate of flow. An open system can have mass transferred in or out of it, and we can measure the rate of mass flow, \( \frac{dm}{dt} \) in units of kg/s. Energy can flow in or out, and the rate of energy transfer is the power, \( P = \frac{dE}{dt} \), measured in watts.\(^3\) The rate of momentum transfer is called force,

\[
F = \frac{dp}{dt} \quad \text{[definition of force]}
\]

The units of force are kg·m/s\(^2\), which can be abbreviated as newtons, 1 N = kg·m/s\(^2\). Newtons are unfortunately not as familiar as watts. A newton is about how much force you’d use to pet a dog. The most powerful rocket engine ever built, the first stage of the Saturn V that sent astronauts to the moon, had a thrust of about 30 million newtons. In one dimension, positive and negative signs indicate the direction of the force — a positive force is one that pushes or pulls in the direction of the positive \( x \) axis.

\(^1\)Walking into a lamppost example 20

Starting from rest, you begin walking, bringing your momentum up to 100 kg·m/s. You walk straight into a lamppost. Why is the momentum change of \(-100\) kg·m/s so much more painful than the change of +100 kg·m/s when you started walking?

The forces are not really constant, but for this type of qualitative discussion we can pretend they are, and approximate \( \frac{dp}{dt} \) as \( \Delta p/\Delta t \). It probably takes you about 1 s to speed up initially, so the ground’s force on you is \( F = \Delta p/\Delta t \approx 100 \) N. Your impact with the lamppost, however, is over in the blink of an eye, say 1/10 s or

\(^3\)Recall that uppercase \( P \) is power, while lowercase \( p \) is momentum.
less. Dividing by this much smaller $\Delta t$ gives a much larger force, perhaps thousands of newtons (with a negative sign).

This is also the principle of airbags in cars. The time required for the airbag to decelerate your head is fairly long; the time it takes your face to travel 20 or 30 cm. Without an airbag, your face would have hit the dashboard, and the time interval would have been the much shorter time taken by your skull to move a couple of centimeters while your face compressed. Note that either way, the same amount of momentum is transferred: the entire momentum of your head.

Force is defined as a derivative, and the derivative of a sum is the sum of the derivatives. Therefore force is additive: when more than one force acts on an object, you add the forces to find out what happens. An important special case is that forces can cancel. Consider your body sitting in a chair as you read this book. Let the positive $x$ axis be upward. The chair’s upward force on you is represented with a positive number, which cancels out with the earth’s downward gravitational force, which is negative. The total rate of momentum transfer into your body is zero, and your body doesn’t change its momentum.

Finding momentum from force example 21

An object of mass $m$ starts at rest at $t = t_0$. A force varying as $F = bt^{-2}$, where $b$ is a constant, begins acting on it. Find the greatest speed it will ever have.

\[
F = \frac{dp}{dt},
\]

\[
dp = F dt
\]

\[
=p = \int F dt + p_0
\]

\[
= -\frac{b}{t} + p_0
\]

where $p_0$ is a constant of integration. The given initial condition is that $p = 0$ at $t = t_0$, so we find that $p_0 = b/t_0$. The negative term gets closer to zero with increasing time, so the maximum momentum is achieved by letting $t$ approach infinity. That is, the object will never stop speeding up, but it will also never surpass a certain speed. In the limit $t \to \infty$, we identify $p_0$ as the momentum that the object will approach asymptotically. The maximum velocity is $v = p_0/m = b/m t_0$.

Discussion Question

Many collisions, like the collision of a bat with a baseball, appear to be instantaneous. Most people also would not imagine the bat and ball as bending or being compressed during the collision. Consider the following possibilities:
(1) The collision is instantaneous.
(2) The collision takes a finite amount of time, during which the ball and bat retain their shapes and remain in contact.
(3) The collision takes a finite amount of time, during which the ball and bat are bending or being compressed.

How can two of these be ruled out based on energy or momentum considerations?

3.2.2 Newton’s laws

Although momentum is the third conserved quantity we’ve encountered, historically it was the first to be discovered. Isaac Newton formulated a complete treatment of mechanical systems in terms of force and momentum. Newton’s theory was based on three laws of motion, which we now think of as consequences of conservation of mass, energy, and momentum.

<table>
<thead>
<tr>
<th>Newton’s laws in one dimension:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Newton’s first law:</strong> If there is no force acting on an object, it stays in the same state of motion.</td>
</tr>
<tr>
<td><strong>Newton’s second law:</strong> The sum of all the forces acting on an object determines the rate at which its momentum changes, ( F_{\text{total}} = \frac{dp}{dt} ).</td>
</tr>
<tr>
<td><strong>Newton’s third law:</strong> Forces occur in opposite pairs. If object A interacts with object B, then A’s force on B and B’s force on A are related by ( F_{AB} = -F_{BA} ).</td>
</tr>
</tbody>
</table>

The second law is the definition of force, which we’ve already encountered. The first law is a special case of the second law — if \( \frac{dp}{dt} \) is zero, then \( p = mv \) is a constant, and since mass is conserved, constant \( p \) implies constant \( v \). The third law is a restatement of conservation of momentum: for two objects interacting, we have constant total momentum, so \( 0 = \frac{d}{dt}(p_A + p_B) = F_{BA} + F_{AB} \).

\[ a = \frac{F}{m} \quad \text{example 22} \]

Many modern textbooks restate Newton’s second law as \( a = \frac{F}{m} \), i.e., as an equation that predicts an object’s acceleration based on the force exerted on it. This is easily derived from Newton’s original form as follows: \( a = \frac{dv}{dt} = \frac{(dp/dt)/m}{m} = \frac{F}{m} \).

\[ \text{Gravitational force related to } g \quad \text{example 23} \]

As a special case of the previous example, consider an object in free fall, and let the \( x \) axis point down. Then \( a = +g \), and \( F = ma = mg \). For example, the gravitational force on a 1 kg mass at the earth’s surface is about 9.8 N. Even if other forces act on the object, and it isn’t in free fall, the gravitational force on it is still the same, and can still be calculated as \( mg \).

---

\[ ^4 \text{This is with the benefit of hindsight. At the time, the word “force” already had certain connotations, and people thought they understood what it meant and how to measure it, e.g., by using a spring scale. From their point of view, } F = \frac{dp}{dt} \text{ was not a definition but a testable — and controversial! — statement.} \]
Two magnets exert forces on each other.

It doesn't make sense for the man to talk about the woman's money canceling out his bar tab, because there is no good reason to combine his debts and her assets.

Newton's third law does not mean that forces always cancel out so that nothing can ever move. If these two ice skaters, initially at rest, push against each other, they will both move.

Changing frames of reference

Suppose we change from one frame reference into another, which is moving relative to the first one at a constant velocity \( u \). If an object of mass \( m \) is moving at velocity \( v \) (which need not be constant), then the effect is to change its momentum from \( mv \) in one frame to \( mv + mu \) in the other. Force is defined as the derivative of momentum with respect to time, and the derivative of a constant is zero, so adding the constant \( mu \) has no effect on the result. We therefore conclude that observers in different inertial frames of reference agree on forces.

Using the third law correctly

If you’ve already accepted Galilean relativity in your heart, then there is nothing really difficult about the first and second laws. The third law, however, is more of a conceptual challenge. The first hurdle is that it is counterintuitive. Is it really true that if a fighter jet collides with a mosquito, the mosquito’s force on the jet is just as strong as the jet’s force on the mosquito? Yes, it is true, but it is hard to believe at first. That amount of force simply has more of an effect on the mosquito, because it has less mass.

A more humane and practical experiment is shown in figure d. A large magnet and a small magnet are weighed separately, and then one magnet is hung from the pan of the top balance so that it is directly above the other magnet. There is an attraction between the two magnets, causing the reading on the top scale to increase and the reading on the bottom scale to decrease. The large magnet is more “powerful” in the sense that it can pick up a heavier paperclip from the same distance, so many people have a strong expectation that one scale’s reading will change by a far different amount than the other. Instead, we find that the two changes are equal in magnitude but opposite in direction, so the upward force of the top magnet on the bottom magnet is of the same magnitude as the downward force of the bottom magnet on the top magnet.

To students, it often sounds as though Newton’s third law implies nothing could ever change its motion, since the two equal and opposite forces would always cancel. As illustrated in figure e, the fallacy arises from assuming that we can add things that it doesn’t make sense to add. It only makes sense to add up forces that are acting on the same object, whereas two forces related to each other by Newton’s third law are always acting on two different objects. If two objects are interacting via a force and no other forces are involved, then both objects will accelerate — in opposite directions, as shown in figure f!

Here are some suggestions for avoiding misapplication of Newton’s third law:

1. Changing frames of reference example 24
2. Using the third law correctly
3. Newton’s third law does not mean that forces always cancel out so that nothing can ever move. If these two ice skaters, initially at rest, push against each other, they will both move.
1. It always relates exactly two forces, not more.

2. The two forces involve exactly two objects, in the pattern A on B, B on A.

3. The two forces are always of the same type, e.g., friction and friction, or gravity and gravity.

Directions of forces

We’ve already seen that momentum, unlike energy, has a direction in space. Since force is defined in terms of momentum, force also has a direction in space. For motion in one dimension, we have to pick a coordinate system, and given that choice, forces and momenta will be positive or negative. We’ve already used signs to represent directions of forces in Newton’s third law, $F_{AB} = -F_{BA}$.

There is, however, a complication with force that we were able to avoid with momentum. If an object is moving on a line, we’re guaranteed that its momentum is in one of two directions: the two directions along the line. But even an object that stays on a line may still be subject to forces that act perpendicularly to the line. For example, suppose a coin is sliding to the right across a table, h, and let’s choose a positive $x$ axis that points to the right. The coin’s motion is along a horizontal line, and its momentum is positive and decreasing. Because the momentum is decreasing, its time derivative $\frac{dp}{dt}$ is negative. This derivative equals the horizontal force of friction $F_1$, and its negative sign tells us that this force on the coin is to the left.

But there are also vertical forces on the coin. The Earth exerts a downward gravitational force $F_2$ on it, and the table makes an upward force $F_3$ that prevents the coin from sinking into the wood. In fact, without these vertical forces the horizontal frictional force wouldn’t exist: surfaces don’t exert friction against one another unless they are being pressed together.

To avoid mathematical complication, we want to postpone the full three-dimensional treatment of force and momentum until section 3.4. For now, we’ll limit ourselves to examples like the coin, in which the motion is confined to a line, and any forces perpendicular to the line cancel each other out.

Discussion Questions

A Criticize the following incorrect statement: “If an object is at rest and the total force on it is zero, it stays at rest. There can also be cases where an object is moving and keeps on moving without having any total force on it, but that can only happen when there’s no friction, like in outer space.”
The table gives laser timing data for Ben Johnson’s 100 m dash at the 1987 World Championship in Rome. (His world record was later revoked because he tested positive for steroids.) How does the total force on him change over the duration of the race?

You hit a tennis ball against a wall. Explain any and all incorrect ideas in the following description of the physics involved: “According to Newton’s third law, there has to be a force opposite to your force on the ball. The opposite force is the ball’s mass, which resists acceleration, and also air resistance.”

Tam Anh grabs Sarah by the hand and tries to pull her. She tries to remain standing without moving. A student analyzes the situation as follows. “If Tam Anh’s force on Sarah is greater than her force on him, he can get her to move. Otherwise, she’ll be able to stay where she is.” What’s wrong with this analysis?

### 3.2.3 What force is not

Violin teachers have to endure their beginning students’ screeching. A frown appears on the woodwind teacher’s face as she watches her student take a breath with an expansion of his ribcage but none in his belly. What makes physics teachers cringe is their students’ verbal statements about forces. Below I have listed several dicta about what force is not.

**Force is not a property of one object.**

A great many of students’ incorrect descriptions of forces could be cured by keeping in mind that a force is an interaction of two objects, not a property of one object.

*Incorrect statement:* “That magnet has a lot of force.”

If the magnet is one millimeter away from a steel ball bearing, they may exert a very strong attraction on each other, but if they were a meter apart, the force would be virtually undetectable. The magnet’s strength can be rated using certain electrical units (ampere $\cdot$ meters$^2$), but not in units of force.

**Force is not a measure of an object’s motion.**

If force is not a property of a single object, then it cannot be used as a measure of the object’s motion.

*Incorrect statement:* “The freight train rumbled down the tracks with awesome force.”

Force is not a measure of motion. If the freight train collides with a stalled cement truck, then some awesome forces will occur, but if it hits a fly the force will be small.

**Force is not energy.**

*Incorrect statement:* “How can my chair be making an upward force on my rear end? It has no power!”

Power is a concept related to energy, e.g., a 100-watt lightbulb uses up 100 joules per second of energy. When you sit in a chair, no energy