from rest with an acceleration of 10 m/s². This is really only a lower limit; if there really was a hole that deep, the fall would actually take a longer time than the one you calculate, both because there is air friction and because gravity gets weaker as you get deeper (at the center of the earth, \( g \) is zero, because the earth is pulling you equally in every direction at once).

22 How many cubic inches are there in a cubic foot? The answer is not 12.

23 Assume a dog’s brain is twice as great in diameter as a cat’s, but each animal’s brain cells are the same size and their brains are the same shape. In addition to being a far better companion and much nicer to come home to, how many times more brain cells does a dog have than a cat? The answer is not 2.

24 The population density of Los Angeles is about 4000 people/km². That of San Francisco is about 6000 people/km². How many times farther away is the average person’s nearest neighbor in LA than in San Francisco? The answer is not 1.5.

25 A hunting dog’s nose has about 10 square inches of active surface. How is this possible, since the dog’s nose is only about 1 in \( \times \) 1 in \( \times \) 1 in = 1 in³? After all, 10 is greater than 1, so how can it fit?

26 Estimate the number of blades of grass on a football field.

27 In a computer memory chip, each bit of information (a 0 or a 1) is stored in a single tiny circuit etched onto the surface of a silicon chip. The circuits cover the surface of the chip like lots in a housing development. A typical chip stores 64 Mb (megabytes) of data, where a byte is 8 bits. Estimate (a) the area of each circuit, and (b) its linear size.

28 Suppose someone built a gigantic apartment building, measuring 10 km \( \times \) 10 km at the base. Estimate how tall the building would have to be to have space in it for the entire world’s population to live.

29 A hamburger chain advertises that it has sold 10 billion Bongo Burgers. Estimate the total mass of feed required to raise the cows used to make the burgers.

30 Estimate the volume of a human body, in cm³.

31 How many cm² is 1 mm²? Solution, p. 1034

32 Compare the light-gathering powers of a 3-cm-diameter telescope and a 30-cm telescope. Solution, p. 1034

33 One step on the Richter scale corresponds to a factor of 100 in terms of the energy absorbed by something on the surface of the Earth, e.g., a house. For instance, a 9.3-magnitude quake would
release 100 times more energy than an 8.3. The energy spreads out from the epicenter as a wave, and for the sake of this problem we'll assume we're dealing with seismic waves that spread out in three dimensions, so that we can visualize them as hemispheres spreading out under the surface of the earth. If a certain 7.6-magnitude earthquake and a certain 5.6-magnitude earthquake produce the same amount of vibration where I live, compare the distances from my house to the two epicenters.

34 In Europe, a piece of paper of the standard size, called A4, is a little narrower and taller than its American counterpart. The ratio of the height to the width is the square root of 2, and this has some useful properties. For instance, if you cut an A4 sheet from left to right, you get two smaller sheets that have the same proportions. You can even buy sheets of this smaller size, and they're called A5. There is a whole series of sizes related in this way, all with the same proportions. (a) Compare an A5 sheet to an A4 in terms of area and linear size. (b) The series of paper sizes starts from an A0 sheet, which has an area of one square meter. Suppose we had a series of boxes defined in a similar way: the B0 box has a volume of one cubic meter, two B1 boxes fit exactly inside an B0 box, and so on. What would be the dimensions of a B0 box?

35 Estimate the mass of one of the hairs in Albert Einstein’s moustache, in units of kg.

36 According to folklore, every time you take a breath, you are inhaling some of the atoms exhaled in Caesar’s last words. Is this true? If so, how many?

37 The Earth’s surface is about 70% water. Mars’s diameter is about half the Earth’s, but it has no surface water. Compare the land areas of the two planets.

38 The traditional Martini glass is shaped like a cone with the point at the bottom. Suppose you make a Martini by pouring vermouth into the glass to a depth of 3 cm, and then adding gin to bring the depth to 6 cm. What are the proportions of gin and vermouth?

39 The central portion of a CD is taken up by the hole and some surrounding clear plastic, and this area is unavailable for storing data. The radius of the central circle is about 35% of the outer radius of the data-storing area. What percentage of the CD’s area is therefore lost?

40 The one-liter cube in the photo has been marked off into smaller cubes, with linear dimensions one tenth those of the big one. What is the volume of each of the small cubes?
41 Estimate the number of man-hours required for building the Great Wall of China. ▶ Solution, p. 1034

42 (a) Using the microscope photo in the figure, estimate the mass of a one cell of the *E. coli* bacterium, which is one of the most common ones in the human intestine. Note the scale at the lower right corner, which is 1 µm. Each of the tubular objects in the column is one cell. (b) The feces in the human intestine are mostly bacteria (some dead, some alive), of which *E. coli* is a large and typical component. Estimate the number of bacteria in your intestines, and compare with the number of human cells in your body, which is believed to be roughly on the order of $10^{13}$. (c) Interpreting your result from part b, what does this tell you about the size of a typical human cell compared to the size of a typical bacterial cell?

43 The figure shows a practical, simple experiment for determining *g* to high precision. Two steel balls are suspended from electromagnets, and are released simultaneously when the electric current is shut off. They fall through unequal heights $\Delta x_1$ and $\Delta x_2$. A computer records the sounds through a microphone as first one ball and then the other strikes the floor. From this recording, we can accurately determine the quantity $T$ defined as $T = \Delta t_2 - \Delta t_1$, i.e., the time lag between the first and second impacts. Note that since the balls do not make any sound when they are released, we have no way of measuring the individual times $\Delta t_2$ and $\Delta t_1$.
(a) Find an equation for $g$ in terms of the measured quantities $T$, $\Delta x_1$ and $\Delta x_2$.
(b) Check the units of your equation.
(c) Check that your equation gives the correct result in the case where $\Delta x_1$ is very close to zero. However, is this case realistic?
(d) What happens when $\Delta x_1 = \Delta x_2$? Discuss this both mathematically and physically.

44 Estimate the number of jellybeans in figure o on p. 44. ▶ Solution, p. 1035

45 Let the function $x$ be defined by $x(t) = Ae^{bt}$, where $t$ has units of seconds and $x$ has units of meters. (For $b < 0$, this could be a fairly accurate model of the motion of a bullet shot into a tank of oil.) Show that the Taylor series of this function makes sense if and only if $A$ and $b$ have certain units.

46 A 2002 paper by Steegmann *et al.* uses data from modern human groups like the Inuit to argue that Neanderthals in Ice Age Europe had to eat up “to 4,480 kcal per day to support strenuous winter foraging and cold resistance costs.” What’s wrong here?

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**Key to symbols:**

- easy
- typical
- challenging
- difficult
- very difficult
An answer check is available at www.lightandmatter.com.
Chapter 1
Conservation of Mass

It took just a moment for that head to fall, but a hundred years might not produce another like it.

Joseph-Louis Lagrange, referring to the execution of Lavoisier on May 8, 1794

1.1 Mass

Change is impossible, claimed the ancient Greek philosopher Parmenides. His work was nonscientific, since he didn’t state his ideas in a form that would allow them to be tested experimentally, but modern science nevertheless has a strong Parmenidean flavor. His main argument that change is an illusion was that something can’t be turned into nothing, and likewise if you have nothing, you can’t turn it into something. To make this into a scientific theory, we have to decide on a way to measure what “something” is, and we can then
check by measurements whether the total amount of “something” in the universe really stays constant. How much “something” is there in a rock? Does a sunbeam count as “something?” Does heat count? Motion? Thoughts and feelings?

If you look at the table of contents of this book, you’ll see that the first four chapters have the word “conservation” in them. In physics, a conservation law is a statement that the total amount of a certain physical quantity always stays the same. This chapter is about conservation of mass. The metric system is designed around a unit of distance, the meter, a unit of mass, the kilogram, and a time unit, the second. Numerical measurement of distance and time probably date back almost as far into prehistory as counting money, but mass is a more modern concept. Until scientists figured out that mass was conserved, it wasn’t obvious that there could be a single, consistent way of measuring an amount of matter, hence jiggers of whiskey and cords of wood. You may wonder why conservation of mass wasn’t discovered until relatively modern times, but it wasn’t obvious, for example, that gases had mass, and that the apparent loss of mass when wood was burned was exactly matched by the mass of the escaping gases.

Once scientists were on the track of the conservation of mass concept, they began looking for a way to define mass in terms of a definite measuring procedure. If they tried such a procedure, and the result was that it led to nonconservation of mass, then they would throw it out and try a different procedure. For instance, we might be tempted to define mass using kitchen measuring cups, i.e., as a measure of volume. Mass would then be perfectly conserved for a process like mixing marbles with peanut butter, but there would be processes like freezing water that led to a net increase in mass, and others like soaking up water with a sponge that caused a decrease. If, with the benefit of hindsight, it seems like the measuring cup definition was just plain silly, then here’s a more subtle example of a wrong definition of mass. Suppose we define it using a bathroom scale, or a more precise device such as a postal scale that works on the same principle of using gravity to compress or twist a spring. The trouble is that gravity is not equally strong all over the surface of the earth, so for instance there would be nonconservation of mass when you brought an object up to the top of a mountain, where gravity is a little weaker.

Although some of the obvious possibilities have problems, there do turn out to be at least two approaches to defining mass that lead to its being a conserved quantity, so we consider these definitions to be “right” in the pragmatic sense that what’s correct is what’s useful.

One definition that works is to use balances, but compensate for the local strength of gravity. This is the method that is used
by scientists who actually specialize in ultraprecise measurements. A standard kilogram, in the form of a platinum-iridium cylinder, is kept in a special shrine in Paris. Copies are made that balance against the standard kilogram in Parisian gravity, and they are then transported to laboratories in other parts of the world, where they are compared with other masses in the local gravity. The quantity defined in this way is called *gravitational mass*.

A second and completely different approach is to measure how hard it is to change an object’s state of motion. This tells us its *inertial mass*. For example, I’d be more willing to stand in the way of an oncoming poodle than in the path of a freight train, because my body will have a harder time convincing the freight train to stop. This is a dictionary-style conceptual definition, but in physics we need to back up a conceptual definition with an operational definition, which is one that spells out the operations required in order to measure the quantity being defined. We can operationalize our definition of inertial mass by throwing a standard kilogram at an object at a speed of 1 m/s (one meter per second) and measuring the recoiling object’s velocity. Suppose we want to measure the mass of a particular block of cement. We put the block in a toy wagon on the sidewalk, and throw a standard kilogram at it. Suppose the standard kilogram hits the wagon, and then drops straight down to the sidewalk, having lost all its velocity, and the wagon and the block inside recoil at a velocity of 0.23 m/s. We then repeat the experiment with the block replaced by various numbers of standard kilograms, and find that we can reproduce the recoil velocity of 0.23 m/s with four standard kilograms in the wagon. We have determined the mass of the block to be four kilograms.\(^1\) Although this definition of inertial mass has an appealing conceptual simplicity, it is obviously not very practical, at least in this crude form. Nevertheless, this method of collision is very much like the methods used for measuring the masses of subatomic particles, which, after all, can’t be put on little postal scales!

Astronauts spending long periods of time in space need to monitor their loss of bone and muscle mass, and here as well, it’s impossible to measure gravitational mass. Since they don’t want to have standard kilograms thrown at them, they use a slightly different technique (figures d and e). They strap themselves to a chair which is attached to a large spring, and measure the time it takes for one cycle of vibration.

\(^1\)You might think intuitively that the recoil velocity should be exactly one fourth of a meter per second, and you’d be right except that the wagon has some mass as well. Our present approach, however, only requires that we give a way to test for equality of masses. To predict the recoil velocity from scratch, we’d need to use conservation of momentum, which is discussed in a later chapter.
1.1.1 Problem-solving techniques

How do we use a conservation law, such as conservation of mass, to solve problems? There are two basic techniques.

As an analogy, consider conservation of money, which makes it illegal for you to create dollar bills using your own inkjet printer. (Most people don’t intentionally destroy their dollar bills, either!) Suppose the police notice that a particular store doesn’t seem to have any customers, but the owner wears lots of gold jewelry and drives a BMW. They suspect that the store is a front for some kind of crime, perhaps counterfeiting. With intensive surveillance, there are two basic approaches they could use in their investigation. One method would be to have undercover agents try to find out how much money goes in the door, and how much money comes back out at the end of the day, perhaps by arranging through some trick to get access to the owner’s briefcase in the morning and evening. If the amount of money that comes out every day is greater than the amount that went in, and if they’re convinced there is no safe on the premises holding a large reservoir of money, then the owner must be counterfeiting. This inflow-equals-outflow technique is useful if we are sure that there is a region of space within which there is no supply of mass that is being built up or depleted.

A stream of water example 1

If you watch water flowing out of the end of a hose, you’ll see that the stream of water is fatter near the mouth of the hose, and skinnier lower down. This is because the water speeds up as it falls. If the cross-sectional area of the stream was equal all along its length, then the rate of flow (kilograms per second) through a lower cross-section would be greater than the rate of flow through a cross-section higher up. Since the flow is steady, the amount of water between the two cross-sections stays constant. Conservation of mass therefore requires that the cross-sectional area of the stream shrink in inverse proportion to the increasing speed of the falling water.

self-check A

Suppose the you point the hose straight up, so that the water is rising rather than falling. What happens as the velocity gets smaller? What happens when the velocity becomes zero? Answer, p. 1054

How can we apply a conservation law, such as conservation of mass, in a situation where mass might be stored up somewhere? To use a crime analogy again, a prison could contain a certain number of prisoners, who are not allowed to flow in or out at will. In physics, this is known as a closed system. A guard might notice that a certain prisoner’s cell is empty, but that doesn’t mean he’s escaped. He could be sick in the infirmary, or hard at work in the shop earning cigarette money. What prisons actually do is to count all their prisoners every day, and make sure today’s total is the same as
yesterday’s. One way of stating a conservation law is that for a closed system, the total amount of stuff (mass, in this chapter) stays constant.

Lavoisier and chemical reactions in a closed system example 2

The French chemist Antoine-Laurent Lavoisier is considered the inventor of the concept of conservation of mass. Before Lavoisier, chemists had never systematically weighed their chemicals to quantify the amount of each substance that was undergoing reactions. They also didn’t completely understand that gases were just another state of matter, and hadn’t tried performing reactions in sealed chambers to determine whether gases were being consumed from or released into the air. For this they had at least one practical excuse, which is that if you perform a gas-releasing reaction in a sealed chamber with no room for expansion, you get an explosion! Lavoisier invented a balance that was capable of measuring milligram masses, and figured out how to do reactions in an upside-down bowl in a basin of water, so that the gases could expand by pushing out some of the water. In a crucial experiment, Lavoisier heated a red mercury compound, which we would now describe as mercury oxide (HgO), in such a sealed chamber. A gas was produced (Lavoisier later named it “oxygen”), driving out some of the water, and the red compound was transformed into silvery liquid mercury metal. The crucial point was that the total mass of the entire apparatus was exactly the same before and after the reaction. Based on many observations of this type, Lavoisier proposed a general law of nature, that mass is always conserved. (In earlier experiments, in which closed systems were not used, chemists had become convinced that there was a mysterious substance, phlogiston, involved in combustion and oxidation reactions, and that phlogiston’s mass could be positive, negative, or zero depending on the situation!)

1.1.2 Delta notation

A convenient notation used throughout physics is \( \Delta \), the uppercase Greek letter delta, which indicates “change in” or “after minus before.” For example, if \( b \) represents how much money you have in the bank, then a deposit of $100 could be represented as \( \Delta b = 100 \). That is, the change in your balance was $100, or the balance after the transaction minus the balance before the transaction equals $100. A withdrawal would be indicated by \( \Delta b < 0 \). We represent “before” and “after” using the subscripts \( i \) (initial) and \( f \) (final), e.g., \( \Delta b = b_f - b_i \). Often the delta notation allows more precision than English words. For instance, “time” can be used to mean a point in time (“now’s the time”), \( t \), or it could mean a period of time (“the whole time, he had spit on his chin”), \( \Delta t \).

This notation is particularly convenient for discussing conserved quantities. The law of conservation of mass can be stated simply as
The two pendulum bobs are constructed with equal gravitational masses. If their inertial masses are also equal, then each pendulum should take exactly the same amount of time per swing.

If the cylinders have slightly unequal ratios of inertial to gravitational mass, their trajectories will be a little different.

A simplified drawing of an Eötvös-style experiment. If the two masses, made out of two different substances, have slightly different ratios of inertial to gravitational mass, then the apparatus will twist slightly as the earth spins.

\[ \Delta m = 0, \text{ where } m \text{ is the total mass of any closed system.} \]

**Self-check B**

If \( x \) represents the location of an object moving in one dimension, then how would positive and negative signs of \( \Delta x \) be interpreted?

**Answer, p. 1054**

**Discussion Questions**

A  If an object had a straight-line \( x - t \) graph with \( \Delta x = 0 \) and \( \Delta t \neq 0 \), what would be true about its velocity? What would this look like on a graph? What about \( \Delta t = 0 \) and \( \Delta x \neq 0 \)?

### 1.2 Equivalence of gravitational and inertial mass

We find experimentally that both gravitational and inertial mass are conserved to a high degree of precision for a great number of processes, including chemical reactions, melting, boiling, soaking up water with a sponge, and rotting of meat and vegetables. Now it’s logically possible that both gravitational and inertial mass are conserved, but that there is no particular relationship between them, in which case we would say that they are separately conserved. On the other hand, the two conservation laws may be redundant, like having one law against murder and another law against killing people!

Here’s an experiment that gets at the issue: stand up now and drop a coin and one of your shoes side by side. I used a 400-gram shoe and a 2-gram penny, and they hit the floor at the same time as far as I could tell by eye. This is an interesting result, but a physicist and an ordinary person will find it interesting for different reasons.

The layperson is surprised, since it would seem logical that heavier objects would always fall faster than light ones. However, it’s fairly easy to prove that if air friction is negligible, any two objects made of the same substance must have identical motion when they fall. For instance, a 2-kg copper mass must exhibit the same falling motion as a 1-kg copper mass, because nothing would be changed by physically joining together two 1-kg copper masses to make a single 2-kg copper mass. Suppose, for example, that they are joined with a dab of glue; the glue isn’t under any strain, because the two masses are doing the same thing side by side. Since the glue isn’t really doing anything, it makes no difference whether the masses fall separately or side by side.

What a physicist finds remarkable about the shoe-and-penny experiment is that it came out the way it did even though the shoe and the penny are made of different substances. There is absolutely no theoretical reason why this should be true. We could say that it

\[ ^2 \text{The argument only fails for objects light enough to be affected appreciably by air friction: a bunch of feathers falls differently if you wad them up because the pattern of air flow is altered by putting them together.} \]
happens because the greater gravitational mass of the shoe is exactly counteracted by its greater inertial mass, which makes it harder for gravity to get it moving, but that just leaves us wondering why inertial mass and gravitational mass are always in proportion to each other. It’s possible that they are only approximately equivalent. Most of the mass of ordinary matter comes from neutrons and protons, and we could imagine, for instance, that neutrons and protons do not have exactly the same ratio of gravitational to inertial mass. This would show up as a different ratio of gravitational to inertial mass for substances containing different proportions of neutrons and protons.

Galileo did the first numerical experiments on this issue in the seventeenth century by rolling balls down inclined planes, although he didn’t think about his results in these terms. A fairly easy way to improve on Galileo’s accuracy is to use pendulums with bobs made of different materials. Suppose, for example, that we construct an aluminum bob and a brass bob, and use a double-pan balance to verify to good precision that their gravitational masses are equal. If we then measure the time required for each pendulum to perform a hundred cycles, we can check whether the results are the same. If their inertial masses are unequal, then the one with a smaller inertial mass will go through each cycle faster, since gravity has an easier time accelerating and decelerating it. With this type of experiment, one can easily verify that gravitational and inertial mass are proportional to each other to an accuracy of $10^{-3}$ or $10^{-4}$.

In 1889, the Hungarian physicist Roland Eötvös used a slightly different approach to verify the equivalence of gravitational and inertial mass for various substances to an accuracy of about $10^{-8}$, and the best such experiment, figure d, improved on even this phenomenal accuracy, bringing it to the $10^{-12}$ level. In all the experiments described so far, the two objects move along similar trajectories: straight lines in the penny-and-shoe and inclined plane experiments, and circular arcs in the pendulum version. The Eötvös-style experiment looks for differences in the objects’ trajectories. The concept can be understood by imagining the following simplified version. Suppose, as in figure b, we roll a brass cylinder off of a tabletop and measure where it hits the floor, and then do the same with an aluminum cylinder, making sure that both of them go over the edge with precisely the same velocity. An object with zero gravitational mass would fly off straight and hit the wall, while an object with zero inertial mass would make a sudden 90-degree turn and drop straight to the floor. If the aluminum and brass cylinders have ordinary, but slightly unequal, ratios of gravitational to inertial mass, then they will follow trajectories that are just slightly different. In other words, if inertial and gravitational mass are not exactly proportional to each other for all substances, then objects made of

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\[^{3}\text{V.B. Braginskii and V.I. Panov, Soviet Physics JETP 34, 463 (1972).}\]
different substances will have different trajectories in the presence of gravity.

A simplified drawing of a practical, high-precision experiment is shown in figure c. Two objects made of different substances are balanced on the ends of a bar, which is suspended at the center from a thin fiber. The whole apparatus moves through space on a complicated, looping trajectory arising from the rotation of the earth superimposed on the earth’s orbital motion around the sun. Both the earth’s gravity and the sun’s gravity act on the two objects. If their inertial masses are not exactly in proportion to their gravitational masses, then they will follow slightly different trajectories through space, which will result in a very slight twisting of the fiber between the daytime, when the sun’s gravity is pulling upward, and the night, when the sun’s gravity is downward. Figure d shows a more realistic picture of the apparatus.

This type of experiment, in which one expects a null result, is a tough way to make a career as a scientist. If your measurement comes out as expected, but with better accuracy than other people had previously achieved, your result is publishable, but won’t be considered earthshattering. On the other hand, if you build the most sensitive experiment ever, and the result comes out contrary to expectations, you’re in a scary situation. You could be right, and earn a place in history, but if the result turns out to be due to a defect in your experiment, then you’ve made a fool of yourself.

1.3 Galilean relativity

I defined inertial mass conceptually as a measure of how hard it is to change an object’s state of motion, the implication being that if you don’t interfere, the object’s motion won’t change. Most people, however, believe that objects in motion have a natural tendency to slow down. Suppose I push my refrigerator to the west for a while at 0.1 m/s, and then stop pushing. The average person would say fridge just naturally stopped moving, but let’s imagine how someone in China would describe the fridge experiment carried out in my house here in California. Due to the rotation of the earth, California is moving to the east at about 400 m/s. A point in China at the same latitude has the same speed, but since China is on the other side of the planet, China’s east is my west. (If you’re finding the three-dimensional visualization difficult, just think of China and California as two freight trains that go past each other, each traveling at 400 m/s.) If I insist on thinking of my dirt as being stationary, then China and its dirt are moving at 800 m/s to my west. From China’s point of view, however, it’s California that is moving 800 m/s in the opposite direction (my east). When I’m pushing the fridge to the west at 0.1 m/s, the observer in China describes its speed as 799.9 m/s. Once I stop pushing, the fridge speeds back up to 800
m/s. From my point of view, the fridge “naturally” slowed down when I stopped pushing, but according to the observer in China, it “naturally” sped up!

What’s really happening here is that there’s a tendency, due to friction, for the fridge to stop moving relative to the floor. In general, only relative motion has physical significance in physics, not absolute motion. It’s not even possible to define absolute motion, since there is no special reference point in the universe that everyone can agree is at rest. Of course if we want to measure motion, we do have to pick some arbitrary reference point which we will say is standing still, and we can then define $x$, $y$, and $z$ coordinates extending out from that point, which we can define as having $x = 0$, $y = 0$, $z = 0$. Setting up such a system is known as choosing a frame of reference. The local dirt is a natural frame of reference for describing a game of basketball, but if the game was taking place on the deck of a moving ocean liner, we would probably pick a frame of reference in which the deck was at rest, and the land was moving.

Galileo was the first scientist to reason along these lines, and we now use the term Galilean relativity to refer to a somewhat modernized version of his principle. Roughly speaking, the principle of Galilean relativity states that the same laws of physics apply in any frame of reference that is moving in a straight line at constant speed. We need to refine this statement, however, since it is not necessarily obvious which frames of reference are going in a straight line at constant speed. A person in a pickup truck pulling away from a stoplight could admit that the car’s velocity is changing, or she could insist that the truck is at rest, and the meter on the dashboard is going up because the asphalt picked that moment to start moving faster and faster backward! Frames of reference are not all created equal, however, and the accelerating truck’s frame of reference is not as good as the asphalt’s. We can tell, because a bowling ball in the back of the truck, as in figure c, appears to behave strangely in the driver’s frame of reference: in her rear-view mirror, she sees the ball, initially at rest, start moving faster and faster toward the back of the truck. This goofy behavior is evidence that there is something wrong with her frame of reference. A person on the sidewalk, however, sees the ball as standing still. In the sidewalk’s frame of reference, the truck pulls away from the ball, and this makes sense, because the truck is burning gas and using up energy to change its state of motion.

We therefore define an inertial frame of reference as one in which we never see objects change their state of motion without any apparent reason. The sidewalk is a pretty good inertial frame, and a car moving relative to the sidewalk at constant speed in a straight line defines a pretty good inertial frame, but a car that is accelerating or turning is not a inertial frame.
The principle of Galilean relativity states that inertial frames exist, and that the same laws of physics apply in all inertial frames of reference, regardless of one frame’s straight-line, constant-speed motion relative to another.\(^4\)

Another way of putting it is that all inertial frames are created equal. We can say whether one inertial frame is in motion or at rest relative to another, but there is no privileged “rest frame.” There is no experiment that comes out any different in laboratories in different inertial frames, so there is no experiment that could tell us which inertial frame is really, truly at rest.

\[\text{The speed of sound} \quad \text{example 3}\]

- The speed of sound in air is only 340 m/s, so unless you live at a near-polar latitude, you’re moving at greater than the speed of sound right now due to the Earth’s rotation. In that case, why don’t we experience exciting phenomena like sonic booms all the time? – It might seem as though you’re unprepared to deal with this question right now, since the only law of physics you know is conservation of mass, and conservation of mass doesn’t tell you anything obviously useful about the speed of sound or sonic booms. Galilean relativity, however, is a blanket statement about all the laws of physics, so in a situation like this, it may let you predict the results of the laws of physics without actually knowing what all the laws are! If the laws of physics predict a certain value for the speed of sound, then they had better predict the speed

\(^4\)The principle of Galilean relativity is extended on page 195.
of the sound relative to the air, not their speed relative to some special “rest frame.” Since the air is moving along with the rotation of the earth, we don’t detect any special phenomena. To get a sonic boom, the source of the sound would have to be moving relative to the air.

The Foucault pendulum example 4
Note that in the example of the bowling ball in the truck, I didn’t claim the sidewalk was exactly a Galilean frame of reference. This is because the sidewalk is moving in a circle due to the rotation of the Earth, and is therefore changing the direction of its motion continuously on a 24-hour cycle. However, the curve of the motion is so gentle that under ordinary conditions we don’t notice that the local dirt’s frame of reference isn’t quite inertial. The first demonstration of the noninertial nature of the earth-fixed frame of reference was by Foucault using a very massive pendulum (figure d) whose oscillations would persist for many hours without becoming imperceptible. Although Foucault did his demonstration in Paris, it’s easier to imagine what would happen at the north pole: the pendulum would keep swinging in the same plane, but the earth would spin underneath it once every 24 hours. To someone standing in the snow, it would appear that the pendulum’s plane of motion was twisting. The effect at latitudes less than 90 degrees turns out to be slower, but otherwise similar. The Foucault pendulum was the first definitive experimental proof that the earth really did spin on its axis, although scientists had been convinced of its rotation for a century based on more indirect evidence about the structure of the solar system.

Although popular belief has Galileo being prosecuted by the Catholic Church for saying the earth rotated on its axis and also orbited the sun, Foucault’s pendulum was still centuries in the future, so Galileo had no hard proof; Galileo’s insights into relative versus absolute motion simply made it more plausible that the world could be spinning without producing dramatic effects, but didn’t disprove the contrary hypothesis that the sun, moon, and stars went around the earth every 24 hours. Furthermore, the Church was much more liberal and enlightened than most people believe. It didn’t (and still doesn’t) require a literal interpretation of the Bible, and one of the Church officials involved in the Galileo affair wrote that “the Bible tells us how to go to heaven, not how the heavens go.” In other words, religion and science should be separate. The actual reason Galileo got in trouble is shrouded in mystery, since Italy in the age of the Medicis was a secretive place where unscrupulous people might settle a score with poison or a false accusation of heresy. What is certain is that Galileo’s satirical style of scientific writing made many enemies among the powerful Jesuit scholars who were his intellectual opponents — he compared one to a snake that doesn’t know its own back is broken. It’s also possible that the Church was far
less upset by his astronomical work than by his support for atomism (discussed further in the next section). Some theologians perceived atomism as contradicting transubstantiation, the Church’s doctrine that the holy bread and wine were literally transformed into the flesh and blood of Christ by the priest’s blessing.

**self-check C**
What is incorrect about the following supposed counterexamples to the principle of inertia?

1. When astronauts blast off in a rocket, their huge velocity does cause a physical effect on their bodies — they get pressed back into their seats, the flesh on their faces gets distorted, and they have a hard time lifting their arms.

2. When you’re driving in a convertible with the top down, the wind in your face is an observable physical effect of your absolute motion.  
Answer, p. 1054

**Solved problem: a bug on a wheel**  
page 71, problem 12

**Discussion Questions**

A  A passenger on a cruise ship finds, while the ship is docked, that he can leap off of the upper deck and just barely make it into the pool on the lower deck. If the ship leaves dock and is cruising rapidly, will this adrenaline junkie still be able to make it?

B  You are a passenger in the open basket hanging under a helium balloon. The balloon is being carried along by the wind at a constant velocity. If you are holding a flag in your hand, will the flag wave? If so, which way? [Based on a question from PSSC Physics.]

C  Aristotle stated that all objects naturally wanted to come to rest, with the unspoken implication that “rest” would be interpreted relative to the surface of the earth. Suppose we could transport Aristotle to the moon, put him in a space suit, and kick him out the door of the spaceship and into the lunar landscape. What would he expect his fate to be in this situation? If intelligent creatures inhabited the moon, and one of them independently came up with the equivalent of Aristotelian physics, what would they think about objects coming to rest?

D  Sally is on an amusement park ride which begins with her chair being hoisted straight up a tower at a constant speed of 60 miles/hour. Despite stern warnings from her father that he’ll take her home the next time she misbehaves, she decides that as a scientific experiment she really needs to release her corndog over the side as she’s on the way up. She does not throw it. She simply sticks it out of the car, lets it go, and watches it against the background of the sky, with no trees or buildings as reference points. What does the corndog’s motion look like as observed by Sally? Does its speed ever appear to her to be zero? What acceleration does she observe it to have: is it ever positive? negative? zero? What would her enraged father answer if asked for a similar description of its motion as it appears to him, standing on the ground?
1.3.1 Applications of calculus

Let’s see how this relates to calculus. If an object is moving in one dimension, we can describe its position with a function \( x(t) \). The derivative \( v = \frac{dx}{dt} \) is called the velocity, and the second derivative \( a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \) is the acceleration. Galilean relativity tells us that there is no detectable effect due to an object’s absolute velocity, since in some other frame of reference, the object’s velocity might be zero. However, an acceleration does have physical consequences.

Observers in different inertial frames of reference will disagree on velocities, but agree on accelerations. Let’s keep it simple by continuing to work in one dimension. One frame of reference uses a coordinate system \( x_1 \), and the other we label \( x_2 \). If the positive \( x_1 \) and \( x_2 \) axes point in the same direction, then in general two inertial frames could be related by an equation of the form \( x_2 = x_1 + b + ut \), where \( u \) is the constant velocity of one frame relative to the other, and the constant \( b \) tells us how far apart the origins of the two coordinate systems were at \( t = 0 \). The velocities are different in the two frames of reference:

\[
\frac{dx_2}{dt} = \frac{dx_1}{dt} + u,
\]

Suppose, for example, frame 1 is defined from the sidewalk, and frame 2 is fixed to a float in a parade that is moving to our left at a velocity \( u = 1 \text{ m/s} \). A dog that is moving to the right with a velocity \( v_1 = \frac{dx_1}{dt} = 3 \text{ m/s} \) in the sidewalk’s frame will appear to be moving at a velocity of \( v_2 = \frac{dx_2}{dt} = \frac{dx_1}{dt} + u = 4 \text{ m/s} \) in the float’s frame.

---

\( i / \) This Air Force doctor volunteered to ride a rocket sled as a medical experiment. The obvious effects on his head and face are not because of the sled’s speed but because of its rapid changes in speed: increasing in (ii) and (iii), and decreasing in (v) and (vi). In (iv) his speed is greatest, but because his speed is not increasing or decreasing very much at this moment, there is little effect on him. (U.S. Air Force)
For acceleration, however, we have
\[
\frac{d^2 x_2}{dt^2} = \frac{d^2 x_1}{dt^2},
\]
since the derivative of the constant \( u \) is zero. Thus an acceleration, unlike a velocity, can have a definite physical significance to all observers in all frames of reference. If this wasn’t true, then there would be no particular reason to define a quantity called acceleration in the first place.

**self-check D**

The figure shows a bottle of beer sitting on a table in the dining car of a train. Does the tilting of the surface tell us about the train’s velocity, or its acceleration? What would a person in the train say about the bottle’s velocity? What about a person standing in a field outside and looking in through the window? What about the acceleration? ▶ Answer, p. 1054

### 1.4 A preview of some modern physics

“Mommy, why do you and Daddy have to go to work?” “To make money, sweetie-pie.” “Why do we need money?” “To buy food.” “Why does food cost money?” When small children ask a chain of “why” questions like this, it usually isn’t too long before their parents end up saying something like, “Because that’s just the way it is,” or, more honestly, “I don’t know the answer.”

The same happens in physics. We may gradually learn to explain things more and more deeply, but there’s always the possibility that a certain observed fact, such as conservation of mass, will never be understood on any deeper level. Science, after all, uses limited methods to achieve limited goals, so the ultimate reason for all existence will always be the province of religion. There is, however, an appealing explanation for conservation of mass, which is atomism, the theory that matter is made of tiny, unchanging particles. The atomic hypothesis dates back to ancient Greece, but the first solid evidence to support it didn’t come until around the eighteenth century, and individual atoms were never detected until about 1900. The atomic theory implies not only conservation of mass, but a couple of other things as well.

First, it implies that the total mass of one particular element is conserved. For instance, lead and gold are both elements, and if we assume that lead atoms can’t be turned into gold atoms, then the total mass of lead and the total mass of gold are separately conserved. It’s as though there was not just a law against pickpocketing, but also a law against surreptitiously moving money from one of the victim’s pockets to the other. It turns out, however, that although chemical reactions never change one type of atom into another, transmutation can happen in nuclear reactions, such as the
ones that created most of the elements in your body out of the primordial hydrogen and helium that condensed out of the aftermath of the Big Bang.

Second, atomism implies that mass is quantized, meaning that only certain values of mass are possible and the ones in between can’t exist. We can have three atoms of gold or four atoms of gold, but not three and a half. Although quantization of mass is a natural consequence of any theory in which matter is made up of tiny particles, it was discovered in the twentieth century that other quantities, such as energy, are quantized as well, which had previously not been suspected.

**self-check E**

Is money quantized?  [Answer, p. 1054]

If atomism is starting to make conservation of mass seem inevitable to you, then it may disturb you to know that Einstein discovered it isn’t really conserved. If you put a 50-gram iron nail in some water, seal the whole thing up, and let it sit on a fantastically precise balance while the nail rusts, you’ll find that the system loses about $6 \times 10^{-12}$ kg of mass by the time the nail has turned completely to rust. This has to do with Einstein’s famous equation $E = mc^2$. Rusting releases heat energy, which then escapes out into the room. Einstein’s equation states that this amount of heat, $E$, is equivalent to a certain amount of mass, $m$. The $c$ in the $c^2$ is the speed of light, which is a large number, and a large amount of energy is therefore equivalent to a very small amount of mass, so you don’t notice nonconservation of mass under ordinary conditions. What is really conserved is not the mass, $m$, but the mass-plus-energy, $E + mc^2$. The point of this discussion is not to get you to do numerical exercises with $E = mc^2$ (at this point you don’t even know what units are used to measure energy), but simply to point out to you the empirical nature of the laws of physics. If a previously accepted theory is contradicted by an experiment, then the theory needs to be changed. This is also a good example of something called the correspondence principle, which is a historical observation about how scientific theories change: when a new scientific theory replaces an old one, the old theory is always contained within the new one as an approximation that works within a certain restricted range of situations. Conservation of mass is an extremely good approximation for all chemical reactions, since chemical reactions never release or consume enough energy to change the total mass by a large percentage. Conservation of mass would not have been accepted for 110 years as a fundamental principle of physics if it hadn’t been verified over and over again by a huge number of accurate experiments.

*This chapter is summarized on page 1071. Notation and terminology are tabulated on pages 1066-1067.*
Problems

The symbols √, ■, etc. are explained on page 72.

1 Thermometers normally use either mercury or alcohol as their working fluid. If the level of the fluid rises or falls, does this violate conservation of mass?

2 The ratios of the masses of different types of atoms were determined a century before anyone knew any actual atomic masses in units of kg. One finds, for example, that when ordinary table salt, NaCl, is melted, the chlorine atoms bubble off as a gas, leaving liquid sodium metal. Suppose the chlorine escapes, so that its mass cannot be directly determined by weighing. Experiments show that when 1.00000 kg of NaCl is treated in this way, the mass of the remaining sodium metal is 0.39337 kg. Based on this information, determine the ratio of the mass of a chlorine atom to that of a sodium atom.

3 An atom of the most common naturally occurring uranium isotope breaks up spontaneously into a thorium atom plus a helium atom. The masses are as follows:

uranium 3.95292849 × 10⁻²⁵ kg
thorium 3.88638748 × 10⁻²⁵ kg
helium 6.646481 × 10⁻²⁷ kg

Each of these experimentally determined masses is uncertain in its last decimal place. Is mass conserved in this process to within the accuracy of the experimental data? How would you interpret this?

4 If two spherical water droplets of radius \( b \) combine to make a single droplet, what is its radius? (Assume that water has constant density.)

5 Make up an experiment that would test whether mass is conserved in an animal’s metabolic processes.

6 The figure shows a hydraulic jack. What is the relationship between the distance traveled by the plunger and the distance traveled by the object being lifted, in terms of the cross-sectional areas?

7 In an example in this chapter, I argued that a stream of water must change its cross-sectional area as it rises or falls. Suppose that the stream of water is confined to a constant-diameter pipe. Which assumption breaks down in this situation?

8 A river with a certain width and depth splits into two parts, each of which has the same width and depth as the original river. What can you say about the speed of the current after the split?

9 The diagram shows a cross-section of a wind tunnel of the
kind used, for example, to test designs of airplanes. Under normal conditions of use, the density of the air remains nearly constant throughout the whole wind tunnel. How can the speed of the air be controlled and calculated? (Diagram by NASA, Glenn Research Center.)

10 A water wave is in a tank that extends horizontally from $x = 0$ to $x = a$, and from $z = 0$ to $z = b$. We assume for simplicity that at a certain moment in time the height $y$ of the water’s surface only depends on $x$, not $z$, so that we can effectively ignore the $z$ coordinate. Under these assumptions, the total volume of the water in the tank is

$$V = b \int_0^a y(x) \, dx.$$ 

Since the density of the water is essentially constant, conservation of mass requires that $V$ is always the same. When the water is calm, we have $y = h$, where $h = V/ab$. If two different wave patterns move into each other, we might imagine that they would add in the sense that $y_{total} - h = (y_1 - h) + (y_2 - h)$. Show that this type of addition is consistent with conservation of mass.

11 The figure shows the position of a falling ball at equal time intervals, depicted in a certain frame of reference. On a similar grid, show how the ball’s motion would appear in a frame of reference that was moving horizontally at a speed of one box per unit time relative to the first frame.

12 The figure shows the motion of a point on the rim of a rolling wheel. (The shape is called a cycloid.) Suppose bug A is riding on the rim of the wheel on a bicycle that is rolling, while bug B is on the spinning wheel of a bike that is sitting upside down on the floor. Bug A is moving along a cycloid, while bug B is moving in a circle. Both wheels are doing the same number of revolutions per minute. Which bug has a harder time holding on, or do they find it equally
Key to symbols:

- easy
- typical
- challenging
- difficult
- very difficult

An answer check is available at www.lightandmatter.com.
Chapter 2
Conservation of Energy

Do you pronounce it Joule’s to rhyme with schools, Joule’s to rhyme with Bowls, or Joule’s to rhyme with Scowls? Whatever you call it, by Joule’s, or Joule’s, or Joule’s, it’s good!

Advising slogan of the Joule brewery. The name, and the corresponding unit of energy, are now usually pronounced so as to rhyme with “school.”

2.1 Energy

2.1.1 The energy concept

You’d probably like to be able to drive your car and light your apartment without having to pay money for gas and electricity, and if you do a little websurfing, you can easily find people who say they have the solution to your problem. This kind of scam has been around for centuries. It used to be known as a perpetual motion machine, but nowadays the con artists’ preferred phrase is “free energy.”¹ A typical “free-energy” machine would be a sealed box that heats your house without needing to be plugged into a wall socket or a gas pipe. Heat comes out, but nothing goes in, and this can go on indefinitely. But an interesting thing happens if you try to check on the advertised performance of the machine. Typically, you’ll find out that either the device is still in development, or it’s back-ordered because so many people have already taken advantage of this Fantastic Opportunity! In a few cases, the magic box exists, but the inventor is only willing to demonstrate very small levels of heat output for short periods of time, in which case there’s probably a tiny hearing-aid battery hidden in there somewhere, or some other trick.

Since nobody has ever succeeded in building a device that creates heat out of nothing, we might also wonder whether any device exists

¹An entertaining account of this form of quackery is given in Voodoo Science: The Road from Foolishness to Fraud, Robert Park, Oxford University Press, 2000. Until reading this book, I hadn’t realized the degree to which pseudoscience had penetrated otherwise respectable scientific organizations like NASA.
Heat energy can be converted to light energy. Very hot objects glow visibly, and even objects that aren’t so hot give off infrared light, a color of light that lies beyond the red end of the visible rainbow. This photo was made with a special camera that records infrared light. The man’s warm skin emits quite a bit of infrared light energy, while his hair, at a lower temperature, emits less.

That can do the opposite, turning heat into nothing. You might think that a refrigerator was such a device, but actually your refrigerator doesn’t destroy the heat in the food. What it really does is to extract some of the heat and bring it out into the room. That’s why it has big radiator coils on the back, which get hot when it’s in operation.

If it’s not possible to destroy or create heat outright, then you might start to suspect that heat was a conserved quantity. This would be a successful rule for explaining certain processes, such as the transfer of heat between a cold Martini and a room-temperature olive: if the olive loses a little heat, then the drink must gain the same amount. It would fail in general, however. Sunlight can heat your skin, for example, and a hot lightbulb filament can cool off by emitting light. Based on these observations, we could revise our proposed conservation law, and say that there is something called heatpluslight, which is conserved. Even this, however, needs to be generalized in order to explain why you can get a painful burn playing baseball when you slide into a base. Now we could call it heatpluslightplusmotion. The word is getting pretty long, and we haven’t even finished the list.

Rather than making the word longer and longer, physicists have hijacked the word “energy” from ordinary usage, and give it a new, specific technical meaning. Just as the Parisian platinum-iridium kilogram defines a specific unit of mass, we need to pick something that defines a definite unit of energy. The metric unit of energy is the joule (J), and we’ll define it as the amount of energy required to heat 0.24 grams of water from 20 to 21 degrees Celsius. (Don’t memorize the numbers.)²

Temperature of a mixture example 1

If 1.0 kg of water at 20°C is mixed with 4.0 kg of water at 30°C, what is the temperature of the mixture?

Let’s assume as an approximation that each degree of temperature change corresponds to the same amount of energy. In other words, we assume \( \Delta E = mc \Delta T \), regardless of whether, as in the definition of the joule, we have \( \Delta T = 21^\circ\text{C}-20^\circ\text{C} \) or, as in the present example, some other combination of initial and final temperatures. To be consistent with the definition of the joule, we must have \( c = (1 \text{ J})/(0.24 \text{ g})/(1^\circ\text{C}) = 4.2 \times 10^3 \text{ J/kg} \cdot ^\circ\text{C} \), which is referred to as the specific heat of water.

²Although the definition refers to the Celsius scale of temperature, it’s not necessary to give an operational definition of the temperature concept in general (which turns out to be quite a tricky thing to do completely rigorously); we only need to establish two specific temperatures that can be reproduced on thermometers that have been calibrated in a standard way. Heat and temperature are discussed in more detail in section 2.4, and in chapter 5. Conceptually, heat is a measure of energy, whereas temperature relates to how concentrated that energy is.
Conservation of energy tells us $\Delta E = 0$, so

$$m_1 c \Delta T_1 + m_2 c \Delta T_2 = 0$$

or

$$\frac{\Delta T_1}{\Delta T_2} = -\frac{m_2}{m_1} = -4.0$$.

If $T_1$ has to change four times as much as $T_2$, and the two final temperatures are equal, then the final temperature must be $28^\circ C$.

Note how only differences in temperature and energy appeared in the preceding example. In other words, we don’t have to make any assumptions about whether there is a temperature at which all an object’s heat energy is removed. Historically, the energy and temperature units were invented before it was shown that there is such a temperature, called absolute zero. There is a scale of temperature, the Kelvin scale, in which the unit of temperature is the same as the Celsius degree, but the zero point is defined as absolute zero. But as long as we only deal with temperature differences, it doesn’t matter whether we use Kelvin or Celsius. Likewise, as long as we deal with differences in heat energy, we don’t normally have to worry about the total amount of heat energy the object has. In standard physics terminology, “heat” is used only to refer to differences, while the total amount is called the object’s “thermal energy.” This distinction is often ignored by scientists in casual speech, and in this book I’ll usually use “heat” for either quantity.

We’re defining energy by adding up things from a list, which we lengthen as needed: heat, light, motion, etc. One objection to this approach is aesthetic: physicists tend to regard complication as a synonym for ugliness. If we have to keep on adding more and more forms of energy to our laundry list, then it’s starting to sound like energy is distressingly complicated. Luckily it turns out that energy is simpler than it seems. Many forms of energy that are apparently unrelated turn out to be manifestations of a small number of forms at the atomic level, and this is the topic of section 2.4.

Discussion Questions

A The ancient Greek philosopher Aristotle said that objects “naturally” tended to slow down, unless there was something pushing on them to keep them moving. What important insight was he missing?

2.1.2 Logical issues

Another possible objection is that the open-ended approach to defining energy might seem like a kind of cheat, since we keep on inventing new forms whenever we need them. If a certain experiment seems to violate conservation of energy, can’t we just invent
As in figure b, an infrared camera distinguishes hot and cold areas. As the bike skids to a stop with its brakes locked, the kinetic energy of the bike and rider is converted into heat in both the floor (top) and the tire (bottom).

Actually all scientific theories are unprovable. A theory can never be proved, because the experiments can only cover a finite number out of the infinitely many situations in which the theory is supposed to apply. Even a million experiments won’t suffice to prove it in the same sense of the word “proof” that is used in mathematics. However, even one experiment that contradicts a theory is sufficient to show that the theory is wrong. A theory that is immune to disproof is a bad theory, because there is no way to test it. For instance, if I say that 23 is the maximum number of angels that can dance on the head of a pin, I haven’t made a properly falsifiable scientific theory, since there’s no method by which anyone could even attempt to prove me wrong based on observations or experiments.

Conservation of energy is testable because new forms of energy are expected to show regular mathematical behavior, and are supposed to be related in a measurable way to observable phenomena. As an example, let’s see how to extend the energy concept to include motion.

2.1.3 Kinetic energy

Energy of motion is called kinetic energy. (The root of the word is the same as the word “cinema” – in French, kinetic energy is “énergie cinétique.”) How does an object’s kinetic energy depend on its mass and velocity? Joule attempted a conceptually simple experiment on his honeymoon in the French-Swiss Alps near Mt. Chamonix, in which he measured the difference in temperature between the top and bottom of a waterfall. The water at the top of the falls has some gravitational energy, which isn’t our subject right now, but as it drops, that gravitational energy is converted into kinetic energy, and then into heat energy due to internal friction in the churning pool at the bottom:

\[
\text{gravitational energy} \rightarrow \text{kinetic energy} \rightarrow \text{heat energy}
\]

In the logical framework of this book’s presentation of energy, the significance of the experiment is that it provides a way to find out how an object’s kinetic energy depends on its mass and velocity. The increase in heat energy should equal the kinetic energy of the water just before impact, so in principle we could measure the water’s mass, velocity, and kinetic energy, and see how they relate to one another.\(^3\)

\(^3\)From Joule’s point of view, the point of the experiment was different. At that time, most physicists believed that heat was a quantity that was conserved...
Although the story is picturesque and memorable, most books that mention the experiment fail to note that it was a failure! The problem was that heat wasn’t the only form of energy being released. In reality, the situation was more like this:

gravitational energy $\rightarrow$ kinetic energy $\rightarrow$ heat energy $+$ sound energy $+$ energy of partial evaporation.

The successful version of the experiment, shown in figures d and f, used a paddlewheel spun by a dropping weight. As with the waterfall experiment, this one involves several types of energy, but the difference is that in this case, they can all be determined and taken into account. (Joule even took the precaution of putting a screen between himself and the can of water, so that the infrared light emitted by his warm body wouldn’t warm it up at all!) The result is

$$ K = \frac{1}{2}mv^2 \quad \text{[kinetic energy]}.$$ 

Whenever you encounter an equation like this for the first time, you should get in the habit of interpreting it. First off, we can tell that by making the mass or velocity greater, we’d get more kinetic energy. That makes sense. Notice, however, that we have mass to the first power, but velocity to the second. Having the whole thing proportional to mass to the first power is necessary on theoretical grounds, since energy is supposed to be additive. The dependence on $v^2$ couldn’t have been predicted, but it is sensible. For instance, suppose we reverse the direction of motion. This would reverse the sign of $v$, because in one dimension we use positive and negative signs to indicate the direction of motion. But since $v^2$ is what appears in the equation, the resulting kinetic energy is unchanged.

Separately from the rest of the things to which we now refer as energy, i.e., mechanical energy. Separate units of measurement had been constructed for heat and mechanical energy, but Joule was trying to show that one could convert back and forth between them, and that it was actually their sum that was conserved, if they were both expressed in consistent units. His main result was the conversion factor that would allow the two sets of units to be reconciled. By showing that the conversion factor came out the same in different types of experiments, he was supporting his assertion that heat was not separately conserved. From Joule’s perspective or from ours, the result is to connect the mysterious, invisible phenomenon of heat with forms of energy that are visible properties of objects, i.e., mechanical energy.

If you’ve had a previous course in physics, you may have seen this presented not as an empirical result but as a theoretical one, derived from Newton’s laws, and in that case you might feel you’re being cheated here. However, I’m going to reverse that reasoning and derive Newton’s laws from the conservation laws in chapter 3. From the modern perspective, conservation laws are more fundamental, because they apply in cases where Newton’s laws don’t.
A realistic drawing of Joule’s apparatus, based on the illustration in his original paper. The paddlewheel is sealed inside the can in the middle. Joule wound up the two 13-kg lead weights and dropped them 1.6 meters, repeating this 20 times to produce a temperature change of only about half a degree Fahrenheit in the water inside the sealed can. He claimed in his paper to be able to measure temperatures to an accuracy of 1/200 of a degree.

What about the factor of 1/2 in front? It comes out to be exactly 1/2 by the design of the metric system. If we’d been using the old-fashioned British engineering system of units (which is no longer used in the U.K.), the equation would have been \( K = (7.44 \times 10^{-2} \text{ Btu} \cdot \text{s}^2/\text{slug} \cdot \text{ft}^2)mv^2 \). The version of the metric system called the SI,\(^5\) in which everything is based on units of kilograms, meters, and seconds, not only has the numerical constant equal to 1/2, but makes it unitless as well. In other words, we can think of the joule as simply an abbreviation, \( 1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2 \). More familiar examples of this type of abbreviation are 1 minute=60 s, and the metric unit of land area, 1 hectare=10000 m\(^2\).

\(^5\)Système International

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Example 2

There used to be two commonly used systems of metric units, referred to as mks and cgs. The mks system, now called the SI, is based on the meter, the kilogram, and the second. The cgs system, which is now obsolete, was based on the centimeter, the gram, and the second. In the cgs system, the unit of energy is not the joule but the erg, \( 1 \text{ erg} = 1 \text{ g} \cdot \text{cm}^2/\text{s}^2 \). How many ergs are in one joule?

The simplest approach is to treat the units as if they were alge-

---

\(^5\)Système International
bra symbols.

\[
1 \text{ J} = 1 \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2}
\]

\[
= 1 \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2} \times \frac{1000 \text{ g}}{1 \text{ kg}} \times \left( \frac{100 \text{ cm}}{1 \text{ m}} \right)^2
\]

\[
= 10^7 \frac{\text{g} \cdot \text{cm}^2}{\text{s}^2}
\]

\[
= 10^7 \text{ erg}
\]

---

Cabin air in a jet airplane example 3

A jet airplane typically cruises at a velocity of 270 m/s. Outside air is continuously pumped into the cabin, but must be cooled off first, both because (1) it heats up due to friction as it enters the engines, and (2) it is heated as a side-effect of being compressed to cabin pressure. Calculate the increase in temperature due to the first effect. The specific heat of dry air is about \(1.0 \times 10^3\) J/kg·°C.

This is easiest to understand in the frame of reference of the plane, in which the air rushing into the engine is stopped, and its kinetic energy converted into heat.\(^6\) Conservation of energy tells us

\[
0 = \Delta E = \Delta K + \Delta E_{\text{heat}}.
\]

In the plane’s frame of reference, the air’s initial velocity is \(v_i = 270\) m/s, and its final velocity is zero, so the change in its kinetic energy is negative,

\[
\Delta K = K_f - K_i
\]

\[
= 0 - \frac{1}{2}mv_i^2
\]

\[
= -\frac{1}{2}mv_i^2.
\]

Assuming that the specific heat of air is roughly independent of temperature (which is why the number was stated with the word “about”), we can substitute into \(0 = \Delta K + \Delta E_{\text{heat}}\), giving

\[
0 = -\frac{1}{2}mv_i^2 + mc\Delta T
\]

\[
\frac{1}{2}v_i^2 = c\Delta T.
\]

Note how the mass cancels out. This is a big advantage of solving problems algebraically first, and waiting until the end to plug in

\(^6\) It’s not at all obvious that the solution would work out in the earth’s frame of reference, although Galilean relativity states that it doesn’t matter which frame we use. Chapter 3 discusses the relationship between conservation of energy and Galilean relativity.
numbers. With a purely numerical approach, we wouldn't even have known what value of \( m \) to pick, or if we'd guessed a value like 1 kg, we wouldn't have known whether our answer depended on that guess.

Solving for \( \Delta T \), and writing \( v \) instead of \( v_i \) for simplicity, we find

\[
\Delta T = \frac{v^2}{2c} \approx 40^\circ C.
\]

The passengers would be boiled alive if not for the refrigeration. The first stage of cooling happens via heat exchangers in the engine struts, but a second stage, using a refrigerator under the floor of the cabin, is also necessary. Running this refrigerator uses up energy, cutting into the fuel efficiency of the airplane, which is why typically only 50% of the cabin's air is replaced in each pumping cycle of 2-3 minutes. Although the airlines prefer to emphasize that this is a much faster recirculation rate than in the ventilation systems of most buildings, people are packed more tightly in an airplane.

2.1.4 Power

Power, \( P \), is defined as the rate of change of energy, \( \frac{dE}{dt} \). Power thus has units of joules per second, which are usually abbreviated as watts, 1 W=1 J/s. Since energy is conserved, we would have \( \frac{dE}{dt} = 0 \) if \( E \) was the total energy of a closed system, and that's not very interesting. What's usually more interesting to discuss is either the power flowing in or out of an open system, or the rate at which energy is being transformed from one form into another. The following is an example of energy flowing into an open system.

\( \text{Heating by a lightbulb} \)

\( \triangleright \) The electric company bills you for energy in units of kilowatt-hours (kilowatts multiplied by hours) rather than in SI units of joules. How many joules is a kilowatt-hour?

\( \triangleright \) 1 kilowatt-hour = (1 kW)(1 hour) = (1000 J/s)(3600 s) = 3.6 MJ.

Now here's an example of energy being transformed from one form into another.

\( \text{Human wattage} \)

\( \triangleright \) Food contains chemical energy (discussed in more detail in section 2.4), and for historical reasons, food energy is normally given in non-SI units of Calories. One Calorie with a capital “C” equals 1000 calories, and 1 calorie is defined as 4.18 J. A typical person consumes 2000 Calories of food in a day, and converts nearly all of that directly to body heat. Compare the person's heat production to the rate of energy consumption of a 100-watt lightbulb.

\( \triangleright \) Strictly speaking, we can't really compute the derivative \( \frac{dE}{dt} \),
since we don’t know how the person’s metabolism ebbs and flows over the course of a day. What we can really compute is $\Delta E / \Delta t$, which is the power averaged over a one-day period.

Converting to joules, we find $\Delta E = 8 \times 10^6$ J for the amount of energy transformed into heat within our bodies in one day. Converting the time interval likewise into SI units, $\Delta t = 9 \times 10^4$ s. Dividing, we find that our power is $90$ J/s = $90$ W, about the same as a lightbulb.

### 2.1.5 Gravitational energy

Gravitational energy, to which I’ve already alluded, is different from heat and kinetic energy in an important way. Heat and kinetic energy are properties of a single object, whereas gravitational energy describes an interaction between two objects. When the skater in figures g and h is at the top, his distance from the bulk of the planet earth is greater. Since we observe his kinetic energy decreasing on the way up, there must be some other form of energy that is increasing. We invent a new form of energy, called gravitational energy, and written $U$ or $U_g$, which depends on the distance between his body and the planet. Where is this energy? It’s not in the skater’s body, and it’s not inside the earth, either, since it takes two to tango. If either object didn’t exist, there wouldn’t be any interaction or any way to measure a distance, so it wouldn’t make sense to talk about a distance-dependent energy. Just as marriage is a relationship between two people, gravitational energy is a relationship between two objects.

There is no precise way to define the distance between the skater and the earth, since both are objects that have finite size. As discussed in more detail in section 2.3, gravity is one of the fundamental forces of nature, a universal attraction between any two particles that have mass. Each atom in the skater’s body is at a definite distance from each atom in the earth, but each of these distances is different. An atom in his foot is only a few centimeters from some of the atoms in the plaster side of the pool, but most of the earth’s atoms are thousands of kilometers away from him. In theory, we might have to add up the contribution to the gravitational energy for every interaction between an atom in the skater’s body and an atom in the earth.

For our present purposes, however, there is a far simpler and more practical way to solve problems. In any region of the earth’s surface, there is a direction called “down,” which we can establish by dropping a rock or hanging a plumb bob. In figure h, the skater is moving up and down in one dimension, and if we did measurements of his kinetic energy, like the made-up data in the figure, we could infer his gravitational energy. As long as we stay within a relatively small range of heights, we find that an object’s gravitational energy increases at a steady rate with height. In other words, the strength
of gravity doesn’t change much if you only move up or down a few meters. We also find that the gravitational energy is proportional to the mass of the object we’re testing. Writing \( y \) for the height, and \( g \) for the overall constant of proportionality, we have

\[
U_g = mgy. \quad \text{[gravitational energy; } y=\text{height; only accurate within a small range of heights]}
\]

The number \( g \), with units of joules per kilogram per meter, is called the gravitational field. It tells us the strength of gravity in a certain region of space. Near the surface of our planet, it has a value of about 9.8 J/kg·m, which is conveniently close to 10 J/kg·m for rough calculations.

**Example 6: Velocity at the bottom of a drop**

If the skater in figure g drops 3 meters from rest, what is his velocity at the bottom of the pool?

Starting from conservation of energy, we have

\[
0 = \Delta E = \Delta K + \Delta U = K_f - K_i + U_f - U_i = \frac{1}{2}mv_f^2 + mgy_f - mgy_i \quad \text{(because } K_i=0) = \frac{1}{2}mv_i^2 + mg\Delta y, \quad (\Delta y <0)
\]

so

\[
v = \sqrt{-2g\Delta y} = \sqrt{-2(10 \text{ J/kg} \cdot \text{m})(-3 \text{ m})} = 8 \text{ m/s} \quad \text{(rounded to one sig. fig.)}
\]

There are a couple of important things to note about this example. First, we were able to massage the equation so that it only involved \( \Delta y \), rather than \( y \) itself. In other words, we don’t need to worry about where \( y = 0 \) is; any coordinate system will work, as long as the positive \( y \) axis points up, not down. This is no accident. Gravitational energy can always be changed by adding a constant onto it, with no effect on the final result, as long as you’re consistent within a given problem.

The other interesting thing is that the mass canceled out: even if the skater gained weight or strapped lead weights to himself, his velocity at the bottom would still be 8 m/s. This isn’t an accident either. This is the same conclusion we reached in section 1.2, based on the equivalence of gravitational and inertial mass. The kinetic energy depends on the inertial mass, while gravitational energy is
related to gravitational mass, but since these two masses are equal, we were able to use a single symbol, \( m \), for them, and cancel them out.

We can see from the equation \( v = \sqrt{-2g \Delta y} \) that a falling object’s velocity isn’t constant. It increases as the object drops farther and farther. What about its acceleration? If we assume that air friction is negligible, the arguments in section 1.2 show that the acceleration can’t depend on the object’s mass, so there isn’t much else the acceleration can depend on besides \( g \). In fact, the acceleration of a falling object equals \(-g\) (in a coordinate system where the positive \( y \) axis points up), as we can easily show using the chain rule:

\[
\left( \frac{dv}{dt} \right) = \left( \frac{dv}{dK} \right) \left( \frac{dK}{dU} \right) \left( \frac{dU}{dy} \right) \left( \frac{dy}{dt} \right)
= \left( \frac{1}{mv} \right) (-1)(mg)(v)
= -g,
\]

where I’ve calculated \( dv/dK \) as \( 1/(dK/dv) \), and \( dK/dU = -1 \) can be found by differentiating \( K + U = \text{(constant)} \) to give \( dK + dU = 0 \).

We can also check that the units of \( g \), \( J/kg\cdot m \), are equivalent to the units of acceleration,

\[
\frac{J}{kg\cdot m} = \frac{kg\cdot m^2/s^2}{kg\cdot m} = \frac{m}{s^2},
\]

and therefore the strength of the gravitational field near the earth’s surface can just as well be stated as \( 10 \text{ m/s}^2 \).

\footnote{Speed after a given time example 7}

\( \triangleright \) An object falls from rest. How fast is it moving after two seconds? Assume that the amount of energy converted to heat by air friction is negligible.

\( \triangleright \) Under the stated assumption, we have \( a = -g \), which can be integrated to give \( v = -gt + \text{constant} \). If we let \( t = 0 \) be the beginning of the fall, then the constant of integration is zero, so at \( t = 2 \text{ s} \) we have \( v = -gt = -(10 \text{ m/s}^2) \times (2 \text{ s}) = 20 \text{ m/s} \).

\footnote{The Vomit Comet example 8}

\( \triangleright \) The U.S. Air Force has an airplane, affectionately known as the Vomit Comet, in which astronaut trainees can experience simulated weightlessness. Oversimplifying a little, imagine that the plane climbs up high, and then drops straight down like a rock. (It actually flies along a parabola.) Since the people are falling with the same acceleration as the plane, the sensation is just like what you’d experience if you went out of the earth’s gravitational

\footnote{There is a mathematical loophole in this argument that would allow the object to hover for a while with zero velocity and zero acceleration. This point is discussed on page 1022.}
field. If the plane can start from 10 km up, what is the maximum amount of time for which the dive can last?

Based on data about acceleration and distance, we want to find time. Acceleration is the second derivative of distance, so if we integrate the acceleration twice with respect to time, we can find how position relates to time. For convenience, let’s pick a coordinate system in which the positive y axis is down, so $a=g$ instead of $-g$.

$$a = g$$

$$v = gt + \text{constant} \quad \text{(integrating)}$$

$$v = gt \quad \text{(starts from rest)}$$

$$y = \frac{1}{2}gt^2 + \text{constant} \quad \text{(integrating again)}$$

Choosing our coordinate system to have $y = 0$ at $t = 0$, we can make the second constant of integration equal zero as well, so

$$t = \sqrt{\frac{2y}{g}}$$

$$= \sqrt{\frac{2 \cdot 10000 \text{ m}}{10 \text{ m/s}^2}}$$

$$= \sqrt{2000 \text{ s}^2}$$

$$= 40 \text{ s} \quad \text{(to one sig. fig.)}$$

Note that if we hadn’t converted the altitude to units of meters, we would have gotten the wrong answer, but we would have been alerted to the problem because the units inside the square root wouldn’t have come out to be s$^2$. In general, it’s a good idea to convert all your data into SI (meter-kilogram-second) units before you do anything with them.

**High road, low road**

In figure i, what can you say based on conservation of energy about the speeds of the balls when the reach point B? What does conservation of energy tell you about which ball will get there first? Assume friction doesn’t convert any mechanical energy to heat or sound energy.

Since friction is assumed to be negligible, there are only two forms of energy involved: kinetic and gravitational. Since both balls start from rest, and both lose the same amount of gravitational energy, they must have the same kinetic energy at the end, and therefore they’re rolling at the same speed when they reach B. (A subtle point is that the balls have kinetic energy both because they’re moving through space and because they’re spinning as they roll. These two types of energy must be in fixed
How much energy is required to raise the submerged box through a height $\Delta y$?

As the box moves up, it invades a volume $V' = b^2 \Delta y$ previously occupied by some of the fluid, and fluid flows into an equal volume that it has vacated on the bottom. Lowering this amount of fluid by a height $b$ reduces the fluid's gravitational energy by $\rho V'gb = \rho gb^3 \Delta y$, so the net change in energy is

$$\Delta E = mg\Delta y - \rho gb^3 \Delta y = (m - \rho V)g\Delta y.$$  

In other words, it's as if the mass of the box had been reduced by an amount equal to the fluid that otherwise would have occupied that volume. This is known as Archimedes' principle, and it is true even if the box is not a cube, although we'll defer the more general proof until page 207 in chapter 3. If the box is less dense than the fluid, then it will float.

If the father and son on the seesaw in figure k start from rest, what will happen?

Note that although the father is twice as massive, he is at half the distance from the fulcrum. If the seesaw was going to start rotating, it would have to be losing gravitational energy in order to gain some kinetic energy. However, there is no way for it to gain or lose gravitational energy by rotating in either direction. The change in gravitational energy would be

$$\Delta U = \Delta U_1 + \Delta U_2 = g(m_1 \Delta y_1 + m_2 \Delta y_2),$$

but $\Delta y_1$ and $\Delta y_2$ have opposite signs and are in the proportion of two to one, since the son moves along a circular arc that covers the same angle as the father's but has half the radius. Therefore $\Delta U = 0$, and there is no way for the seesaw to trade gravitational energy for kinetic.
The seesaw example demonstrates the principle of the lever, which is one of the basic mechanical building blocks known as simple machines. As discussed in more detail in chapters 3 and 4, the principle applies even when the interactions involved aren’t gravitational.

Note that although a lever makes it easier to lift a heavy weight, it also decreases the distance traveled by the load. By reversing the lever, we can make the load travel a greater distance, at the expense of increasing the amount of force required. The human muscular-skeletal system uses reversed levers of this kind, which allows us to move more rapidly, and also makes our bodies more compact, at the expense of brute strength. A piano uses reversed levers so that a small amount of motion of the key produces a longer swing of the hammer. Another interesting example is the hydraulic jack shown in figure n. The analysis in terms of gravitational energy is exactly the same as for the seesaw, except that the relationship between $\Delta y_1$ and $\Delta y_2$ is now determined not by geometry but by conservation of mass: since water is highly incompressible, conservation of mass is approximately the same as a requirement of constant volume, which can only be satisfied if the distance traveled by each piston is in inverse proportion to its cross-sectional area.

**Discussion Questions**

A Hydroelectric power (water flowing over a dam to spin turbines) appears to be completely free. Does this violate conservation of energy? If not, then what is the ultimate source of the electrical energy produced by a hydroelectric plant?

B You throw a steel ball up in the air. How can you prove based on conservation of energy that it has the same speed when it falls back into your hand? What if you threw a feather up? Is energy not conserved in this case?

C Figure m shows a pendulum that is released at A and caught by a peg as it passes through the vertical, B. To what height will the bob rise on the right?

D What is wrong with the following definitions of $g$?
(a) “$g$ is gravity.”
(b) “$g$ is the speed of a falling object.”
(c) “$g$ is how hard gravity pulls on things.”

2.1.6 Equilibrium and stability

The seesaw in figure k is in equilibrium, meaning that if it starts out being at rest, it will stay put. This is known as a neutral equilibrium, since the seesaw has no preferred position to which it will return if we disturb it. If we move it to a different position and release it, it will stay at rest there as well. If we put it in motion, it will simply continue in motion until one person’s feet hit the ground.

Most objects around you are in stable equilibria, like the black
block in figure o/3. Even if the block is moved or set in motion, it will oscillate about the equilibrium position. The pictures are like graphs of $y$ versus $x$, but since the gravitational energy $U = mgy$ is proportional to $y$, we can just as well think of them as graphs of $U$ versus $x$. The block’s stable equilibrium position is where the function $U(x)$ has a local minimum. The book you’re reading right now is in equilibrium, but gravitational energy isn’t the only form of energy involved. To move it upward, we’d have to supply gravitational energy, but downward motion would require a different kind of energy, in order to compress the table more. (As we’ll see in section 2.4, this is electrical energy due to interactions between atoms within the table.)

A differentiable function’s local extrema occur where its derivative is zero. A position where $dU/dx$ is zero can be a stable (3), neutral (2), or unstable equilibrium, (4). An unstable equilibrium is like a pencil balanced on its tip. Although it could theoretically remain balanced there forever, in reality it will topple due to any tiny perturbation, such as an air current or a vibration from a passing truck. This is a technical, mathematical definition of instability, which is more restrictive than the way the word is used in ordinary speech. Most people would describe a domino standing upright as being unstable, but in technical usage it would be considered stable, because a certain finite amount of energy is required to tip it over, and perturbations smaller than that would only cause it to oscillate around its equilibrium position.

The domino is also an interesting example because it has two local minima, one in which it is upright, and another in which it is lying flat. A local minimum that is not the global minimum, as in figure o/5, is referred to as a metastable equilibrium.

Example 12.

Figure p shows a special-purpose one-block funicular railroad near Hill and Fourth Streets in Los Angeles, California, used for getting passengers up and down a very steep hill. It has two cars attached to a single loop of cable, arranged so that while one car goes up, the other comes down. They pass each other in the middle. Since one car’s gravitational energy is increasing while
the other's is decreasing, the system is in neutral equilibrium. If there were no frictional heating, exactly zero energy would be required in order to operate the system. A similar counterweighting principle is used in aerial tramways in mountain resorts, and in elevators (with a solid weight, rather than a second car, as counterweight).
The U-shaped tube in figure q has cross-sectional area $A$, and the density of the water inside is $\rho$. Find the gravitational energy as a function of the quantity $y$ shown in the figure, and show that there is an equilibrium at $y=0$.

The question is a little ambiguous, since gravitational energy is only well defined up to an additive constant. To fix this constant, let’s define $U$ to be zero when $y=0$. The difference between $U(y)$ and $U(0)$ is the energy that would be required to lift a water column of height $y$ out of the right side, and place it above the dashed line, on the left side, raising it through a height $y$. This water column has height $y$ and cross-sectional area $A$, so its volume is $Ay$, its mass is $\rho Ay$, and the energy required is $mgy=(\rho Ay)gy=\rho gAy^2$. We then have $U(y) = U(0) + \rho gAy^2 = \rho gAy^2$.

To find equilibria, we look for places where the derivative $dU/dy = 2\rho gAy$ equals 0. As we’d expect intuitively, the only equilibrium occurs at $y=0$. The second derivative test shows that this is a local minimum (not a maximum or a point of inflection), so this is a stable equilibrium.

2.1.7 Predicting the direction of motion

Kinetic energy doesn’t depend on the direction of motion. Sometimes this is helpful, as in the high road-low road example (p. 84, example 9), where we were able to predict that the balls would have the same final speeds, even though they followed different paths and were moving in different directions at the end. In general, however, the two conservation laws we’ve encountered so far aren’t enough to predict an object’s path through space, for which we need conservation of momentum (chapter 3), and the mathematical technique of vectors. Before we develop those ideas in their full generality, however, it will be helpful to do a couple of simple examples, including one that we’ll get a lot of mileage out of in section 2.3.

Suppose we observe an air hockey puck gliding frictionlessly to the right at a velocity $v$, and we want to predict its future motion. Since there is no friction, no kinetic energy is converted to heat. The only form of energy involved is kinetic energy, so conservation of energy, $\Delta E = 0$, becomes simply $\Delta K = 0$. There’s no particular reason for the puck to do anything but continue moving to the right at constant speed, but it would be equally consistent with conservation of energy if it spontaneously decided to reverse its direction of motion, changing its velocity to $-v$. Either way, we’d have $\Delta K = 0$. There is, however, a way to tell which motion is physical and which is unphysical. Suppose we consider the whole thing again in the frame of reference that is initially moving right along with the puck. In this frame, the puck starts out with $K = 0$. What we originally described as a reversal of its velocity from $v$ to $-v$ is, in this
new frame of reference, a change from zero velocity to $-2v$, which would violate conservation of energy. In other words, the physically possible motion conserves energy in all frames of reference, but the unphysical motion only conserves energy in one special frame of reference.

For our second example, we consider a car driving off the edge of a cliff (r). For simplicity, we assume that air friction is negligible, so only kinetic and gravitational energy are involved. Does the car follow trajectory 1, familiar from Road Runner cartoons, trajectory 2, a parabola, or 3, a diagonal line? All three could be consistent with conservation of energy, in the ground’s frame of reference. For instance, the car would have constant gravitational energy along the initial horizontal segment of trajectory 1, so during that time it would have to maintain constant kinetic energy as well. Only a parabola, however, is consistent with conservation of energy combined with Galilean relativity. Consider the frame of reference that is moving horizontally at the same speed as that with which the car went over the edge. In this frame of reference, the cliff slides out from under the initially motionless car. The car can’t just hover for a while, so trajectory 1 is out. Repeating the same math as in example 8 on p. 83, we have

$$x^* = 0, \quad y^* = \frac{1}{2} gt^2$$

in this frame of reference, where the stars indicate coordinates measured in the moving frame of reference. These coordinates are related to the ground-fixed coordinates $(x, y)$ by the equations

$$x = x^* + vt \quad \text{and} \quad y = y^*,$$

where $v$ is the velocity of one frame with respect to the other. We therefore have

$$x = vt, \quad y = \frac{1}{2} gt^2,$$

in our original frame of reference. Eliminating $t$, we can see that this has the form of a parabola:

$$y = \frac{g}{2v^2} x^2.$$

**self-check A**

What would the car’s motion be like in the * frame of reference if it followed trajectory 3?  

Answer, p. 1054
2.2 Numerical techniques

Engineering majors are a majority of the students in the kind of physics course for which this book is designed, so most likely you fall into that category. Although you surely recognize that physics is an important part of your training, if you’ve had any exposure to how engineers really work, you’re probably skeptical about the flavor of problem-solving taught in most science courses. You realize that not very many practical engineering calculations fall into the narrow range of problems for which an exact solution can be calculated with a piece of paper and a sharp pencil. Real-life problems are usually complicated, and typically they need to be solved by number-crunching on a computer, although we can often gain insight by working simple approximations that have algebraic solutions. Not only is numerical problem-solving more useful in real life, it’s also educational; as a beginning physics student, I really only felt like I understood projectile motion after I had worked it both ways, using algebra and then a computer program. (This was back in the days when 64 kilobytes of memory was considered a lot.)

In this section, we’ll start by seeing how to apply numerical techniques to some simple problems for which we know the answer in “closed form,” i.e., a single algebraic expression without any calculus or infinite sums. After that, we’ll solve a problem that would have made you world-famous if you could have done it in the seventeenth century using paper and a quill pen! Before you continue, you should read Appendix 1 on page 1020 that introduces you to the Python programming language.

First let’s solve the trivial problem of finding how much time it takes an object moving at speed \( v \) to travel a straight-line distance \( \text{dist} \). This closed-form answer is, of course, \( \text{dist}/v \), but the point is to introduce the techniques we can use to solve other problems of this type. The basic idea is to divide the distance up into \( n \) equal parts, and add up the times required to traverse all the parts. The following Python function does the job. Note that you shouldn’t type in the line numbers on the left, and you don’t need to type in the comments, either. I’ve omitted the prompts >>> and ... in order to save space.

```python
1 import math
2 def time1(dist,v,n):
3     x=0  # Initialize the position.
4     dx = dist/n  # Divide dist into n equal parts.
5     t=0  # Initialize the time.
6     for i in range(n):
7         x = x+dx  # Change x.
8         dt=dx/v  # time=distance/speed
9         t=t+dt  # Keep track of elapsed time.
10     return t
```
How long does it take to move 1 meter at a constant speed of 1 m/s? If we do this,

```python
>>> print(time1(1.0,1.0,10))  # dist, v, n
0.99999999999999989
```

Python produces the expected answer by dividing the distance into ten equal 0.1-meter segments, and adding up the ten 0.1-second times required to traverse each one. Since the object moves at constant speed, it doesn’t even matter whether we set \( n \) to 10, 1, or a million:

```python
>>> print(time1(1.0,1.0,1))  # dist, v, n
1.0
```

Now let’s do an example where the answer isn’t obvious to people who don’t know calculus: how long does it take an object to fall through a height \( h \), starting from rest? We know from example 8 on page 83 that the exact answer, found using calculus, is \( \sqrt{2h/g} \).

Let’s see if we can reproduce that answer numerically. The main difference between this program and the previous one is that now the velocity isn’t constant, so we need to update it as we go along. Conservation of energy gives \( mgh = \frac{1}{2}mv^2 + mgy \) for the velocity \( v \) at height \( y \), so \( v = -\sqrt{2g(h-y)} \). (We choose the negative root because the object is moving down, and our coordinate system has the positive \( y \) axis pointing up.)

```python
import math
def time2(h,n):
    g=9.8  # gravitational field
    y=h    # Initialize the height.
    v=0    # Initialize the velocity.
    dy = -h/n  # Divide h into n equal parts.
    t=0    # Initialize the time.
    for i in range(n):
        y = y+dy  # Change y. (Note dy<0.)
        v = -math.sqrt(2*g*(h-y))  # from cons. of energy
        dt=dy/v  # dy and v are <0, so dt is >0
        t=t+dt  # Keep track of elapsed time.
    return t
```

For \( h=1.0 \) m, the closed-form result is \( \sqrt{2 \cdot 1.0 \text{ m}/9.8 \text{ m/s}^2} = 0.45 \text{ s} \).

With the drop split up into only 10 equal height intervals, the numerical technique provides a pretty lousy approximation:
But by increasing \( n \) to ten thousand, we get an answer that’s as close as we need, given the limited accuracy of the raw data:

>>> print(time2(1.0,10000))  # h, n
0.44846664060793945

A subtle point is that we changed \( y \) in line 9, and then on line 10 we calculated \( v \), which depends on \( y \). Since \( y \) is only changing by a ten-thousandth of a meter with each step, you might think this wouldn’t make much of a difference, and you’d be almost right, except for one small problem: if we swapped lines 9 and 10, then the very first time through the loop, we’d have \( v=0 \), which would produce a division-by-zero error when we calculated \( dt \)! Actually what would make the most sense would be to calculate the velocity at height \( y \) and the velocity at height \( y+dy \) (recalling that \( dy \) is negative), average them together, and use that value of \( y \) to calculate the best estimate of the velocity between those two points. Since the acceleration is constant in the present example, this modification results in a program that gives an exact result even for \( n=1 \):

```python
1 import math
2 def time3(h,n):
3     g=9.8
4     y=h
5     v=0
6     dy = -h/n
7     t=0
8     for i in range(n):
9         y_old = y
10         y = y+dy
11         v_old = math.sqrt(2*g*(h-y_old))
12         v = math.sqrt(2*g*(h-y))
13         v_avg = -(v_old+v)/2.
14         dt=dy/v_avg
15         t=t+dt
16     return t

>>> print(time3(1.0,1))  # h, n
0.45175395145262565
```

Now we’re ready to attack a problem that challenged the best minds of Europe back in the days when there were no computers. In 1696, the mathematician Johann Bernoulli posed the following
famous question. Starting from rest, an object slides frictionlessly over a curve joining the point \((a, b)\) to the point \((0, 0)\). Of all the possible shapes that such a curve could have, which one gets the object to its destination in the least possible time, and how much time does it take? The optimal curve is called the brachistochrone, from the Greek “short time.” The solution to the brachistochrone problem evaded Bernoulli himself, as well as Leibniz, who had been one of the inventors of calculus. The English physicist Isaac Newton, however, stayed up late one night after a day’s work running the royal mint, and, according to legend, produced an algebraic solution at four in the morning. He then published it anonymously, but Bernoulli is said to have remarked that when he read it, he knew instantly from the style that it was Newton — he could “tell the lion from the mark of his claw.”

Rather than attempting an exact algebraic solution, as Newton did, we’ll produce a numerical result for the shape of the curve and the minimum time, in the special case of \(a=1.0\) m and \(b=1.0\) m. Intuitively, we want to start with a fairly steep drop, since any speed we can build up at the start will help us throughout the rest of the motion. On the other hand, it’s possible to go too far with this idea: if we drop straight down for the whole vertical distance, and then do a right-angle turn to cover the horizontal distance, the resulting time of 0.68 s is quite a bit longer than the optimal result, the reason being that the path is unnecessarily long. There are infinitely many possible curves for which we could calculate the time, but let’s look at third-order polynomials,

\[
y = c_1 x + c_2 x^2 + c_3 x^3,
\]

where we require \(c_3 = (b - c_1 a - c_2 a^2)/a^3\) in order to make the curve pass through the point \((a, b)\). The Python program, below, is not much different from what we’ve done before. The function only asks for \(c_1\) and \(c_2\), and calculates \(c_3\) internally at line 4. Since the motion is two-dimensional, we have to calculate the distance between one point and the next using the Pythagorean theorem, at line 16.

```python
import math
def timeb(a,b,c1,c2,n):
g=9.8
c3 = (b-c1*a-c2*a**2)/(a**3)
x=a
y=b
dx = -a/n
t=0
for i in range(n):
y_old = y
x = x+dx
y = c1*x+c2*x**2+c3*x**3
```

![Approximations to the brachistochrone curve using a third-order polynomial (solid line), and a seventh-order polynomial (dashed). The latter only improves the time by four milliseconds.](image)
As a first guess, we could try a straight diagonal line, \( y = x \), which corresponds to setting \( c_1 = 1 \), and all the other coefficients to zero. The result is a fairly long time:

>>> a=1.
>>> b=1.
>>> n=10000
>>> c1=1.
>>> c2=0.
>>> print(timeb(a,b,c1,c2,n))
0.63887656499994161

What we really need is a curve that’s very steep on the right, and flatter on the left, so it would actually make more sense to try \( y = x^3 \):

>>> c1=0.
>>> c2=0.
>>> print(timeb(a,b,c1,c2,n))
0.59458339947087069

This is a significant improvement, and turns out to be only a hundredth of a second off of the shortest possible time! It’s possible, although not very educational or entertaining, to find better approximations to the brachistochrone curve by fiddling around with the coefficients of the polynomial by hand. The real point of this discussion was to give an example of a nontrivial problem that can be attacked successfully with numerical techniques. I found the first approximation shown in figure a,

\[
y = (0.62)x + (-0.93)x^2 + (1.31)x^3
\]

by using the program listed in appendix 2 on page 1023 to search automatically for the optimal curve. The seventh-order approximation shown in the figure came from a straightforward extension of the same program.
2.3 Gravitational phenomena

Cruise your radio dial today and try to find any popular song that would have been imaginable without Louis Armstrong. By introducing solo improvisation into jazz, Armstrong took apart the jigsaw puzzle of popular music and fit the pieces back together in a different way. In the same way, Newton reassembled our view of the universe. Consider the titles of some recent physics books written for the general reader: The God Particle, Dreams of a Final Theory. When the subatomic particle called the neutrino was recently proven for the first time to have mass, specialists in cosmology began discussing seriously what effect this would have on calculations of the evolution of the universe from the Big Bang to its present state. Without the English physicist Isaac Newton, such attempts at universal understanding would not merely have seemed ambitious, they simply would not have occurred to anyone.

This section is about Newton’s theory of gravity, which he used to explain the motion of the planets as they orbited the sun. Newton tosses off a general treatment of motion in the first 20 pages of his Mathematical Principles of Natural Philosophy, and then spends the next 130 discussing the motion of the planets. Clearly he saw this as the crucial scientific focus of his work. Why? Because in it he showed that the same laws of nature applied to the heavens as to the earth, and that the gravitational interaction that made an apple fall was the same as the as the one that kept the earth’s motion from carrying it away from the sun.

2.3.1 Kepler’s laws

Newton wouldn’t have been able to figure out why the planets move the way they do if it hadn’t been for the astronomer Tycho Brahe (1546-1601) and his protege Johannes Kepler (1571-1630), who together came up with the first simple and accurate description of how the planets actually do move. The difficulty of their task is suggested by the figure below, which shows how the relatively simple orbital motions of the earth and Mars combine so that as seen from earth Mars appears to be staggering in loops like a drunken sailor.

Brahe, the last of the great naked-eye astronomers, collected extensive data on the motions of the planets over a period of many years, taking the giant step from the previous observations’ accuracy of about 10 minutes of arc (10/60 of a degree) to an unprecedented 1 minute. The quality of his work is all the more remarkable considering that his observatory consisted of four giant brass protractors mounted upright in his castle in Denmark. Four different observers would simultaneously measure the position of a planet in order to check for mistakes and reduce random errors.

With Brahe’s death, it fell to his former assistant Kepler to try
As the earth and Mars revolve around the sun at different rates, the combined effect of their motions makes Mars appear to trace a strange, looped path across the background of the distant stars.

to make some sense out of the volumes of data. After 900 pages of calculations and many false starts and dead-end ideas, Kepler finally synthesized the data into the following three laws:

**Kepler’s elliptical orbit law:** The planets orbit the sun in elliptical orbits with the sun at one focus.

**Kepler’s equal-area law:** The line connecting a planet to the sun sweeps out equal areas in equal amounts of time.

**Kepler’s law of periods:** The time required for a planet to orbit the sun, called its period, \( T \), is proportional to the long axis of the ellipse raised to the 3/2 power. The constant of proportionality is the same for all the planets.

Although the planets’ orbits are ellipses rather than circles, most are very close to being circular. The earth’s orbit, for instance, is only flattened by 1.7% relative to a circle. In the special case of a planet in a circular orbit, the two foci (plural of “focus”) coincide at the center of the circle, and Kepler’s elliptical orbit law thus says that the circle is centered on the sun. The equal-area law implies that a planet in a circular orbit moves around the sun with constant speed. For a circular orbit, the law of periods then amounts to a statement that the time for one orbit is proportional to \( r^{3/2} \), where \( r \) is the radius. If all the planets were moving in their orbits at the same speed, then the time for one orbit would simply depend on the circumference of the circle, so it would only be proportional to \( r \) to the first power. The more drastic dependence on \( r^{3/2} \) means that the outer planets must be moving more slowly than the inner planets.
Our main focus in this section will be to use the law of periods to deduce the general equation for gravitational energy. The equal-area law turns out to be a statement on conservation of angular momentum, which is discussed in chapter 4. We’ll demonstrate the elliptical orbit law numerically in chapter 3, and analytically in chapter 4.

2.3.2 Circular orbits

Kepler’s laws say that planets move along elliptical paths (with circles as a special case), which would seem to contradict the proof on page 90 that objects moving under the influence of gravity have parabolic trajectories. Kepler was right. The parabolic path was really only an approximation, based on the assumption that the gravitational field is constant, and that vertical lines are all parallel. In figure e, trajectory 1 is an ellipse, but it gets chopped off when the cannonball hits the earth, and the small piece of it that is above ground is nearly indistinguishable from a parabola. Our goal is to connect the previous calculation of parabolic trajectories, \( y = \frac{g}{2v^2}x^2 \), with Kepler’s data for planets orbiting the sun in nearly circular orbits. Let’s start by thinking in terms of an orbit that circles the earth, like orbit 2 in figure e. It’s more natural now to choose a coordinate system with its origin at the center of the earth, so the parabolic approximation becomes \( y = r - \frac{g}{2v^2}x^2 \), where \( r \) is the distance from the center of the earth. For small values of \( x \), i.e., when the cannonball hasn’t traveled very far from the muzzle of the gun, the parabola is still a good approximation to the actual circular orbit, defined by the Pythagorean theorem, \( r^2 = x^2 + y^2 \), or \( y = r\sqrt{1 - x^2/r^2} \). For small values of \( x \), we can use the approximation \( \sqrt{1 + \epsilon} \approx 1 + \epsilon/2 \) to find \( y \approx r - \frac{1}{2r}x^2 \). Setting this equal to the equation of the parabola, we have \( g/2v^2 = (1/2r) \), or

\[
v = \sqrt{gr} \quad \text{[condition for a circular orbit].}
\]

To get a feel for what this all means, let’s calculate the velocity required for a satellite in a circular low-earth orbit. Real low-earth-orbit satellites are only a few hundred km up, so for purposes of rough estimation we can take \( r \) to be the radius of the earth, and \( g \) is not much less than its value on the earth’s surface, 10 m/s\(^2\). Taking numerical data from Appendix 4, we have

\[
v = \sqrt{gr} \]
\[
= \sqrt{(10 \text{ m/s}^2)(6.4 \times 10^3 \text{ km})} 
= \sqrt{(10 \text{ m/s}^2)(6.4 \times 10^6 \text{ m})} 
= \sqrt{6.4 \times 10^7 \text{ m}^2/\text{s}^2} 
= 8000 \text{ m/s}
\]
(about twenty times the speed of sound).

In one second, the satellite moves 8000 m horizontally. During this time, it drops the same distance any other object would: about 5 m. But a drop of 5 m over a horizontal distance of 8000 m is just enough to keep it at the same altitude above the earth’s curved surface.

2.3.3 The sun’s gravitational field

We can now use the circular orbit condition \( v = \sqrt{gr} \), combined with Kepler’s law of periods, \( T \propto r^{3/2} \) for circular orbits, to determine how the sun’s gravitational field falls off with distance.\(^8\) From there, it will be just a hop, skip, and a jump to get to a universal description of gravitational interactions.

The velocity of a planet in a circular orbit is proportional to \( r/T \), so

\[
\begin{align*}
\frac{r}{T} &\propto \sqrt{gr} \\
\frac{r}{r^{3/2}} &\propto \sqrt{gr} \\
g &\propto \frac{1}{r^2}
\end{align*}
\]

If gravity behaves systematically, then we can expect the same to be true for the gravitational field created by any object, not just the sun.

There is a subtle point here, which is that so far, \( r \) has just meant the radius of a circular orbit, but what we have come up with smells more like an equation that tells us the strength of the gravitational field made by some object (the sun) if we know how far we are from the object. In other words, we could reinterpret \( r \) as the distance from the sun.

2.3.4 Gravitational energy in general

We now want to find an equation for the gravitational energy of any two masses that attract each other from a distance \( r \). We assume that \( r \) is large enough compared to the distance between the objects so that we don’t really have to worry about whether \( r \) is measured from center to center or in some other way. This would be a good approximation for describing the solar system, for example, since the sun and planets are small compared to the distances between them — that’s why you see Venus (the “evening star”) with your bare eyes as a dot, not a disk.

The equation we seek is going to give the gravitational energy, \( U \), as a function of \( m_1 \), \( m_2 \), and \( r \). We already know from expe-

\(^8\)There is a hidden assumption here, which is that the sun doesn’t move. Actually the sun wobbles a little because of the planets’ gravitational interactions with it, but the wobble is small due to the sun’s large mass, so it’s a pretty good approximation to assume the sun is stationary. Chapter 3 provides the tools to analyze this sort of thing completely correctly — see p. 144.
The gravitational energy $U = \frac{Gm_1 m_2}{r}$ graphed as a function of $r$.

Experience with gravity near the earth’s surface that $U$ is proportional to the mass of the object that interacts with the earth gravitationally, so it makes sense to assume the relationship is symmetric: $U$ is presumably proportional to the product $m_1 m_2$. We can no longer assume $\Delta U \propto \Delta r$, as in the earth’s-surface equation $\Delta U = mg\Delta y$, since we are trying to construct an equation that would be valid for all values of $r$, and $g$ depends on $r$. We can, however, consider an infinitesimally small change in distance $dr$, for which we’ll have $dU = m_2 g_1 dr$, where $g_1$ is the gravitational field created by $m_1$. (We could just as well have written this as $dU = m_1 g_2 dr$, since we’re not assuming either mass is “special” or “active.”) Integrating this equation, we have

$$\int dU = \int m_2 g_1 dr$$

$$U = m_2 \int g_1 dr$$

$$U \propto m_1 m_2 \int \frac{1}{r^2} dr$$

$$U \propto -\frac{m_1 m_2}{r},$$

where we’re free to take the constant of integration to be equal to zero, since gravitational energy is never a well-defined quantity in absolute terms. Writing $G$ for the constant of proportionality, we have the following fundamental description of gravitational interactions:

$$U = -\frac{Gm_1 m_2}{r} \quad [\text{gravitational energy of two masses separated by a distance } r]$$

We’ll refer to this as Newton’s law of gravity, although in reality he stated it in an entirely different form, which turns out to be mathematically equivalent to this one.

Let’s interpret his result. First, don’t get hung up on the fact that it’s negative, since it’s only differences in gravitational energy that have physical significance. The graph in figure f could be shifted up or down without having any physical effect. The slope of this graph relates to the strength of the gravitational field. For instance, suppose figure f is a graph of the gravitational energy of an asteroid interacting with the sun. If the asteroid drops straight toward the sun, from A to B, the decrease in gravitational energy is very small, so it won’t speed up very much during that motion. Points C and D, however, are in a region where the graph’s slope is much greater. As the asteroid moves from C to D, it loses a lot of gravitational energy, and therefore speeds up considerably. This is due to the stronger gravitational field.