Design of the Cushions on a Pool Table: An Elegant and Nontrivial Problem in Classical Mechanics

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One can easily get the impression that classical mechanics is a dead science, and that classical physics problems either are trivial and have closed-form solutions or are ugly and need to be solved by numerical simulation. A counterexample is the problem of a pool ball rebounding from a cushion, which is nontrivial, but nontrivial in interesting ways. If the cushions were vertical walls, then the static frictional force from the tabletop would not produce enough torque to reverse the direction of the ball's spin: a ball that came in on a natural roll (rolling without slipping) would be slipping when it rebounded, and would skid for a while before settling down again to a natural roll. On a real pool table, the point of contact between the ball and the cusion is above the center of the ball by a height b. The original designers of the modern cushion probably experimented with various values of b, and picked one that maximized the distance rolled by a rebounding ball.

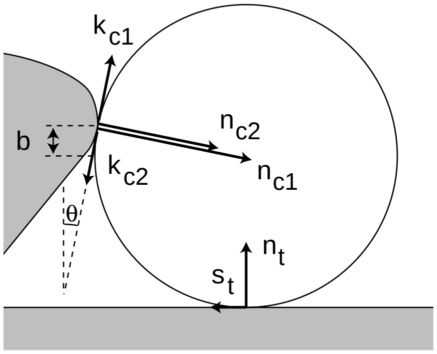
Calculating b has been a popular textbook and exam problem, but the explicit or implicit assumptions are usually physically wrong, and lead to a value of b that is too big compared to the one found on real pool tables. Real pool tables have $b/r \approx 0.26$, which turns out to be surprisingly hard to explain theoretically. First we need to understand how the results of the ball's collision with the cushion relate to its rebound distance. The rebound distance is finite because of rolling resistance, which can be modeled[1] as $F_r = \mu_r F_N$, where F_N is the normal force between the tabletop and the ball. Typical values[1] of μ_r are on the order of 0.01. This is much smaller than μ_k, the coefficient of kinetic friction. In the limit of $\mu_r/\mu_k \to 0$, any slipping by the rebounding ball exacts too high a price in energy, and the maximum rebound distance is achieved with a design in which the ball rolls without slipping before, during, and after the collision. On the other hand, in the limiting case of $\mu_k \to 0$, there is no point in investing any kinetic energy in rotation, and the optimal design is one that kills the spin of the ball. For a real pool table, we are much closer to $\mu_r/\mu_k \to 0$. A straightforward but lengthy calculation shows that this effect leads to a fractional change in b that is on the order of μ_r/μ_k, so we neglect the correction:

Assumption 1: In the optimum design, the ball is always rolling without slipping, i.e. the frictional force between the ball and the tabletop is always static friction.

Under normal conditions, the ball doesn't hop, either:

Assumption 2: The ball doesn't hop.

Under these two assumptions, the frictional force between the cushion and the ball is always kinetic; if both frictional forces were static for any finite time, then the ball wouldn't be moving! The kinetic friction reverses its direction at the moment when the ball is instantaneously at rest.



Now let's define some notation. We're going to solve the problem using conservation of momentum and angular momentum, so rather than defining symbols for the actual forces (which vary over the course of the collision), we define symbols for the six *momentum transfers* involved, as shown in the figure. The subscript 1 refers to impulses that occur before the ball stops, and 2 is for impulses that occur after the ball starts to rebound.

To get a solution, we need a model of the frictional forces. Let's try the standard textbook model.

Assumption 3: The standard textbook model of friction applies.

In particular, we neglect rolling resistance during the collision, since it isn't part of the standard textbook model. In the standard model, the surfaces have no memory. We'll come back to this point later.

Since the collision is brief, we also have:

Assumption 4: The impulse due to gravity is zero.

Conservation of momentum gives

$$\Delta p_x = (n_{c1}+n_{c2}) \cos\theta + (k_{c1}-k_{c2}) \sin\theta - s_t \quad , \qquad [1]$$

and assumptions 2 and 4 give

$$0 = n_t - (n_{c1}+n_{c2}) \sin\theta + (k_{c1}-k_{c2}) \cos\theta \quad . \qquad [2]$$

Assumption 1 tells us that the total angular momentum change and total momentum change have to be related by $\Delta L=(2r/5)\Delta p_x$, where the factor of 2/5 comes from the moment of inertia of a uniform sphere. Equating this to the sum of the angular impulses acting on the ball,

$$(2r/5)\Delta p_x = r(s_t+k_{c1}-k_{c2}) \quad . \qquad [3]$$

Using the standard model of friction,

$$k_{c1} = \mu_k n_{c1} \quad \text{, and} \qquad [4a]$$

$$k_{c2} = \mu_k n_{c2} \quad . \qquad [4b]$$

For the static friction force s_t, the standard model of friction gives the inequality $s_t<\mu_s n_t$. We'll solve for the minimum value of *b*, which corresponds to the maximum value of s_t:

$$s_t = \mu_s n_t \qquad [4c]$$

If the collision was completely elastic, then the second half of the motion would be a time-reversed version of the first half, and we'd have $n_{c1}=n_{c2}$. In reality, the collision is slightly inelastic due to kinetic friction and the dissipation of heat in the compression of the ball and the cushion. It's convenient to parametrize the inelasticity of the collision with the parameter α defined by

$$\alpha = 1 - n_{c2}/n_{c1} \quad ,$$

so that α is zero for a perfectly elastic collision, and $\alpha\approx 1$ for a maximally inelastic one. We expect α to be small.

Eliminating all the unknown impulses from equations 1-4 results in the following transcendental equation:

$$\left(2-\alpha+\frac{7}{2}\mu_s\mu_k\alpha\right)\cos\theta = \left(\frac{7}{2}\mu_s\left(2-\alpha\right)-\mu_k\alpha\right)\sin\theta+\frac{5}{2}\mu_k\alpha \quad ,$$

which implicitly relates $b = r \sin \theta$ to the other quantities. If we take the observed value of θ=15°, and guess μ_s=0.3 and μ_k=0.2, we find α=1.2, which is clearly unphysical!

We observe that the collision is quite elastic (small α), and that θ=15° seems to result in rolling without slipping on the rebound. This is inconsistent with the results of the model. Which assumption was wrong? I suspect it's assumption 3, that the standard model of friction applies. I suspect that the kinetic frictional force k_{c2} is smaller than predicted by the standard model of friction; the compression of the felt during the incoming part of the collision may leave it in a physical state in which it makes less than the usual amount of friction during the outgoing phase. A similar effect occurs in rolling resistance: the felt, having been compressed by the leading edge of the ball, is left in a physical state where it makes less than the usual amount of normal force on the trailing edge.

My students and I are currently working on measuring the frictional coefficients for pool balls on a pool table. It would also be interesting to determine the effective "spring constant" of the bumber, which would allow an approximate determination of the duration of a typical collision, leading to a numerical test of assumption 4, about neglecting gravity. It's also possible that assumption 1 is violated, and the ball does slip a little on the rebound. It might be possible to test this using video capture. Likewise, video measurements might allow one to determine how elastic the collision is.

References

1. Shepard, **Amateur Physics for the Amateur Pool Player**, http://www.playpool.com/apapp.