The Schrödinger equation gives

\[ E \cdot \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + U \cdot \Psi \]

\[ = -\frac{\hbar^2}{2m} \cdot \left( k_x^2 + k_y^2 + k_z^2 \right) \Psi + U \cdot \Psi \]

\[ E - U = \frac{\hbar^2}{2m} \left( k_x^2 + k_y^2 + k_z^2 \right), \]

which can be satisfied since we’re in a classically allowed region with \( E - U > 0, \) and the right-hand side is manifestly positive.

**Exact treatment of the ground state of hydrogen example 5**

The general treatment of the hydrogen atom for all values of \( n \) is beyond the mathematical scope of this book, but it’s fairly straightforward to verify it for a particular \( n, \) especially given a lucky guess as to what functional form to try for the wavefunction. The form that works for the ground state is

\[ \Psi = u e^{-r/a}, \]

where \( r = \sqrt{x^2 + y^2 + z^2} \) is the electron’s distance from the proton, and \( u \) provides for normalization. We showed in example 9, p. 55 that the Laplacian of this function is

\[ \nabla^2 \Psi = \left( -\frac{2}{ar} + \frac{1}{a^2} \right) \Psi. \]

The Schrödinger equation gives

\[ E \cdot \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + U \cdot \Psi \]

\[ = \frac{\hbar^2}{2m} \left( \frac{2}{ar} - \frac{1}{a^2} \right) \Psi - \frac{ke^2}{r} \cdot \Psi \]

If we require this equation to hold for all \( r, \) then we must have equality for both the terms of the form \((\text{constant}) \times \Psi\) and for those of the form \((\text{constant}/r) \times \Psi.\) That means

\[ E = -\frac{\hbar^2}{2ma^2} \]

and

\[ 0 = \frac{\hbar^2}{mar} - \frac{ke^2}{r}. \]

These two equations can be solved for the unknowns \( a \) and \( E, \) giving

\[ a = \frac{\hbar^2}{mk e^2} \]

and

\[ E = -\frac{mk^2 e^4}{2\hbar^2}, \]

where the result for the energy agrees with the equation given on p. 386. The calculation of the normalization constant \( u \) is relegated to homework problem 3.
Wave phases in the hydrogen molecule  

In example 6 on page 378, I argued that the existence of the H$_2$ molecule could essentially be explained by a particle-in-a-box argument: the molecule is a bigger box than an individual atom, so each electron’s wavelength can be longer, its kinetic energy lower. Now that we’re in possession of a mathematical expression for the wavefunction of the hydrogen atom in its ground state, we can make this argument a little more rigorous and detailed. Suppose that two hydrogen atoms are in a relatively cool sample of monoatomic hydrogen gas. Because the gas is cool, we can assume that the atoms are in their ground states. Now suppose that the two atoms approach one another. Making use again of the assumption that the gas is cool, it is reasonable to imagine that the atoms approach one another slowly. Now the atoms come a little closer, but still far enough apart that the region between them is classically forbidden. Each electron can tunnel through this classically forbidden region, but the tunneling probability is small. Each one is now found with, say, 99% probability in its original home, but with 1% probability in the other nucleus. Each electron is now in a state consisting of a superposition of the ground state of its own atom with the ground state of the other atom. There are two peaks in the superposed wavefunction, but one is a much bigger peak than the other.

An interesting question now arises. What are the relative phases of the two electrons? As discussed on page 370, the absolute phase of an electron’s wavefunction is not really a meaningful concept. Suppose atom A contains electron Alice, and B electron Bob. Just before the collision, Alice may have wondered, “Is my phase positive right now, or is it negative? But of course I shouldn’t ask myself such silly questions,” she adds sheepishly.

But relative phases are well defined. As the two atoms draw closer and closer together, the tunneling probability rises, and eventually gets so high that each electron is spending essentially
50% of its time in each atom. It's now reasonable to imagine that either one of two possibilities could obtain. Alice's wavefunction could either look like $\hbar/1$, with the two peaks in phase with one another, or it could look like $\hbar/2$, with opposite phases. Because relative phases of wavefunctions are well defined, states 1 and 2 are physically distinguishable.\(^2\) In particular, the kinetic energy of state 2 is much higher (by about 5 eV); roughly speaking, it is like the two-hump wave pattern of the particle in a box, as opposed to 1, which looks roughly like the one-hump pattern with a much longer wavelength. Not only that, but an electron in state 1 has a large probability of being found in the central region, where it has a large negative electrical energy due to its interaction with both protons. State 2, on the other hand, has a low probability of existing in that region. (This effect also equals about 5 eV.) Thus state 1 represents the true ground-state wavefunction of the $\text{H}_2$ molecule, and putting both Alice and Bob in that state results in a lower energy than their total energy when separated, so the molecule is bound, and will not fly apart spontaneously.

State $\hbar/3$, on the other hand, is not physically distinguishable from $\hbar/2$, nor is $\hbar/4$ from $\hbar/1$. Alice may say to Bob, "Isn't it wonderful that we're in state 1 or 4? I love being stable like this." But she knows it's not meaningful to ask herself at a given moment which state she's in, 1 or 4.

### 17.5 Use of complex numbers

In a classically forbidden region, a particle's total energy, $U + K$, is less than its $U$, so its $K$ must be negative. If we want to keep believing in the equation $K = p^2/2m$, then apparently the momentum of the particle is the square root of a negative number. This is a symptom of the fact that the Schrödinger equation fails to describe all of nature unless the wavefunction and various other quantities are allowed to be complex numbers. In particular it is not possible to describe traveling waves correctly without using complex wavefunctions. Complex numbers were reviewed in subsection 5.7.2, p. 125.

This may seem like nonsense, since real numbers are the only ones that are, well, real! Quantum mechanics can always be related to the real world, however, because its structure is such that the results of measurements always come out to be real numbers.

\(^2\)The reader who has studied chemistry may find it helpful to make contact with the terminology and notation used by chemists. The state represented by pictures 1 and 4 is known as a $\sigma$ orbital, which is a type of "bonding orbital." The state in 2 and 3 is a $\sigma^*$, a kind of "antibonding orbital." Note that although we will not discuss electron spin or the Pauli exclusion principle until sec. 18.5, p. 432, those considerations have no effect on this example, since the two electrons can have opposite spins.
1. Oscillations can go back and forth, but it’s also possible for them to move along a path that bites its own tail, like a circle. Photons act like one, electrons like the other.

2. Back-and-forth oscillations can naturally be described by a segment taken from the real number line, and we visualize the corresponding type of wave as a sine wave. Oscillations around a closed path relate more naturally to the complex number system. The complex number system has rotation built into its structure, e.g., the sequence 1, $i$, $i^2$, $i^3$, \ldots rotates around the unit circle in 90-degree increments.

3. The double slit experiment embodies the one and only mystery of quantum physics. Either type of wave can undergo double-slit interference. For example, we may describe an electron as having non-real momentum in classically forbidden regions, but its average momentum will always come out to be real (the imaginary parts average out to zero), and it can never transfer a non-real quantity of momentum to another particle.

A complete investigation of these issues is beyond the scope of this book, and this is why we have normally limited ourselves to standing waves, which can be described with real-valued wavefunctions. Figure i gives a visual depiction of the difference between real and complex wavefunctions. The following remarks may also be helpful.

Neither of the graphs in i/2 should be interpreted as a path traveled by something. This isn’t anything mystical about quantum physics. It’s just an ordinary fact about waves, which we first encountered in sec. 2.1, p. 32, where we saw the distinction between the motion of a wave and the motion of a wave pattern. In both examples in i/2, the wave pattern is moving in a straight line to the right.

The helical graph in i/2 shows a complex wavefunction whose value rotates around a circle in the complex plane with a frequency $f$. 
related to its energy by $E = hf$. As it does so, its squared magnitude $|\Psi|^2$ stays the same, so the corresponding probability stays constant. Which direction does it rotate? This direction is purely a matter of convention, since the distinction between the symbols $i$ and $-i$ is arbitrary — both are equally valid as square roots of $-1$. We can, for example, arbitrarily say that electrons with positive energies have wavefunctions whose phases rotate counterclockwise, and as long as we follow that rule consistently within a given calculation, everything will work. Note that it is not possible to define anything like a right-hand rule here, because the complex plane shown in the right-hand side of $i/2$ doesn’t represent two dimensions of physical space; unlike a screw going into a piece of wood, an electron doesn’t have a direction of rotation that depends on its direction of travel.

Superposition of complex wavefunctions example 7

The right side of figure $i/3$ is a cartoonish representation of double-slit interference; it depicts the situation at the center, where symmetry guarantees that the interference is constructive. Suppose that at some off-center point, the two wavefunctions being superposed are $\Psi_1 = b$ and $\Psi_2 = bi$, where $b$ is a real number with units. Compare the probability of finding the electron at this position with what it would have been if the superposition had been purely constructive, $b + b = 2b$.

The probability per unit volume is proportional to the square of the magnitude of the total wavefunction, so we have

$$\frac{P_{\text{off center}}}{P_{\text{center}}} = \frac{|b + bi|^2}{|b + b|^2} = \frac{b^2 + 1}{2b^2 + 0} = \frac{1}{2}.$$

Figure j shows a method for visualizing complex wavefunctions. The idea is to use colors to represent complex numbers, according to the arbitrary convention defined in figure j/1. Brightness indicates magnitude, and the rainbow hue shows the argument. Because this representation can’t be understood in a black and white printed book, the figure is also reproduced on the back cover of printed copies. To avoid any confusion, note that the use of rainbow colors does not mean that we are representing actual visible light. In fact, we will be using these visual conventions to represent the wavefunctions of a material particle such as an electron. It is arbitrary that we use red for positive real numbers and blue-green for negative numbers, and that we pick a handedness for the diagram such that going from red toward yellow means going counterclockwise. Although physically the rainbow is a linear spectrum, we are not representing physical colors here, and we are exploiting the fact that the human brain tends to perceive color as a circle rather than a line, with violet and red being perceptually similar. One of the limitations of this representation is that brightness is limited, so we can’t represent complex numbers with arbitrarily large magnitudes.
1. A representation of complex numbers using color and brightness. 2. A wave traveling toward the right. 3. A wave traveling toward the left. 4. A standing wave formed by superposition of waves 2 and 3. 5. A two-dimensional standing wave. 6. A double-slit diffraction pattern.

Figure j/2 shows a traveling wave as it propagates to the right. The standard convention in physics is that for a wave moving in a certain direction, the phase in the forward direction is farther counterclockwise in the complex plane, and you can verify for yourself that this is the case by comparing with the convention defined by j/1. The function being plotted here is $\Psi = e^{ikx}$, where $k = 2\pi/\lambda$ is the wavenumber. For the use of the complex exponential, see sec. 5.7.3, p.129; it simply represents a point on the unit circle in the complex plane. The wavelength $\lambda$ is a constant and can be measured, for example, from one yellow point to the next. The wavelength is not different at different points on the figure, because we are using the colors merely as a visual encoding of the complex numbers — so, for example, a red point on the figure is not a point where the wave has a longer wavelength than it does at a blue point.

Figure j/3 represents a wave traveling to the left.

Figure j/4 shows a standing wave created by superimposing the traveling waves from j/2 and j/3, $\Psi_4 = (\Psi_2 + \Psi_3)/2$. (The reason for the factor of 2 is simply that otherwise some portions of $\Psi_4$ would have magnitudes too great to be represented using the available range of brightness.) All points on this wave have real values, represented by red and blue-green. We made the superposition real by an appropriate choice of the phases of $\Psi_2$ and $\Psi_3$. This is always possible to do when we have a standing wave, but it is only possible for a standing wave, and this is the reason for all of the disclaimers in the captions of previous figures in which I took the liberty of
representing a traveling wave as a sine-wave graph.

Figure j/5 shows a two-dimensional standing wave of a particle in a box, and j/6 shows a double-slit interference pattern. (In the latter, I’ve cheated by making the amplitude of the wave on the right-hand half of the picture much greater than it would actually be.)

\[ A \text{ paradox resolved} \]

Consider the following paradox. Suppose we have an electron that is traveling wave, and its wavefunction looks like a wave-train consisting of 5 cycles of a sine wave. Call the distance between the leading and trailing edges of the wave-train \( L \), so that \( \lambda = L/5 \). By sketching the wave, you can easily check that there are 11 points where its value equals zero. Therefore at a particular moment in time, there are 11 points where a detector has zero probability of detecting the electron.

But now consider how this would look in a frame of reference where the electron is moving more slowly, at one fifth of the speed we saw in the original frame. In this frame, \( L \) is the same, but \( \lambda \) is five times greater, because \( \lambda = h/p \). Therefore in this frame we see only one cycle in the wave-train. Now there are only 3 points where the probability of detection is zero. But how can this be? All observers, regardless of their frames of reference, should agree on whether a particular detector detects the electron.

The resolution to this paradox is that it starts from the assumption that we can depict a traveling wave as a real-valued sine wave, which is zero in certain places. Actually, we can’t. It has to be a complex number with a rotating phase angle in the complex plane, as in figure j/2, and a constant magnitude.

\[ A \text{ discussion question} \]

The zero level of interaction energy \( U \) is arbitrary, e.g., it’s equally valid to pick the zero of gravitational energy to be on the floor of your lab or at the ceiling. Suppose we’re doing the double-slit experiment, i/3, with electrons. We define the zero-level of \( U \) so that the total energy \( E = U + K \) of each electron is positive. and we observe a certain interference pattern like the one in figure i on p. 351. What happens if we then redefine the zero-level of \( U \) so that the electrons have \( E < 0 \)?
The top panel of the figure shows a series of snapshots in the motion of two pulses on a coil spring, one negative and one positive, as they move toward one another and superpose. The final image is very close to the moment at which the two pulses cancel completely. The following discussion is simpler if we consider infinite sine waves rather than pulses. How can the cancellation of two such mechanical waves be reconciled with conservation of energy? What about the case of colliding electromagnetic waves?

Quantum-mechanically, the issue isn’t conservation of energy, it’s conservation of probability, i.e., if there’s initially a 100% probability that a particle exists somewhere, we don’t want the probability to be more than or less than 100% at some later time. What happens when the colliding waves have real-valued wavefunctions $\Psi$? Now consider the sketches of complex-valued wave pulses shown in the bottom panel of the figure as they are getting ready to collide.

### 17.6 Linearity of the Schrödinger equation

Some mathematical relationships and operations are linear, and some are not. For example, $2 \times (3 + 2)$ is the same as $2 \times 3 + 2 \times 2$, but $\sqrt{1 + 1} \neq \sqrt{1} + \sqrt{1}$. Differentiation is a linear operation, $(f + g)' = f' + g'$. The Schrödinger equation is built out of derivatives, so it is linear as well. That is, if $\Psi_1$ and $\Psi_2$ are both solutions of the Schrödinger equation, then so is $\Psi_1 + \Psi_2$. Linearity normally implies linearity with respect both to addition and to multiplication by a scalar. For example, if $\Psi$ is a solution, then so is $\Psi + \Psi + \Psi$, which is the same as $3\Psi$.

Linearity guarantees that the phase of a wavefunction makes no difference as to its validity as a solution to the Schrödinger equation. If $\sin kx$ is a solution, then so is the sine wave $-\sin kx$ with the opposite phase. This fact is logically interdependent with the fact that, as discussed on p. 370, the phase of a wavefunction is unobservable. For measuring devices and humans are material objects that can be described by wavefunctions. So suppose, for example, that we flip the phase of all the particles inside the entire laboratory. By linearity, the evolution of this measurement process is still a valid solution of the Schrödinger equation.

The Schrödinger equation is a wave equation, and its linearity implies that the waves obey the principle of superposition. In most cases in nature, we find that the principle of superposition for waves is at best an approximation. For example, if the amplitude of a tsunami is so huge that the trough of the wave reaches all the way down to the ocean floor, exposing the rocks and sand as it passes overhead, then clearly there is no way to double the amplitude of the wave and still get something that obeys the laws of physics. Even at less extreme amplitudes, superposition is only an approximation for water waves, and so for example it is only approximately true that when two sets of ripples intersect on the surface of a pond, they
pass through without “seeing” each other.

It is therefore natural to ask whether the apparent linearity of the Schrödinger equation is only an approximation to some more precise, nonlinear theory. This is not currently believed to be the case. If we are to make sense of Schrödinger’s cat (sec. 15.4, p. 358), then the experimenter who sees a live cat and the one who sees a dead cat must remain oblivious to their other selves, like the ripples on the pond that intersect without “seeing” each other. Attempts to create slightly nonlinear versions of standard quantum mechanics have been shown to have implausible physical properties, such as allowing the propagation of signals faster than $c$. (This is known as Gisin’s theorem. The original paper, “Weinberg’s non-linear quantum mechanics and supraluminal communications,” is surprisingly readable and nonmathematical.)

The linearity of the Schrödinger equation is what allows us to talk about its solutions as vectors in a vector space (p. 155). For example, if $\Psi_1$ represents an unstable nucleus that has not yet gamma decayed, and $\Psi_2$ is its state after the decay, then any superposition $\alpha \Psi_1 + \beta \Psi_2$, with real or complex coefficients $\alpha$ and $\beta$, is a possible wavefunction, and we can notate this as a vector, $\binom{\alpha}{\beta}$, in a two-dimensional vector space.

People wrestling with the weirdness of Schrödinger’s cat sometimes say that it’s impossible to have a superposition of the live cat and the dead cat. Not true. The linearity of the Schrödinger equation guarantees that we can take any wavefunctions $\Psi_1$ and $\Psi_2$ and superpose them, and this is exactly what happens, according to quantum physics, when we do the experiment. What is impossible, for all practical purposes, is to observe any type of interaction between the live and dead cats, such as wave interference, which we estimated on p. 369 and found to be undetectable, due to the extremely small wavelength.

17.7 The inner product and observables

17.7.1 The inner product

In sec. 6.6, p. 156, we discussed the idea of an inner product, which is an operation on a vector space that acts like the dot product, i.e., it takes two vectors as inputs and gives back a scalar as an output. We also saw that any time we had a way of measuring magnitudes, we automatically got an inner product for free. In quantum mechanics, we do have a way of measuring magnitudes, which is the total probability of a given wavefunction. We even have a convenient notation for this, the bra-ket notation introduced in sec. 16.1.3, p. 372. In example 3, we had a wavefunction $|\psi\rangle$, and the fact of its being normalized was written as $\langle \psi | \psi \rangle = 1$. 

Section 17.7 The inner product and observables 409
\langle u|v \rangle = \langle v|u \rangle^* \\
\langle u|\alpha v + \beta w \rangle = \alpha \langle u|v \rangle + \beta \langle u|w \rangle \\
\langle \alpha u + \beta v|w \rangle = \alpha^* \langle u|w \rangle + \beta^* \langle v|w \rangle.

This is a statement that the total probability for something to happen must be 1, but we can also think of it as a statement that the “magnitude” of $\sqrt{\mathcal{U}}$ has a certain value. Because we have a way of defining magnitudes of wavefunctions, we automatically get an inner product.

If we’re dealing with wavefunctions that are expressed as functions of position, then it’s pretty clear how to define an appropriate inner product: $\langle u|v \rangle = \int u^*v \, dx$. We need to use the complex conjugate $u^*$ rather than just $u$, for the following reason. When we take the inner product of a wavefunction with itself, it has a probability interpretation. Probabilities are always real and positive. So we need to set things up in such a way that $\langle u|u \rangle$ is always real and positive. When we take $\langle u|u \rangle$, the thing inside the integral is $u^*u$, which is the same as $|u|^2$, and this is indeed real and positive, so when we integrate it we get an answer that is real and positive.

Note the similarity between the expression $\int u^*v \, dx$ and the expression $u_x v_x + u_y v_y + u_z v_z$ for a dot product: the integral is a continuous sum, and the dot product is a discrete sum.

We recall from our previous discussion (p. 156) that an inner product can generally be described as a measure of how similar two vectors are. For example, in the Euclidean plane, vectors that are perpendicular to each other have a dot product of zero, which tells you that they lie along lines that are completely different. When two vectors have an inner product of zero, we say that they are orthogonal.

Physically, two wavefunctions have a zero inner product if and only if they are completely distinguishable from each other by the measurement of some observable. By analogy with vectors in Euclidean space, we say that the two wavefunctions are orthogonal. For example, $\langle \psi_1|\psi_2 \rangle = 0$, as can be verified from the integral $\int_0^\pi \sin x \sin 2x \, dx = 0$. These states are also distinguishable by measuring either their momentum or their energy.

Suppose that $u$ and $v$ are both properly normalized wavefunctions. If $|\langle u|v \rangle| = 1$, then the states are identical.\(^3\) If $\langle u|v \rangle = 0$, then $u$ and $v$ are completely distinguishable from one another. There is also the intermediate case where $\langle u|v \rangle$ has a magnitude greater than 0 but less than 1. In this case, we could say that $u$ is a mixture of $v$ plus some other state $w$ that is distinguishable from $v$, i.e., that

$$
|u\rangle = \alpha|v\rangle + \beta|w\rangle.
$$

where $\langle v|w \rangle = 0$. We then have

$$
\langle u|v \rangle = (\alpha \langle v \rangle + \beta \langle w \rangle)v = \alpha.
$$

\(^3\)If the inner product is, for example, $-1$, then the wavefunctions differ only by an unobservable difference in phase, so they really describe the same state.
Some examples of interpretation of the inner product.

<table>
<thead>
<tr>
<th>Inner Product</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \uparrow \uparrow</td>
<td>\uparrow \uparrow \rangle$</td>
</tr>
<tr>
<td>$\langle \downarrow \uparrow</td>
<td>\uparrow \uparrow \rangle$</td>
</tr>
<tr>
<td>$\langle \uparrow \uparrow</td>
<td>\downarrow \downarrow \rangle$</td>
</tr>
</tbody>
</table>

Now suppose that we make measurements capable of determining whether or not the system is in the state $v$. If the system is prepared in state $u$, and we make these measurements on it, then by the linearity of the Schrödinger equation, the result is that the measuring apparatus or observer ends up in a Schrödinger’s-cat state that looks like

$$\alpha|\text{observed } v\rangle + \beta|\text{observed } w\rangle.$$ 

We interpret squares of amplitudes as probabilities, so

$$P = |\alpha|^2 = |\langle u|v \rangle|^2$$

gives us the probability that we will have observed the state to be $v$. This final leap in the logic, to a probability interpretation, has felt mysterious to several generations of physicists, but recent work has clarified the situation somewhat.

### 17.7.2 Observables

When I was in college, my stepmother attempted to make me more hip, more artistic, and less uncool. I would go over for a visit sometimes on the weekend, and she would sit me down, pour us each a glass of wine, and draw me out. In one of these conversations, I insisted that love didn’t exist, because there was no way to measure it. The physicist’s way of thinking, which by that time I had already started to adopt, is that we should distinguish carefully between things that are observables and things that aren’t. One of the basic principles of quantum mechanics is that $\langle u|v \rangle = 0$ if and only if there is some observable that perfectly distinguishes state $u$ from state $v$. (We may actually need more than one observable.)

This makes it important to be clear on what really is an observable and what isn’t. Informally, what we mean by an observable is that:

1. we can tell by looking at the state what value it has, or at least assign probabilities to values;
2. it has a single value; and

3. (optionally) it has some counterpart in classical physics.

The following are all observables:

• position

• momentum

• energy

• angular momentum (ch. 18).

The following are not:

• phase and normalization (sec. 16.1.4)

• time (because many systems, such as an atom, are too simple to act as clocks).

A couple of interesting borderline cases are wavelength and angle. Classically, an electron doesn’t have a wavelength, so we would usually choose to talk about its momentum as an observable, not its wavelength (although they relate via \( p = h/\lambda \), so it really doesn’t make much difference). For an electron in an atom, thinking in a simplified picture in two dimensions, we could talk about its angle in the plane, but this causes problems because angles are not really single-valued, e.g., 0 is the same as 360°.

The fact that position is an observable but time is not is one of the things that makes it difficult to reconcile quantum mechanics with relativity, which considers time to be just another dimension. This difficulty has been reconciled for special relativity, but not for general relativity.

For those with some background in linear algebra, it may be helpful to connect this to the idea of eigenvalues and eigenvectors. In quantum mechanics, we associate with an observable such as momentum \( p \) some linear operator \( \mathcal{O}_p \). In this example, \( \mathcal{O}_p \) is basically the derivative operator (multiplied by some constant factors). If a particle is in a state of definite momentum, then it is an eigenvector of \( \mathcal{O}_p \) with eigenvalue \( p \). The basic motivation for using this style of definition is that if \( \Psi \) is an eigenvector, then \( c\Psi \) is also an eigenvector, with the same eigenvalue; therefore this mathematical setup automatically keeps phase and normalization from being observables.
17.8 Time evolution and unitarity

17.8.1 The simplest cases of time evolution

So far we have not said too much about how a wavefunction changes with time, except that a state of definite energy $E$ has some frequency given by $E = \hbar \omega$. As a simple example, consider a particle in a box. Let’s say that it’s initially in the ground state, which is a standing wave like $\bigcirc$. It has some energy, so we can determine its frequency and period. Based on experience with standing waves on a string, we expect that after half a period it will look like $\bigcirc$. The problem is how it gets from the frown to the smile. If it behaved like a wave on a string, it would go from $\bigcirc$ to $\bigcirc$ to $\bigcirc$. But passing through the flat intermediate state won’t work, because of the fundamental structure of quantum mechanics, which includes the principle of state fundamentalism (p. 334). The state, represented by our picture of the wavefunction, is supposed to be all there is to know about the system. But if someone presented us with the zero wavefunction $\bigcirc$, we would have no information at all about the state.

So the wavefunction’s amplitude can’t oscillate back and forth along the real number line between positive and negative values, as it would for a wave on a string. What it actually does is to spin around in a circle in the complex plane. That is, instead of going like

$$1 \times \bigcirc \quad 0 \times \bigcirc \quad -1 \times \bigcirc, \quad \text{[wrong]}$$

it goes like

$$1 \times \bigcirc \quad -i \times \bigcirc \quad -1 \times \bigcirc. \quad \text{[right]}$$

(We have $-i$ here rather than $i$ because the convention is to have it spin clockwise.) This is enough to define the time-dependence of any state that has a definite energy: the energy tells us a frequency $\omega$, and then we know that the time-evolution of the state is simply that it spins its phase like $e^{-i\omega t}$, as in Euler’s formula (sec. 5.7.3, p. 129).

Another simple example is a traveling wave with a definite energy, figure m. Frozen at one moment in time, such a wave looks like $e^{ikx}$, which we visualize as a repeating rainbow. If we let this wave travel, then as we saw in ch. 2, its rigidly gliding motion should be described by a function whose input is of the form $kx - \omega t$. In other words, we have $\Psi = e^{i(kx-\omega t)}$. An observer who is watching the wave go by will say that a certain point with a fixed phase, say $\Psi = 1$, which is red in the figure, is moving to the right at a certain speed. But an observer who just stays in one place as the wave washes over her will say that $\Psi$ is spinning around in a circle in the complex plane. To see this more explicitly, we can break up the exponential into separate factors, $\Psi = e^{ikx}e^{-i\omega t}$. The time-dependent factor of
The ammonia molecule, in states that are inverted relative to one another.

\[ e^{-i\omega t} \] is just what we described a moment ago: a phase spinning clockwise in the complex plane.

### 17.8.2 The two-state system

As the next step up in sophistication we could consider a system that has two possible states. A simple example is the ammonia molecule in figure n. If the molecule is isolated and has no angular momentum, then it can never get from one orientation to the other by rotating. It can, however, flip from one to the other by turning itself inside out like an umbrella in a strong wind. Classically, if we initially put the molecule in one orientation, then it wouldn’t have enough energy to get through the intermediate flat configuration. But quantum-mechanically it can get through by tunneling. Therefore if we prepare the system in a pure \(|1\rangle\) state at some time, we expect that in the future it will be in some mixture \(a_1|1\rangle + a_2|2\rangle\), and if we check it there will be some probability \(|a_2|^2\) of finding that it’s switched.

Now this implies that \(|1\rangle\) and \(|2\rangle\) cannot be states of definite energy, because if they were, then after we prepared the system in an initial state with \(|a_1| = 1\) and \(|a_2| = 0\), all that would happen would be that the complex number \(a_1\) would spin its phase clockwise around the unit circle at some frequency given by \(E = \hbar \omega\). The probability \(|a_1|^2\) would stay the same, and indeed every observable of the system would stay the same, because absolute phases don’t mean anything. This is not the case — we expect by the totalitarian principle (p. 334) that the system can tunnel from 1 to 2. Therefore \(|1\rangle\) and \(|2\rangle\) must not be states of definite energy. The actual states of definite energy are standing wave patterns that look like \(\sim\) (the ground state) and \(\sim\) (the excited state). Because these states have slightly different energies, they oscillate at slightly different frequencies. If we initially put the system in the state \(\langle \sim | + | \sim \rangle\), then there is constructive interference on the left and destructive interference on the right. This is what it would mean to prepare the molecule in the state \(|1\rangle\). State \(|1\rangle\) is this superposition of the two energy states. Because the frequencies are unequal, eventually the two waves will flip their relative phases, and we will have state \(|2\rangle\). This is in fact the most general possible behavior for any quantum-mechanical system with two states: it can at most just oscillate between them. The oscillation can be complete (like the ammonia’s \(1 \leftrightarrow 2\)), nonexistent (if the system is in a state of definite energy), or partial (if the system is in an unequal mixture of two energy states).

### 17.8.3 The time-dependent Schrödinger equation

When a system has not just one or two but many possible states, the time-evolution is given by an equation known as the time-dependent Schrödinger equation, which we will not write down explicitly here. For example, a nonrelativistic electron in free space...
has infinitely many possible states, since there are no boundary conditions constraining its wave pattern. In this example the time-dependent Schrödinger equation becomes a certain wave equation, which is dispersive (sec. 16.2, p. 373), i.e., different wavelengths move at different speeds. It’s not hard to see why this is, since $p = h/\lambda$, and $p = mv$ for nonrelativistic motion.\(^4\)

For standing waves, the time-dependent Schrödinger equation is equivalent to the simpler time-independent version. Both versions are perfectly linear (sec. 17.6, p. 408).

17.8.4 Unitarity

In discussion question B on p. 408, we considered two traveling waves that collided head-on and superposed, and we convinced ourselves that probability would be conserved. It’s possible to prove that the time-evolution of the wavefunction always results in conservation of probability.

To put this in real-world terms, suppose that your history teacher calls on you in class and asks you what happened on December 7, 1941. With a straight face, you answer, “Professor, I can guarantee that something happened on that day.” “That’s correct. And what about August 6, 1945?” “Yes, something also happened on that day.” In other words, if we have a properly normalized wavefunction at a certain time, then we expect it to remain properly normalized at all later times.

The time evolution is also completely deterministic, so that if we know $\Psi$ initially, we can always predict it in the future. We can also “predict” backward in time, so that the system’s history can always be recovered from knowledge of its present state. Thus there is never any loss of information over time.

Summarizing, we have the following important principle:

**Unitary evolution of the wavefunction**
The wavefunction evolves over time in a deterministic and unitary manner, meaning that probability is conserved and information is never lost.

The word “unitary” is defined more precisely in linear algebra \(^2\)420.

Since we think of quantum mechanics as being all about randomness, this determinism may seem surprising. But determinism in the time-evolution of the wavefunction isn’t the same as determinism in the results of experiments as perceived and recorded by a human brain. Suppose that you prepare a uranium atom in its ground state, then wait one half-life and observe whether or not

\(^4\)But as discussed in sec. 16.2, it is necessary to distinguish phase velocity from group velocity.
it has decayed, as in the thought experiment of Schrödinger’s cat (p. 358). There is no uncertainty or randomness about the wavefunction of the whole system (atom plus you) at the end. We know for sure what it looks like. It consists of an equal superposition of two states, one in which the atom has decayed and your brain has observed that fact, and one in which the atom has not yet decayed and that fact is instead recorded in your brain.

As a possible example of a violation of unitarity, in an exotic context, consider the disappearance of matter into a black hole. If I throw my secret teenage diary into a black hole, then it contributes a little bit to the black hole’s mass, but the embarrassing information on the pages is lost forever. This loss of information seems to imply nonunitarity. This is one of several arguments suggesting that quantum mechanics cannot fully handle the gravitational force. Thus although physicists currently seem to possess a completely successful theory of gravity (Einstein’s theory of general relativity) and a completely successful theory of the microscopic world (quantum mechanics), the two theories are irreconcilable, and we can only make educated guesses, for example, about the behavior of a hypothetical microscopic black hole.

A standing wave example 9

As discussed on p. 413, a standing wave \( \bigcirc \) will evolve, after a quarter of a period, into \( -i \times \bigcirc \). The probability density depends only on the squared magnitude of the wavefunction, and is therefore unchanged over this evolution. Thus if the initial wave was normalized, then so is the final one, since multiplying a complex number by \( -i \) doesn’t change its magnitude, only its argument (phase). This is consistent with unitarity.

Unitarity also requires that information is never lost, and we can verify this. Suppose that we see the wavefunction \( -i \times \bigcirc \) at the later time. We can then infer what the wavefunction must have been a quarter-period earlier: it must have been \( \bigcirc \).

Any state of definite energy example 10

The reasoning of example 9 immediately carries over to any state of definite energy and any time interval, since the time evolution just amounts to spinning the phase in the complex plane at frequency \( \omega \), but this has no effect on any probabilities.

A mixture of energies example 11

Suppose we have a particle in a box with the initial wavefunction \( \bigcirc + \bigcirc \).

The momentum of the short-wavelength part is twice the momentum of the long-wavelength one, so assuming the motion is non-relativistic, the two energies differ by a factor of four. Therefore this is not a state of definite energy.

The energies of the two parts differ by a factor of four, so by \( E = \)
\( \hbar \omega \) so do their frequencies. As an example, let’s consider the evolution of this state over a time equal to half the period of the low-energy part, which is two full periods of the high-energy part. After this time, we have

\[ \psi \psi^\dagger. \]

A brute-force verification of the conservation of probability is possible, but involves a somewhat messy and tedious calculation that \( \int_0^{\pi} (A \sin x + \sin 2x)^2 \, dx \) is the same as \( \int_0^{\pi} (A \sin x - \sin 2x)^2 \, dx \). (If you graph the two functions in a utility such as the online app desmos, you should be able to see easily that this holds visually.)

A much quicker and easier method is to use the linearity of the inner product. For the total initial probability, we have

\[ \langle \psi \psi^\dagger \rangle = \langle \psi \psi^\dagger \rangle + \langle \psi \psi^\dagger \rangle. \]

But since the states \( \psi \) and \( \psi^\dagger \) are perfectly distinguishable by measurement of an observable (energy), it follows that \( \langle \psi \psi^\dagger \rangle = 0 \) and \( \langle \psi \psi^\dagger \rangle = 0 \). Therefore the total initial probability simplifies to the expression

\[ \langle \psi \psi^\dagger \rangle. \]

This has a simple interpretation: if we measure the energy, then the first term is the probability that we’ll find the lower energy, and the second one is the probability of the higher energy. If we assume that the state initially had the right normalization, then the two terms in the expression add up to 1.

If we now go through similar reasoning for the wavefunction at the later time, we get

\[ \langle \psi \psi^\dagger \rangle. \]

But by linearity the first term can be massaged into the form \( \langle -\psi \psi^\dagger \rangle = (-1)^2 \langle \psi \psi^\dagger \rangle = \langle \psi \psi^\dagger \rangle \), as before. Probability has been conserved, as required by unitarity.

Because example 11 didn’t appeal to any specific properties of the states \( \psi \) and \( \psi^\dagger \), it can easily be extended to a proof of unitarity for a very large class of states: anything that we can form by mixing a bunch of states of definite energy.

---

1Radioactive decay example 12

In sec. 14.5, p. 333, I argued that the exponential decay law for radioactivity followed from very general ideas about quantum mechanics. The argument was that an unstable nucleus would be likely to be found in its ground state, and since the ground state is a single state, it has no way of retaining any memory. Therefore, if it’s survived up until a certain time \( t \), then its probability of surviving for some additional time interval \( dt \) can’t depend on \( t \). We...
now have enough of a picture of quantum mechanics to allow a
more detailed understanding of this point.

An initial stab at the problem shows that there is more subtlety to
it than might have been imagined. It seems natural to describe
the system in terms of two states: an undecayed state and a
decayed one. But as we saw in sec. 17.8.2, p. 414, exponential
decay is not even a possibility for a two-state system in quantum
mechanics: the most general possible behavior for such a system
is an oscillation back and forth between the two states.

Of course when an atom emits a photon or a nucleus undergoes
radioactive decay, there are not just two states but infinitely many.
For example, if a nucleus alpha decays, the energy of the alpha
is fixed by conservation of energy, but its momentum vector $p$
can point in any one of an infinite number of directions.

Let’s assume that we can describe the initial, undecayed nucleus
as a single state $|0\rangle$, its ground state. We label each possible final
state, which describes both the state of the daughter nucleus and
the state of the alpha, by the alpha particle’s momentum, $|p\rangle$. If
the nucleus is initially in the state

$$|0\rangle,$$

then because alpha decay is possible (the totalitarian principle
again), after some time interval $\Delta t$ the system will be in a mixture
of decayed and undecayed states,

$$a|0\rangle + \ldots,$$

where $a$ is some number and $\ldots$ denotes an infinite sum over the
various $|p\rangle$ states. The probability of decay after this amount of
time is $|a|^2$, and unitarity requires that $|a| < 1$. What will happen
after a second time interval $\Delta t$? Quantum mechanics is per-
fectly linear, so the rule for carrying the state forward in time by
$\Delta t$ must be a linear function. Let’s call this function $L$, so that
$L(|0\rangle) = a|0\rangle + \ldots$ Now suppose as well that alpha decay is irreversible. After all, it seems unlikely that the alpha would bounce
off of something, come back, and rejoin the daughter nucleus to
rebuild the original parent. Let’s see what happens when we do
a second time step by computing $L(L(|0\rangle))$. By linearity, we get
$L(a|0\rangle + \ldots) = aL(|0\rangle) + L(\ldots)$, or

$$a^2|0\rangle + \ldots$$

Here the new $\ldots$ is different from the old one, and we’ve made use
of the assumption of irreversibility by assuming that $L(\ldots)$ doesn’t
contribute to the amplitude of $|0\rangle$. We have exponential decay,
with the decay probability going like $1$, $|a|^2$, $|a|^4$, $\ldots$. This result
depended on two assumptions: (1) that it was valid to measure
the undecayed state as a single state, and (2) that the decay was irreversible.

There is a further interesting application of unitarity here, which is that exponential decay cannot be exact at all times. For if it were, then we could extrapolate back in time to before the start, and we would find that the decay probability at \( t = -\Delta t \), one clock tick before the start, was \(|a|^{-2}\). This would be greater than 1, which would violate unitarity. What's going on here? In reality, the original nucleus had to be formed somehow, probably in nuclear reactions inside a supernova billions of years ago. That process of formation was not clean and tidy, and the nucleus only later settled down into the behavior described above.

Figure o shows an example of what happens when we start off with this kind of messy state. This is a computer simulation of the time-dependent Schrödinger equation for a particle in one dimension, which we take as a toy model. The particle is like the alpha particle, and the potential \( U \) shown in the graph represents the effect of the attractive and repulsive forces of the rest of the nucleus.

The initial state at \( t = 0 \) is not a state of definite energy — a state of definite energy is one that doesn’t actually physically evolve, it just spins its phase. We can see that this state looks like a noisy mixture of a bunch of different wavelengths. The shorter wavelengths have higher energies. As soon as we start the simulation, a bunch of these waves fly off in both directions. We see them escaping at \( t = 1 \). But the lower-energy waves are too low in energy to be classically allowed to escape over the barriers. They can only escape by tunneling, which is a slower process. By \( t = 1 \), the shape of the part of the wave remaining inside the well has turned into a bell shape. This bell shape is what we were talking about before when we wrote \(|0\rangle\). At \( t = 2 \) the rate of decay sensed by a detector near the edge of the graph is still fluctuating a lot. But by \( t = 3 \) the system has settled down to a steady process of decay in which the system acts as we assumed in our simplified model: the undecayed part is well modeled by a single state, and the decay is irreversible. (Beyond this time, the outgoing waves get too small to see easily on the graphs, but they still exist.)
Linear algebra defines unitarity as follows. A unitary transformation is one that preserves inner products. That is, $O$ is unitary if $\langle Ou|Ov \rangle = \langle u|v \rangle$. This is similar to the way in which rotations preserve dot products in Euclidean geometry. You may have heard about the idea of an orthogonal matrix in a math course, with the classic example being a rotation matrix in two or three dimensions. A unitary matrix is the same concept, but generalized to linear algebra over the complex numbers. The linearity of the Schrödinger equation guarantees that the evolution of the wavefunction from one time to another time can be represented by a linear operator $O$. We require this to be unitary, and it is unitary for the Schrödinger equation.
Problems

Key
✓ A computerized answer check is available online.
★ A difficult problem.

1 (a) A distance scale is shown below the wavefunction illustrated in figure g on page 401. Compare this with the order-of-magnitude estimate derived in section 16.6.2, p. 385, for the radius \( r \) at which the wavefunction begins tailing off. Was the estimate on the right order of magnitude?

(b) Although we normally say the moon orbits the earth, actually they both orbit around their common center of mass, which is below the earth’s surface but not at its center. The same is true of the hydrogen atom. Does the center of mass lie inside the proton, or outside it?

2 The wavefunction of the electron in the ground state of a hydrogen atom, shown in the top left of figure j on p. 436, is

\[
\Psi = \pi^{−1/2} a^{−3/2} e^{−r/a},
\]

where \( r \) is the distance from the proton, and \( a = h^2/kme^2 = 5.3 \times 10^{-11} \) m is a constant that sets the size of the wave. The figure doesn’t show the proton; let’s take the proton to be a sphere with a radius of \( b = 0.5 \) fm.

(a) Reproduce figure j in a rough sketch, and indicate, relative to the size of your sketch, some idea of how big \( a \) and \( b \) are.

(b) Calculate symbolically, without plugging in numbers, the probability that at any moment, the electron is inside the proton. [Hint: Does it matter if you plug in \( r = 0 \) or \( r = b \) in the equation for the wavefunction?] ✓

(c) Calculate the probability numerically. ✓

(d) Based on the equation for the wavefunction, is it valid to think of a hydrogen atom as having a finite size? Can \( a \) be interpreted as the size of the atom, beyond which there is nothing? Or is there any limit on how far the electron can be from the proton?

3 Show that the wavefunction given in problem 2 is properly normalized.
In classical mechanics, an interaction energy of the form
\[ U(x) = \frac{1}{2} k x^2 \]
gives a harmonic oscillator: the particle moves back and forth at a frequency \( \omega = \sqrt{k/m} \). This form for \( U(x) \) is often a good approximation for an individual atom in a solid, which can vibrate around its equilibrium position at \( x = 0 \). (For simplicity, we restrict our treatment to one dimension, and we treat the atom as a single particle rather than as a nucleus surrounded by electrons). The atom, however, should be treated quantum-mechanically, not classically. It will have a wave function. We expect this wave function to have one or more peaks in the classically allowed region, and we expect it to tail off in the classically forbidden regions to the right and left. Since the shape of \( U(x) \) is a parabola, not a series of flat steps as in figure d on page 397, the wavy part in the middle will not be a sine wave, and the tails will not be exponentials.

(a) Show that there is a solution to the Schrödinger equation of the form
\[ \Psi(x) = e^{-bx^2}, \]
and relate \( b \) to \( k \), \( m \), and \( \hbar \). To do this, calculate the second derivative, plug the result into the Schrödinger equation, and then find what value of \( b \) would make the equation valid for all values of \( x \). This wavefunction turns out to be the ground state. Note that this wavefunction is not properly normalized — don’t worry about that.

(b) Sketch a graph showing what this wavefunction looks like.

(c) Let’s interpret \( b \). If you changed \( b \), how would the wavefunction look different? Demonstrate by sketching two graphs, one for a smaller value of \( b \), and one for a larger value.

(d) Making \( k \) greater means making the atom more tightly bound. Mathematically, what happens to the value of \( b \) in your result from part a if you make \( k \) greater? Does this make sense physically when you compare with part c?

Show that a wavefunction of the form \( \Psi = e^{b y} \sin(ax) \) is a possible solution of the Schrödinger equation in two dimensions, with a constant potential \( U \). Can we tell whether it would apply to a classically allowed region, or a classically forbidden one?

This problem generalizes the one-dimensional result from problem 16-5, p. 389.

Find the energy levels of a particle in a three-dimensional rectangular box with sides of length \( a \), \( b \), and \( c \).
7 Americium-241 is an artificial isotope used in smoke detectors. It undergoes alpha decay, with a half-life of 432 years. As discussed in example 1 on page 397, alpha decay can be understood as a tunneling process, and although the barrier is not rectangular in shape, the equation for the tunneling probability on page 397 can still be used as a rough guide to our thinking. For americium-241, the tunneling probability is about $1 \times 10^{-29}$. Suppose that this nucleus were to decay by emitting a helium-3 nucleus instead of an alpha particle (helium-4). Estimate the relevant tunneling probability, assuming that the total energy $E$ remains the same. This higher probability is contrary to the empirical observation that this nucleus is not observed to decay by $^3\text{He}$ emission with any significant probability, and in general $^3\text{He}$ emission is almost unknown in nature; this is mainly because the $^3\text{He}$ nucleus is far less stable than the helium-4 nucleus, and the difference in binding energy reduces the energy available for the decay.

8 The wavefunction $\Psi$ of an electron is a complex number. Make up an example of a value for the wavefunction that is not a real number, and consider the following expressions: $\Psi^2$, $|\Psi|^2$, $|\Psi|^2$. Which of these would it make sense to interpret as a probability density? All of them? Some? Only one? \(\triangleright\) Solution, p. 454

9 In problem 4 on p. 422, you showed that a wavefunction of the form

$$\Psi_0(x) = e^{-x^2/2}$$

was a solution of the Schrödinger equation for the quantum harmonic oscillator in one dimensions. (We ignore units, and the factor of $1/2$ in the exponent is just a convention.) It represents the ground state. The wavefunction of the first excited state is

$$\Psi_1(x) = xe^{-x^2/2},$$

with the same value of $b$.

(a) Show that these states are orthogonal in the sense defined on p. 410.
(b) What is an observable that would distinguish them?
Consider the wavefunctions $\Psi_1 = \bigwedge$ and $\Psi_2 = \bigvee$ for a particle in a one-dimensional box. Suppose we have the superposition $\Psi = A(2\Psi_1 + \Psi_2)$.

(a) If $\Psi$ is to be properly normalized, what is $|A|$?  
(b) Sketch the wavefunction.
(c) Suppose you can measure the position of the particle very accurately. What is the probability that the particle will be found in the left half of the box?  
(d) Instead of measuring position, suppose you measure the energy of the state. What is the probability that you’ll measure the ground state energy?  
(e) Suppose that the wavefunction had been $\Phi = A(2\Psi_1 - \Psi_2)$. Which of your answers to parts a-d would remain the same, and which would change? (You need not redo the work for the ones that would change. Just give your reasoning as to whether they would or would not.) [Problem by B. Shotwell.]
Chapter 18
Quantization of angular momentum

18.1 Quantization of angular momentum

Angular momentum is quantized in quantum physics. As an example, consider a quantum wave-particle confined to a circle, like a wave in a circular moat surrounding a castle. A sine wave in such a “quantum moat” cannot have any old wavelength, because an integer number of wavelengths must fit around the circumference, $C$, of the moat. The larger this integer is, the shorter the wavelength, and a shorter wavelength relates to greater momentum and angular momentum. Since this integer is related to angular momentum, we use the symbol $\ell$ for it:

$$\lambda = C/\ell.$$ 

The angular momentum is

$$L = rp.$$ 

Here, $r = C/2\pi$, and $p = h/\lambda = h\ell/C$, so

$$L = \frac{C}{2\pi} \cdot \frac{h\ell}{C} = \frac{h}{2\pi}\ell.$$ 

In the example of the quantum moat, angular momentum is quantized in units of $h/2\pi$. This makes $h/2\pi$ a pretty important number, so we define the abbreviation $\hbar = h/2\pi$. This symbol is read “h-bar.”

In fact, this is a completely general fact in quantum physics, not just a fact about the quantum moat:

**Quantization of angular momentum**

The angular momentum of a particle due to its motion through space is quantized in units of $\hbar$. 

---

a / 1. Eight wavelengths fit around this circle ($\ell = 8$). This is a standing wave. 2. A traveling wave with $\ell = 8$, depicted according to the color conventions defined in figure j, p. 406.

b / In the benzene molecule, the valence electrons of the carbon atoms occupy quantum states similar to the one in figure a (with smaller values of $\ell$, not $\ell = 8$).
A depiction of the wavefunction of a certain state in the hydrogen atom. The square is a plane slicing through the center of the atom, so that only two of the three dimensions are shown, say $x$ and $y$, but not $z$. The up-down direction in the picture doesn’t represent $z$, it represents $\Psi$.

**self-check A**

What is the angular momentum of the wavefunction shown in figure c?

> Answer, p. 458

### 18.2 Three dimensions

Our discussion of quantum-mechanical angular momentum has so far been limited to rotation in a plane, for which we can simply use positive and negative signs to indicate clockwise and counterclockwise directions of rotation. An atom, however, is unavoidably three-dimensional. We recall from the classical treatment of angular momentum in three-dimensions that the angular momentum of a particle is defined as the vector cross product $\mathbf{r} \times \mathbf{p}$. For an object like a spinning wheel, this vector points along the axis, in the direction given by a right-hand rule.

There is a basic problem here: the angular momentum of the electron in an atom depends on both its distance $r$ from the proton and its momentum $\mathbf{p}$, so in order to know its angular momentum precisely it would seem we would need to know both its position and its momentum simultaneously with good accuracy. This, however, seems forbidden by the Heisenberg uncertainty principle.

Actually the uncertainty principle does place limits on what can be known about a particle’s angular momentum vector, but it does not prevent us from knowing its magnitude as an exact integer multiple of $\hbar$. The reason is that in three dimensions, there are really three separate uncertainty principles:

\[
\Delta p_x \Delta x \gtrsim \hbar \\
\Delta p_y \Delta y \gtrsim \hbar \\
\Delta p_z \Delta z \gtrsim \hbar
\]

Now consider a particle, d/1, that is moving along the $x$ axis at
position $x$ and with momentum $p_x$. We may not be able to know both $x$ and $p_x$ with unlimited accuracy, but we can still know the particle's angular momentum about the origin exactly: it is zero, because the particle is moving directly away from the origin.

Suppose, on the other hand, a particle finds itself, $d/2$, at a position $x$ along the $x$ axis, and it is moving parallel to the $y$ axis with momentum $p_y$. It has angular momentum $xp_y$ about the $z$ axis, and again we can know its angular momentum with unlimited accuracy, because the uncertainty principle only relates $x$ to $p_x$ and $y$ to $p_y$. It does not relate $x$ to $p_y$.

As shown by these examples, the uncertainty principle does not restrict the accuracy of our knowledge of angular momenta as severely as might be imagined. However, it does prevent us from knowing all three components of an angular momentum vector simultaneously. The most general statement about this is the following theorem:

**The angular momentum vector in quantum physics**

The most that can be known about a (nonzero) orbital angular momentum vector is its magnitude and one of its three vector components. Both are quantized in units of $\hbar$.

To see why this is true, consider the example wavefunction shown in figure e. This is the like the quantum moat of figure a, p. 425, but extended to one more dimension. If we slice the sphere in any plane perpendicular to the $z$ axis, we get an 8-cycle circular rainbow exactly like figure a. This is required because $L_z = 8\hbar$. But if we take a slice perpendicular to some other axis, such as the $y$ axis, we don't get a circular rainbow as we would for a state with a definite value of $L_y$. It is obviously not possible to get circular rainbows for slices perpendicular to more than one axis. For those with a taste for rigor, a more careful mathematical argument is given in note 439.

As a consequence of this fact, we find that when the magnitude of the angular momentum is $L = \ell\hbar$, the total number of states with that value of $L$ is $2\ell + 1$. For example, when $L = 2\hbar$, we have 5 possible values of $L_z$: $-2$, $-1$, 0, 1, and 2. In the language of linear algebra (p. 156), these five states could be used as a basis for the five-dimensional space of states with $L = 2\hbar$. We could just as easily have chosen some other axis besides $z$. This would have just been a different choice of basis, which is arbitrary.
18.3 Quantum numbers

18.3.1 Completeness

The three states inside the box are a complete set of quantum numbers for \( \ell = 1 \). Other states with \( \ell = 1 \), such as the one on the right, are not really new: they can be expressed as superpositions of the original three we chose.

For a given \( \ell \), consider the set of states with all the possible values of the angular momentum’s component along some fixed axis. This set of states is complete, meaning that they encompass all the possible states with this \( \ell \).

For example, figure f shows wavefunctions with \( \ell = 1 \) that are solutions of the Schrödinger equation for a particle that is confined to the surface of a sphere. Although the formulae for these wavefunctions are not particularly complicated,\(^1\) they are not our main focus here, so to help with getting a feel for the idea of completeness, I have simply selected three points on the sphere at which to give numerical samples of the value of the wavefunction. These are the top (where the sphere is intersected by the positive \( z \) axis), left (\( x \)), and front (\( y \)). (Although the wavefunctions are shown using the color conventions defined in figure j, p. 406, these numerical samples should make the example understandable if you’re looking at a black and white copy of the book.)

Suppose we arbitrarily choose the \( z \) axis as the one along which to quantize the component of the angular momentum. With this choice, we have three possible values for \( \ell_z \): \(-1\), \(0\), and \(1\). These three states are shown in the three boxes surrounded by the black rectangle. This set of three states is complete.

Consider, for example, the fourth state, shown on the right outside the box. This state is clearly identifiable as a copy of the \( \ell_z = 0 \) state, rotated by 90 degrees counterclockwise, so it is the \( \ell_x = 0 \) state. We might imagine that this would be an entirely new prize to be added to our stamp collection. But it is actually not a state that we didn’t possess before. We can obtain it as the sum of the \( \ell_z = -1 \) and \( \ell_z = 1 \) states, divided by an appropriate normalization factor. Although I’m avoiding making this example an exercise in

\(^1\)They are \( \Psi_{1,-1} = \sin \theta e^{-i\phi} \), \( \Psi_{10} = \sqrt{2} \cos \theta \), and \( \Psi_{11} = \sin \theta e^{i\phi} \), where \( \theta \) is the angle measured down from the \( z \) axis, and \( \phi \) is the angle running counterclockwise around the \( z \) axis. These functions are called spherical harmonics.
manipulating formulae, it is easy to check that the sum does work out properly at the three sample points.

### 18.3.2 Sets of compatible quantum numbers

In sec. 16.1.3, p. 372, we discussed the idea of a quantum number, which is simply a label for a state. For example, a particle in a box with a wavefunction like $\psi$ can be labeled with the quantum number $N = 2$, which is an energy label. Sometimes a single quantum number isn’t enough to give a complete label for a state. For example, if our state was an infinite plane wave in free space, rather than a standing wave in a box, then giving its energy wouldn’t be enough information to completely describe the state. Different states could have the same energy but be propagating in different directions. For this reason, we will often want to use sets of quantum numbers.

Figure g shows some examples in which we can completely describe a wavefunction by giving a set of quantum numbers. It is important that the quantum numbers we use in describing a state be compatible. By analogy, “Bond, James, 007” would be a clear and consistent definition of the famous fictional spy, but in general this identification scheme would not work, because although almost everyone has a first and last name, most people do not have a license to kill with a corresponding double-oh number.

![Diagram](image)

The laser beam in the figure is a state described according to
its definite values \( p_x \) and \( y \), so we have the vanishing uncertainties \( \Delta p_x = 0 \) and \( \Delta y = 0 \). Since the Heisenberg uncertainty principle doesn’t talk about an \( x \) momentum in relation to a \( y \) position, this is OK. If we had been in doubt about whether this violated the uncertainty principle, we would have been reassured by our ability to draw the picture.

It is also possible to have \textit{incompatible} quantum numbers. The combination of \( p_x \) with \( x \) would be an incompatible set of quantum numbers, because a state can’t have a definite \( p_x \) and also a definite \( x \). If we try to draw such a wave, we fail. \( L_x \) and \( L_z \) would also be an incompatible set.

18.3.3 \textbf{Complete and compatible sets of quantum numbers}

Let’s summarize. Just as we expect everyone to have a first and last name, we expect there to be a complete and compatible set of quantum numbers for any given quantum-mechanical system. Completeness means that we have enough quantum numbers to uniquely describe every possible state of the system, although we may need to describe a state as a superposition, as with the state \( \ell_x = 0 \) in figure f on p. 428. Compatibility means that when we specify a set of quantum numbers, we aren’t making a set of demands that can’t be met.

18.4 The Stern-Gerlach experiment

In 1921, Otto Stern proposed an experiment about angular momentum, shown in figure h on p. 430, that his boss at the University of Frankfurt and many of his colleagues were certain wouldn’t work. At this time, quantization of angular momentum had been proposed by Niels Bohr, but most physicists, if they had heard of it at all, thought of the idea as a philosophical metaphor or a mathematical trick that just happened to give correct results. World War I was over, hyperinflation was getting under way in Germany (a paper mark was worth a few percent of its prewar value), and the Nazi coup was still in the future, so that Stern, a Jew, had not yet been forced to flee to America. Because of the difficult economic situation, Stern and his colleague Walther Gerlach scraped up some of the funds to carry out the experiment from US banker Henry Goldman, cofounder of the investment house Goldman-Sachs.

The entire apparatus was sealed inside a vacuum chamber with the best vacuum obtainable at the time. A sample of silver was heated to 1000°C, evaporating it. The atoms leaving the oven encountered two narrow slits, so that what emerged was a beam with a width of only 0.03 mm, or about a third of the width of a human hair. The atoms then encountered a magnetic field. Because the atoms were electrically neutral, we would normally expect them to be unaffected by a magnetic field. But in the planetary model of
the atom, we imagine the electrons as orbiting in circles like little current loops, which would give the atom a magnetic dipole moment \( m \). Even if we are sophisticated enough about quantum mechanics not to believe in the circular orbits, it is reasonable to imagine that such a dipole moment would exist. When a dipole encounters a nonuniform field, it experiences a force.² The rapidly varying magnetic field for this experiment was provided by a pair of specially shaped magnet poles (not shown in the figure).

Because electrons have charge, we expect the motion of an electron to give it a magnetic dipole moment \( m \). But they also have mass, so for exactly the same reasons, we expect there to be some angular momentum \( L \) as well. The analogy is in fact mathematically exact, so that \( m \propto L \). Therefore this experiment with dipoles and magnetic fields is actually a probe of the behavior of angular momentum at the atomic level. Luckily for Stern and Gerlach, who had no modern knowledge of atomic structure, the silver atoms that they chose to use do happen to have nonzero total \( L \), and therefore nonzero \( m \). The atoms come out of the oven with random orientations.

The details of the electromagnetism are a little complicated (2439), but the result is that the beam should be deflected in the \( x \) direction by an amount proportional to \( L_x \), the \( x \) component of its angular momentum. Classically, we would expect the following. Because the orientations of the atoms are random as they enter the magnetic field, they will have every possible value of \( L_x \) ranging from \(-|L|\) to \(+|L|\), and therefore we expect that when the magnetic field is turned on, the effect should be to smear out the image on the glass plate from a vertical line to a somewhat wider oval. The atoms are dispersed from left to right along a certain scale of measurement according to their random value of \( L_x \). The spectrometer is a device for determining \( L_x \), a continuously varying number.

But that’s all the classical theory. Quantum mechanically, \( L_x \) is quantized, so that only certain very specific values of the deflection occur. Therefore we expect to see well separated vertical bands on the glass plate corresponding to the quantized values of \( L_x \). This is approximately what is seen in figure h, although the field rapidly weakens outside the \( x-y \) plane, so we get the slightly more complicated pattern like a sideways lipstick kiss. The spin of the silver atom is clearly quantized, and it apparently has two possible values.

**Discussion questions**

A Could the Stern-Gerlach experiment be carried out with a beam of electrons?

²This is easier to see in the case of an electric dipole in a nonuniform electric field. If the dipole consists of charges \(+q\) and \( -q \) at opposite ends of a stick, then the nonuniform field will make unequal forces on them, and the total force will be nonzero.
A few weeks after the Stern-Gerlach experiment’s results became public, Einstein and Ehrenfest carried out the following reasoning, which seemed to them to make the results inexplicable. Before a particular silver atom enters the magnetic field, its magnetic moment \( \mathbf{m} \) is randomly oriented. Once it enters the magnetic field, it has an energy \( \mathbf{m} \cdot \mathbf{B} \). Unless there is a mechanism for the transfer of energy in or out of the atom, this energy can’t change, and therefore the magnetic moment can only precess about the \( \mathbf{B} \) vector, but the angle between \( \mathbf{m} \) and \( \mathbf{B} \) must remain the same. Therefore the atom cannot align itself with the field. (They considered various mechanisms of energy loss, such as collisions and radiation, and concluded that all of them were too slow by orders of magnitude to have an effect during the atom’s time of flight.) It seemed to them that as soon as the atom left the oven, it was somehow required to have anticipated the direction of the field and picked one of two orientations with respect to it. How can this paradox be resolved?

Suppose we send a beam of oxygen molecules, with \( L = \hbar \), through a Stern-Gerlach spectrometer, throwing away the emerging parts with \( \ell_x = -1 \) and +1 to make a beam of the pure \( \ell_x = 0 \) state. Now we let this beam pass through a second spectrometer that is identical but oriented along the \( z \) axis. Can we produce a beam in which every molecule has both \( \ell_x = 0 \) and \( \ell_z = +1 \)? Hint: See the example in fig. f, p. 428.

### 18.5 Intrinsic spin

#### 18.5.1 Experimental evidence

We observe two values of \( \ell_x \) (the two “lips”) in the Stern-Gerlach experiment. Why two? For a fixed value of \( \ell \), we have seen that the number of values of \( \ell_x \) is \( 2\ell + 1 \). If we set \( 2\ell + 1 \) equal to 2, we get \( \ell = 1/2 \). If so, then we would have to conclude from these results that a silver atom has spin 1/2, so that \( L_x \) takes on the two values \(-\hbar/2 \) and \(+\hbar/2 \). Although it took about five years for the experiment to be interpreted completely correctly, we now think of this “spin one half” as being the angular momentum of one of the electrons in the silver atom — we can think of it as the outermost electron (see example 2, p. 433).

This may seem paradoxical because the quantum moat (p. 425), for instance, gave only angular momenta that were integer multiples of \( \hbar \), not half-units, and I claimed that angular momentum was always quantized in units of \( \hbar \), not just in the case of the quantum moat. That whole discussion, however, assumed that the angular momentum would come from the motion of a particle through space. The \( \hbar/2 \) angular momentum of the electron is simply a property of the particle, like its charge or its mass. It has nothing to do with whether the electron is moving or not, and it does not come from any internal motion within the electron. Nobody has ever succeeded in finding any internal structure inside the electron, and even if there was internal structure, it would be mathematically impossible for it to result in a half-unit of angular momentum.
We simply have to accept this $\hbar/2$ angular momentum, called the “spin” of the electron — Mother Nature rubs our noses in it as an observed fact. Protons and neutrons have the same $\hbar/2$ spin, while photons have an intrinsic spin of $h$. In general, half-integer spins are typical of material particles. Integral values are found for the particles that carry forces: photons, which embody the electric and magnetic fields of force, as well as the more exotic messengers of the nuclear and gravitational forces. The photon is particularly important: it has spin 1.

As was the case with ordinary angular momentum, we can describe spin angular momentum in terms of its magnitude, and its component along a given axis. We write $s$ and $s_z$ for these quantities, expressed in units of $\hbar$, so an electron has $s = 1/2$ and $s_z = +1/2$ or $-1/2$.

18.5.2 Odds and evens, and how they add up

From grade-school arithmetic, we have the rules

\[
\begin{align*}
\text{even} + \text{even} &= \text{even} \\
\text{odd} + \text{even} &= \text{odd} \\
\text{odd} + \text{odd} &= \text{even}.
\end{align*}
\]

Thus we know that 123456789 + 987654321 is even, without having to actually compute the result. Dividing by two gives similar relationships for integer and half-integer angular momenta. For example, a half-integer plus an integer gives a half-integer, and therefore when we add the intrinsic spin 1/2 of an electron to any additional, integer spin that the electron has from its motion through space, we get a half-integer angular momentum. That is, the total angular momentum of an electron will always be a half-integer. Similarly, when we add the intrinsic spin 1 of a photon to its angular momentum due to its integral motion through space, we will always get an integer. Thus the integer or half-integer character of any particle’s total angular momentum (spin + motion) is determined entirely by the particle’s spin.

These relationships tell us things about the spins we can make by putting together different particles to make bigger particles, and they also tell us things about decay processes.

1 Spin of the helium atom example 1
A helium-4 atom consists of two protons, two neutrons, and two electrons. A proton, a neutron, and an electron each have spin 1/2. Since the atom is a composite of six particles, each of which has half-integer spin, the atom as a whole has an integer angular momentum.

1 Silver atoms in the Stern-Gerlach experiment example 2
The silver atoms used in the Stern-Gerlach experiment had an odd number of protons (47), an even number of neutrons (two
isotopes), and an odd number of electrons (47). The result is
that the atom as a whole has an integer spin. However, only the
electrons contribute significantly to the magnetic dipole moment
of an atom (≥439), so the experiment only probed their angular
momentum, which was a half-integer value because 47 is odd. In
principle this could be as high as 47/2, but in the ground state it
turns out to be only 1/2, which can be interpreted as the intrinsic
spin of one of the electrons. The other 46 electrons' orbital and
intrinsic angular momentum end up canceling out.

\[ E \text{mission of a photon from an atom} \]

An atom can emit light,

\[ \text{atom} \rightarrow \text{atom} + \text{photon}. \]

This works in terms of angular momentum because the photon's
spin 1 is an integer. Thus, regardless of whether the atom's an-
gular momentum is an integer or a half-integer, the process is
allowed by conservation of angular momentum. If the atom's an-
gular momentum is an integer, then we have integer = integer+1,
and if it's a half-integer, half-integer = half-integer + 1; either of
these is possible. If not for this logic, it would be impossible for
matter to emit light. In general, if we want a particle such as
a photon to pop into existence like this, it must have an integer
spin.

\[ \text{Beta decay} \]

When a free neutron undergoes beta decay, we have

\[ \text{n} \rightarrow \text{p} + \text{e}^- + \bar{\nu}. \]

All four of these particles have spin 1/2, so the angular momenta
go like

\[ \text{half-integer} \rightarrow \text{half-integer} + \text{half-integer} + \text{half-integer}, \]

which is possible, e.g., \( 1/2 = 3/2 - 5/2 + 3/2 \). Because the
neutrino has almost no interaction with normal matter, it normally
flies off undetected, and the reaction was originally thought to be

\[ \text{n} \rightarrow \text{p} + \text{e}^-. \]

With hindsight, this is impossible, because we can never have

\[ \text{half-integer} \rightarrow \text{half-integer} + \text{half-integer}. \]

The reasoning holds not just for the beta decay of a free neu-
tron, but for any beta decay: a neutrino or antineutrino must be
emitted in order to conserve angular momentum. But historically,
this was not understood at first, and when Enrico Fermi proposed
the existence of the neutrino in 1934, the journal to which he first
submitted his paper rejected it as “too remote from reality.”
18.5.3 Inner product

We’ve been thinking of an inner product like $\langle \Psi_1 | \Psi_2 \rangle$ as a measure of the overlap or similarity between two wave patterns, calculated using an integral like $\int \ldots dx$. This is fine for integer angular momenta that particles have because they’re moving through space, so that, e.g., for states in the “quantum moat,” $\langle \ell = 0 | \ell = 1 \rangle = 0$ (ex. 18, p. 442). This makes sense because states are supposed to have an inner product of zero if they are perfectly distinguishable by measuring some observable like $\ell$. We expect the same familiar behavior for a spin 1/2 that can exist in states $\uparrow$ and $\downarrow$, so that $\langle \uparrow | \downarrow \rangle = 0$. Inner products for half-integer spins just won’t be interpretable visually as overlaps of wave patterns, and that we won’t need to calculate them as integrals.

18.5.4 Classification of states in hydrogen

In sec. 2.7.2, p. 52, we discussed the idea of degeneracy in the context of a classical wave. For standing waves on a square membrane, we have patterns like $\boxed{\ 1 \ 3 \ 5 \ 7 \ }$ and $\boxed{\ 2 \ 4 \ 6 \ 8 \ }$, which have the same frequency $f$. We say that these patterns are two-fold degenerate. In a quantum-mechanical context, $E = hf$, so degeneracy also implies that states have the same energy.

Because the energy of states in the hydrogen atom only depends on $n$, we have degeneracies, and these degeneracies get doubled because the electron’s spin can have two values (example, fig. i). Getting the count right has big implications: in our example, the 8-fold degeneracy is the reason that the second row of the period table has eight chemical elements.

The degeneracy of the different $\ell_z$ and $s_z$ states follows from symmetry, as in our original example of degeneracy on p. 52, and is therefore exact. The degeneracy with respect to different values of $\ell$ for the same $n$ is not at all obvious, and is in fact not exact when effects such as relativity are taken into account. We refer to this as an “accidental” degeneracy. The very high level of degeneracy in the hydrogen atom means that when you observe it the hydrogen spectrum in your lab course, there is a great deal of structure that is effectively hidden from you. Historically, physicists were fooled by the apparent simplicity of the spectrum, and more than 70 years passed between the measurement of the spectrum and the time when the degeneracies were fully recognized and understood.

Figure j on page 436 shows the lowest-energy states of the hydrogen atom. The left-hand column of graphs displays the wavefunctions in the $x - y$ plane, and the right-hand column shows the probability distribution in a three-dimensional representation. The ground-state wavefunction, with $n = 1$, $\ell = 0$, was calculated in example 5, p. 401.

Example: the degeneracy of the $n = 2$ state in hydrogen.

The $n = 2$ energy level would be 4-fold degenerate if we didn’t know about spin (one $\ell = 0$ state plus three $\ell = 1$ states), but the electron’s spin makes it 8-fold degenerate.

i / Counting up a degeneracy.
The three states of the hydrogen atom having the lowest energies. Taking into account electron spin, the number of states is actually doubled.

Discussion questions

A. The quantum number \( n \) is defined as the number of radii at which the wavefunction is zero, including \( r = \infty \). Relate this to the features of figure j. Based on the definition, why can’t there be an \( n = 0 \) state?

B. Relate the features of the wavefunction plots in figure j to the corresponding features of the probability distribution pictures.

C. How can you tell from the wavefunction plots in figure j which ones have which angular momenta?

D. Criticize the following incorrect statement: “The \( \ell = 8 \) wavefunction in figure c has a shorter wavelength in the center because in the center the electron is in a higher energy level.”

E. Discuss the implications of the fact that the probability cloud in of the \( n = 2, \ell = 1 \) state is split into two parts.
18.6 The Pauli exclusion principle

What about other atoms besides hydrogen? It would seem that things would get much more complex with the addition of a second electron. A hydrogen atom only has one particle that moves around much, since the nucleus is so heavy and nearly immobile. Helium, with two, would be a mess. Instead of a wavefunction whose square tells us the probability of finding a single electron at any given location in space, a helium atom would need to have a wavefunction whose square would tell us the probability of finding two electrons at any given combination of points. Ouch! In addition, we would have the extra complication of the electrical interaction between the two electrons, rather than being able to imagine everything in terms of an electron moving in a static field of force created by the nucleus alone.

Despite all this, it turns out that we can get a surprisingly good description of many-electron atoms simply by assuming the electrons can occupy the same standing-wave patterns that exist in a hydrogen atom. The ground state of helium, for example, would have both electrons in states that are very similar to the \( n = 1 \) states of hydrogen. The second-lowest-energy state of helium would have one electron in an \( n = 1 \) state, and the other in an \( n = 2 \) states. The relatively complex spectra of elements heavier than hydrogen can be understood as arising from the great number of possible combinations of states for the electrons.

A surprising thing happens, however, with lithium, the three-electron atom. We would expect the ground state of this atom to be one in which all three electrons settle down into \( n = 1 \) states. What really happens is that two electrons go into \( n = 1 \) states, but the third stays up in an \( n = 2 \) state. This is a consequence of a new principle of physics:

**The Pauli Exclusion Principle**

Two electrons can never occupy the same state. If one electron has wavefunction \( u \), and another electron \( v \), then \( \langle u|v \rangle = 0 \).

There are two \( n = 1 \) states, one with \( s_z = +1/2 \) and one with \( s_z = -1/2 \), but there is no third \( n = 1 \) state for lithium’s third electron to occupy, so it is forced to go into an \( n = 2 \) state.

It can be proved mathematically that the Pauli exclusion principle applies to any type of particle that has half-integer spin. Thus two neutrons can never occupy the same state, and likewise for two protons. Such particles are referred to as fermions, after Enrico Fermi, and, broadly speaking, fundamental fermions are the particles that matter is made of.

Photons, however, are immune to the exclusion principle be-
cause their spin is an integer. Particles with integer spins are called bosons, after Satyendra Nath Bose. Bosons are generally the force-carriers in physics. Classically, we can say that radio signals work because electrical and magnetic forces propagate from the transmitter to the receiver, but quantum-mechanically, we would say that photons are what is being transmitted. Similarly, the strong nuclear force is transmitted by a type of bosons known as gluons, and the weak nuclear force by particles called the W and Z bosons.

This is a brief and incomplete sketch of what is known as the standard model of particle physics, which describes 17 types of fundamental particles, their properties and interactions.

To make the picture neat and tidy, we would like gravity to be like the other forces. Gravitational forces should be carried by some boson, which we would call the graviton. We can even tell some properties of the graviton: it should be massless, electrically neutral, and have spin $\hbar$. However, there are fundamental reasons why our theories of quantum mechanics break down when we try to apply them to gravity. It is almost certainly true that gravitons exist, but we can’t calculate much of anything about their detailed behavior. It is probably impractical as well to detect them directly using any foreseeable human technology — people designing hypothetical detectors end up talking about devices the size of an entire solar system. It is possible, however, that very sensitive and clever experiments could detect some effects that would give us a clue about how to reconcile gravity with quantum mechanics.
Notes for chapter 18

427  \(L_z\) incompatible with \(L_x\)

**Theorem:** On the sphere, if a wavefunction has definite values of both \(L_z\) and \(L_x\), then it is a wavefunction that is constant everywhere, so \(L = 0\).

**Lemma 1:** If the component of \(\ell_A\) along a certain axis \(A\) has a definite value and is nonzero, then (a) \(\Psi = 0\) at the poles, and (b) \(\Psi\) is of the form \(A e^{i\ell_A \phi}\) on any circle in a plane perpendicular to the axis. Part a holds because \(L = 0\) if \(r_\perp = 0\). For b, see p. 425.

**Lemma 2:** If the component of \(\mathbf{L}\) along a certain axis has a definite value and is zero, then \(\Psi\) is constant in any plane perpendicular to that axis. This follows from lemma 1 in the case where \(\ell_A = 0\).

**Case I:** \(\ell_z\) and \(\ell_x\) are both nonzero. We have \(\Psi = 0\) at the poles along both the \(x\) axis and the \(z\) axis. The \(z\)-axis pole is a point on the great circle perpendicular to the \(x\) axis, and vice versa, so applying 1b, \(A = 0\) and \(\Psi\) vanishes on both of these great circles. But now if we apply 1b along any slice perpendicular to either axis, we get \(\Psi = 0\) everywhere on that slice, so \(\Psi = 0\) everywhere.

**Case II:** \(\ell_z\) and \(\ell_x\) are both zero. By lemma 2, \(\Psi\) is a constant everywhere.

**Case III:** One component is zero and the other nonzero. Let \(\ell_z\) be the one that is zero. By 1a, \(\Psi = 0\) at the \(x\)-axis pole, so by 2, \(\Psi = 0\) on the great circle perpendicular to \(z\). But then 1b tells us that \(\Psi = 0\) everywhere.

431  Details of the electromagnetism involved in the Stern-Gerlach experiment

In this example, the forces in the \(x\) and \(z\) directions would be \(F_x = m \cdot (\partial \mathbf{B} / \partial x)\) and \(F_z = m \cdot (\partial \mathbf{B} / \partial z)\). (Because of Gauss’s law for magnetism, these two derivatives are not independent — we have \(\partial B_x / \partial x + \partial B_z / \partial z = 0\).)

Classically, we would expect the following. Each atom has an energy \(m \cdot \mathbf{B}\) due to its interaction with the magnetic field, and this energy is conserved, so that the component \(m_z\) stays constant. However, there is a torque \(m \times \mathbf{B}\), and this causes the direction of the atom’s angular momentum to precess, i.e., wobble like a top, with its angular momentum forming a cone centered on the \(x\) axis. This precession is extremely fast, carrying out about \(10^{10}\) wobbles per second, so that the atom precesses about \(10^6\) times while traveling the 3.5 cm length of the spectrometer. So even though the forces \(F_x\) and \(F_z\) are typically about the same size, the rapid precession causes \(F_z\) to average out to nearly zero, and only a deflection in the \(x\) direction is expected.

Although the discussion of precession above is really classical rather than quantum-mechanical, the result of \(F_z\) averaging out to zero turns out to be approximately right if the field is strong.

As a side issue (example 2, p. 433), there is the question of whether the Stern-Gerlach experiment should also be sensitive to the angular momentum of the nucleus. It isn’t very sensitive to this, because we have the analogy that mass is to angular momentum as charge is to the magnetic dipole moment, so that at a crude level of analysis, ignoring things like factors of two, we expect that for a fixed angular momentum, the dipole moment is proportional to the charge-to-mass ratio of the particle. For this reason, the magnetic dipole moment contributed by one of the protons is thousands of times smaller than that created by one of the electrons. The neutrons are electrically neutral, so we would not expect them to contribute at all. They actually contain electrically charged particles — quarks — but their contributions to the magnetic dipole moment are still small, for the same reason as in the case of the protons.
Problems

Key
√ A computerized answer check is available online.
★ A difficult problem.

1. Estimate the angular momentum of a spinning basketball, in units of $\hbar$. Explain how this result relates to the correspondence principle.

2. The ground state of the nucleus boron-10 ($^{10}$B) has spin $s = 3$, and is therefore degenerate.
   (a) List the $s_z$ states and check that the number of states you get is $2s + 1$.
   (b) Based on the spin, is the ground state of $^{10}$B a fermion, or is it a boson?
   (c) $^{10}$B has four excited states that are bound, and these have various spins. Could some of these states be fermionic and some bosonic?

3. Hydrogen-2 ($^{2}$H) is referred to as deuterium. It contains one proton and one neutron. Its ground state is its only bound state, and in this state the neutron and proton have the following quantum numbers:

   neutron: $\ell = 0, \ell_z = 0, s = 1/2, s_z = 1/2$
   proton: $\ell = 0, \ell_z = 0, s = 1/2, s_z = 1/2$.

   Here the $z$ axis has been chosen parallel to the total angular momentum, and the total angular momentum is 1. Another state in which we could put the system is this one:

   neutron: $\ell = 0, \ell_z = 0, s = 1/2, s_z = 1/2$
   proton: $\ell = 0, \ell_z = 0, s = 1/2, s_z = -1/2$.

   This state’s total angular momentum is 0. (This state is observed to be unbound, but we’re not concerned in this problem with whether states are bound or unbound.)

   Suppose that our system instead consisted of two neutrons and no protons at all. Could you put them in the spin-1 state? In the spin-0 state? (Don’t worry about whether these states are bound.)
This problem builds on the results of problems 16-5 (p. 389) and 17-6 (p. 422).

Suppose we have a three-dimensional box of dimensions $L \times L \times L/2$. Let the box be oriented so that the shorter dimension is along the $z$ direction. For convenience, define the quantity $\epsilon = \frac{\hbar^2}{8mL^2}$, which has units of energy.

(a) What are the five lowest energies allowed in this box, expressed in terms of $\epsilon$? Give the quantum numbers for each energy, and find the degeneracy (p. 435) of each.

(b) Suppose we put five electrons in this box such that they have the lowest possible total energy. (Keep in mind that there is a limit to how many electrons can have the same spatial wavefunction.) What is the total energy of this state?

(c) What are the two lowest-energy photons that can excite one of the five electrons (from the situation described in part b) to an excited state?

[Problem by B. Shotwell.]
Exercise 18: The quantum moat

Consider the following three states for a particle confined to a circle, like a wave in the moat surrounding a castle.

\[ |\Psi_0\rangle = \frac{1}{\sqrt{3}} \cdot A_0 \]
\[ |\Psi_1\rangle = \frac{1}{\sqrt{3}} \cdot A_1 \]
\[ |\Psi_2\rangle = \frac{1}{\sqrt{3}} \cdot A_2 \]

We show each wavefunction first using the color conventions introduced on p. 406, and then using numbers. The numerical description is discretized, i.e., we only consider the values of the wavefunction at found points, evenly spaced around the circle. In this discretized representation, we define the inner product using a sum rather than an integral. We can refer to the wavefunctions by notations such as \( |\Psi_0\rangle \) or simply \( |0\rangle \). The \( A \)'s are normalization constants.

1. In this representation, how would a wavefunction like \( \langle \Psi_1 | \) differ from \( |\Psi_1\rangle \)?
2. Which of these are states of definite angular momentum?
3. The instructor will demonstrate the determination of the magnitude \( |A_0| \) of the normalization constant for \( |0\rangle \).
4. The students determine the magnitudes of \( A_1 \) and \( A_2 \).
5. Suppose that we first prepare the particle in state 2, and then we measure its angular momentum. The instructor will use the discretized wavefunction to approximate \( |\langle 0|2 \rangle|^2 \), which can be interpreted as the probability that the measurement results in \( \ell = 0 \).
6. The students compute \( |\langle 1|2 \rangle|^2 \) and give a similar interpretation.\(^3\)
7. How would \( \langle 1|2 \rangle \) compare with \( \langle 2|1 \rangle \)?
8. Compute \( |\langle 0|1 \rangle|^2 \) and interpret the result.

\(^3\)This is the first of our results in which the discretization actually introduces a nonzero error. For comparison, you should find that your result is reasonably close to the exact \( |A_1 A_2 \int_0^\pi \exp(-i\pi x/2) \, dx|^2 = 2/\pi^2 \approx 0.20 \).
Hints

Hints for chapter 2
Page 58, problem 6:
How could you change the values of $x$ and $t$ so that the value of $y$ would remain the same? What would this represent physically?

Page 61, problem 19:
(a) The most straightforward approach is to apply the equation $\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2}$. Although this equation was developed in the main text in the context of a straight string with a curvy wave on it, it works just as well for a circular loop; the left-hand side is simply the inward acceleration of any point on the rope. Note, however, that we’ve been assuming the string was (at least approximately) parallel to the $x$ axis, which will only be true if you choose a specific value of $x$. You need to get an equation for $y$ in terms of $x$ in order to evaluate the right-hand side.

Hints for chapter 5
Page 140, problem 1:
The answers to the two parts are not the same.

Page 144, problem 18:
There are various ways of doing this, but one easy and natural approach is to change the base of the exponent to $e$ using the same method that we would use for real numbers.

Hints for chapter 11
Page 255, problem 26:
Expand $\sin \theta$ in a Taylor series around $\theta = 90^\circ$.

Solutions to selected problems

Solutions for chapter 2
Page 58, problem 7:
We have $\omega = 2\pi f = 630$ Hz and $k = 2\pi/\lambda = 2\pi f/v = 0.66$ m$^{-1}$. The wave’s equation is

$$u = \sin(kx - \omega t + \delta),$$

which, if we like, we can make into an explicit numerical equation,

$$u = \sin \left[(0.66 \text{ m}^{-1})x - (630 \text{ s}^{-1})t + \delta \right].$$

Page 60, problem 15:
(a) The quantity $x - y$ vanishes along the line $y = x$ lying in the first quadrant at a 45-degree angle between the axes. Squaring produces a trough parallel to this line, with a parabolic cross-section. Geometrically, the Laplacian can be interpreted as a measure of how much the value of $f$ at a point differs from its average value on a small circle centered on that point. The trough is concave up, so we can predict that the Laplacian will be positive everywhere.

(b) The zero result is clearly wrong because it disagrees with our conclusion from part a that the Laplacian is positive. A correct calculation gives $\frac{\partial^2 (x - y)^2}{\partial x^2} + \frac{\partial^2 (x - y)^2}{\partial y^2} = 4$.

Remark: The mistake described in the question is a common one, and is apparently based on
the idea that the notation $\nabla^2$ must mean applying an operator $\nabla$ twice. For those with some exposure to vector calculus, it may be of interest to note that the Laplacian is equivalent to the divergence of the gradient, which can be notated either $\text{div}(\text{grad } f)$ or $\nabla \cdot (\nabla f)$. The important thing to recognize is that the gradient, notated $\text{grad } f$ or $\nabla f$, outputs a vector, not a scalar like the quantity $Q$ defined in this problem.

Solutions for chapter 3

Page 83, problem 6:
(a) The Poynting vectors cancel.
(b) The electric fields cancel, while the magnetic field doubles. Since the total electric field is zero, $E \times B = 0$, and the Poynting vector is zero.
(c) Both methods give zero. This makes sense physically because we interpret the Poynting vector as a measure of the flow of energy. Energy is flowing in and out of the page at equal rates, so there is zero total flow.

Page 84, problem 9:
Although this is not a plane wave, if we take any small section of it, such as one of the squares in the figure, it can be approximated as a plane wave. Therefore we expect the electric and magnetic fields to be like those in a plane wave: perpendicular to each other and with $E = cB$. Since they are perpendicular to each other, the cross product occurring in the expression for the Poynting vector is equal to the product of the magnitudes $EB$, and we must have $EB \propto r^{-2}$. Because $E = cB$, the two fields must have the same dependence on $r$, and this means that we must have both $E \propto r^{-1}$ and $B \propto r^{-1}$. This is somewhat counterintuitive; it tells us that radiation fields fall off more slowly than the static field of a point source.

Solutions for chapter 5

Page 143, problem 12:

$$\sin(a + b) = \frac{e^{i(a+b)} - e^{-i(a+b)}}{2i}$$

$$= \frac{e^{-ia}e^{ib} - e^{-ia}e^{-ib}}{2i}$$

$$= \frac{[(\cos a + i \sin a)(\cos b + i \sin b) - (\cos a - i \sin a)(\cos b - i \sin b)]}{2i}$$

$$= \cos a \sin b + \sin a \cos b$$

By a similar computation, we find $\cos(a + b) = \cos a \cos b - \sin a \sin b$.

Page 143, problem 13:
If $z^3 = 1$, then we know that $|z| = 1$, since cubing $z$ cubes its magnitude. Cubing $z$ triples its argument, so the argument of $z$ must be a number that, when tripled, is equivalent to an angle of zero. There are three possibilities: $0 \times 3 = 0$, $(2\pi/3) \times 3 = 2\pi$, and $(4\pi/3) \times 3 = 4\pi$. (Other possibilities, such as $(32\pi/3)$, are equivalent to one of these.) The solutions are:

$$z = 1, \ e^{2\pi i/3}, \ e^{4\pi i/3}$$

Page 143, problem 14:
This function would be represented by the complex number 1, which lies on the positive real axis, one unit to the right of the origin. In this system of analogies, differentiation is represented by multiplication by $i\omega$, which here is $2i$. Taking a fourth derivative is represented by multiplying
four times by \(2i\), i.e., we take our original point, 1, and make it into \(1 \cdot (2i)^4 = 16\). Satisfying the differential equation then amounts to having \(16 - 16i = 0\), which is true.

**Page 145, problem 22:**
We have \(n = \sin \phi / \sin \theta\). Doing implicit differentiation, we find \(dn = -\sin \phi (\cos \theta / \sin^2 \theta) \, d\theta\), which can be rewritten as \(dn = -n \cot \theta \, d\theta\). This can be minimized by making \(\theta\) as big as possible. To make \(\theta\) as big as possible, we want \(\phi\) to be as close as possible to 90 degrees, i.e., almost grazing the surface of the tank.

This result makes sense, because we’re depending on refraction in order to get a measurement of \(n\). At \(\phi = 0\), we get \(\theta = 0\), which provides no information at all about the index of refraction — the error bars become infinite. The amount of refraction increases as the angles get bigger.

**Solutions for chapter 6**

**Page 159, problem 3:**
As in the example, we have as our starting point the relations \(m^2 = E^2 - p^2\) and \(v = p/E\). Here we want to eliminate \(p\), so we substitute \(p = vE\) into the definition of mass, which gives \(m^2 = E^2 - v^2 E^2\). Solving this for \(E\) gives an expression that can be written most compactly as \(E = m\gamma\).

**Page 159, problem 5:**
\(\{\hat{x}\}\) is not a basis, because there are vectors such as \(\hat{y}\) that we can’t form as a linear combination (i.e., scalar multiple) of \(\hat{x}\). \(\{\hat{x}, \hat{y}\}\) is the standard basis for this vector space. \(\{\hat{x}, \hat{x} + \hat{y}\}\) also works as a basis, because the two vectors are linearly independent, and it’s easy to check that any vector in the plane can be formed as a linear superposition of them. \(\{\hat{x}, \hat{y}, \hat{x} + \hat{y}\}\) is not a basis, because these three vectors are not linearly independent.

**Page 160, problem 6:**
(a) The sketch for \(\ell\) will be a 45-degree line through the origin, while \(r\) will be only the part of that line in the first quadrant. Of the two, only \(\ell\) is a vector space. The set \(r\) isn’t a vector space, because it doesn’t have additive inverses.
(b) We have \((1/2)(\pi + \pi) = 0\), but \((1/2)\pi + (1/2)\pi = \pi\).

**Page 160, problem 7:**
To do anything useful with these expressions describing units, we need to be able to talk about things like dividing meters by seconds to get meters per second. Thus “addition” needs to be multiplication, which corresponds to adding the exponents. Scalar “multiplication” actually has to be exponentiation, e.g., “multiplying” units of meters by the scalar 2 should give square meters.

**Solutions for chapter 7**

**Page 180, problem 9:**
(a) Roughly speaking, the thermal energy is \(\sim k_B T\) (where \(k_B\) is the Boltzmann constant), and we need this to be on the same order of magnitude as \(ke^2/r\) (where \(k\) is the Coulomb constant). For this type of rough estimate it’s not especially crucial to get all the factors of two right, but let’s do so anyway. Each proton’s average kinetic energy due to motion along a particular axis is \((1/2)k_B T\). If two protons are colliding along a certain line in the center-of-mass frame, then their average combined kinetic energy due to motion along that axis is \(2(1/2)k_B T = k_B T\). So in fact the factors of 2 cancel. We have \(T = ke^2/k_{Br}\).
(b) The units are \(K = (J/m)(C^2)/(J/K)\cdot m\), which does work out.
(c) The numerical result is \(\sim 10^{10}\) K, which as suggested is much higher than the temperature
Solutions for chapter 8

Page 197, problem 1:
(a) We have

\[ dP = \rho g \, dy \]
\[ \Delta P = \int \rho g \, dy, \]

and since we’re taking water to be incompressible, and \( g \) doesn’t change very much over 11 km of height, we can treat \( \rho \) and \( g \) as constants and take them outside the integral.

\[ \Delta P = \rho g \Delta y \]
\[ = (1.0 \text{ g/cm}^3)(9.8 \text{ m/s}^2)(11.0 \text{ km}) \]
\[ = (1.0 \times 10^3 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(1.10 \times 10^4 \text{ m}) \]
\[ = 1.0 \times 10^8 \text{ Pa} \]
\[ = 1.0 \times 10^3 \text{ atm.} \]

The precision of the result is limited to a few percent, due to the compressibility of the water, so we have at most two significant figures. If the change in pressure were exactly a thousand atmospheres, then the pressure at the bottom would be 1001 atmospheres; however, this distinction is not relevant at the level of approximation we’re attempting here.

(b) Since the air in the bubble is in thermal contact with the water, it’s reasonable to assume that it keeps the same temperature the whole time. The ideal gas law is \( PV = nkT \), and rewriting this as a proportionality gives

\[ V \propto P^{-1}, \]

or

\[ \frac{V_f}{V_i} = \left( \frac{P_f}{P_i} \right)^{-1} \approx 10^3. \]

Since the volume is proportional to the cube of the linear dimensions, the growth in radius is about a factor of 10.

Page 197, problem 2:
(a) If the expression \( 1 + by \) is to make sense, then \( by \) has to be unitless, so \( b \) has units of \( \text{m}^{-1} \). The input to the exponential function also has to be unitless, so \( k \) also has of \( \text{m}^{-1} \). The only factor with units on the right-hand side is \( P_o \), so \( P_o \) must have units of pressure, or Pa.

(b)

\[ dP = \rho g \, dy \]
\[ \rho = \frac{1}{g} \frac{dP}{dy} \]
\[ = \frac{P_o}{g} e^{-ky}(-k - kby + b) \]

(c) The three terms inside the parentheses on the right all have units of \( \text{m}^{-1} \), so it makes sense to add them, and the factor in parentheses has those units. The units of the result from b then
look like

\[
\frac{\text{kg}}{\text{m}^3} = \frac{\text{Pa}}{\text{m/s}^2} \cdot \text{m}^{-1}
\]

\[
= \frac{\text{N/m}^2}{\text{m}^2/\text{s}^2}
\]

\[
= \frac{\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}}{\text{m}^2/\text{s}^2},
\]

which checks out.

**Solutions for chapter 9**

**Page 209, problem 3:**
If the full-sized brick A undergoes some process, such as heating it with a blowtorch, then we want to be able to apply the equation \( \Delta S = \frac{Q}{T} \) to either the whole brick or half of it, which would be identical to B. When we redefine the boundary of the system to contain only half of the brick, the quantities \( \Delta S \) and \( Q \) are each half as big, because entropy and energy are additive quantities. \( T \), meanwhile, stays the same, because temperature isn’t additive — two cups of coffee aren’t twice as hot as one. These changes to the variables leave the equation consistent, since each side has been divided by 2.

**Solutions for chapter 10**

**Page 228, problem 2:**
Because the surfaces are flat, you get specular reflection. In specular reflection, all the reflected rays go in one direction. Unless the plane is directly overhead, that direction won’t be the right direction to make the rays come back to the radar station.

This is different from a normal plane, which has complicated, bumpy surfaces. These surfaces give diffuse reflection, which spreads the reflected rays randomly in more or less every possible direction.

**Page 228, problem 3:**
It spells “bonk.”

**Solutions for chapter 11**

**Page 250, problem 1:**
For a flat mirror, \( d_i \) and \( d_o \) are equal, so the magnification is 1, i.e., the image is the same size
as the object.

**Page 250, problem 3:**
See the ray diagram below. Decreasing $\theta_o$ decreases $\theta_i$, so the equation $\theta_f = \pm \theta_i + \pm \theta_o$ must have opposite signs on the right. Since $\theta_o$ is bigger than $\theta_i$, the only way to get a positive $\theta_f$ is if the signs are $\theta_f = -\theta_i + \theta_o$. This gives $1/f = -1/d_i + 1/d_o$.

**Page 250, problem 5:**
(a) The object distance is less than the focal length, so the image is virtual: because the object is so close, the cone of rays is diverging too strongly for the mirror to bring it back to a focus. (b) Now the object distance is greater than the focal length, so the image is real. (c),(d) A diverging mirror can only make virtual images.

**Page 250, problem 6:**
(a) In problem #2 we found that the equation relating the object and image distances was of the form $1/f = -1/d_i + 1/d_o$. Let’s make $f = 1.00$ m. To get a virtual image we need $d_o < f$, so let $d_o = 0.50$ m. Solving for $d_i$, we find $d_i = 1/(1/d_o - 1/f) = 1.00$ m. The magnification is $M = d_i/d_o = 2.00$. If we change $d_o$ to 0.55 m, the magnification becomes 2.22. The magnification changes somewhat with distance, so the store’s ad must be assuming you’ll use the mirror at a certain distance. It can’t have a magnification of 5 at all distances. (b) Theoretically yes, but in practical terms no. If you go through a calculation similar to the one in part a, you’ll find that the images of both planets are formed at almost exactly the same $d_i$, $d_i = f$, since $1/d_o$ is pretty close to zero for any astronomical object. The more distant planet has an image half as big ($M = d_i/d_o$, and $d_o$ is doubled), but we’re talking about angular magnification here, so what we care about is the angular size of the image compared to the angular size of the object. The more distant planet has half the angular size, but its image has half the angular size as well, so the angular magnification is the same. If you think about it, it wouldn’t make much sense for the angular magnification to depend on the planet’s distance — if it did, then determining astronomical distances would be much easier than it actually is!

**Page 251, problem 7:**
(a) This occurs when the $d_i$ is infinite. Let’s say it’s a converging mirror creating a virtual image, as in problems 2 and 3. Then we’d get an infinite $d_i$ if we put $d_o = f$, i.e., the object is at the focal point of the mirror. The image is infinitely large, but it’s also infinitely far away, so its angular size isn’t infinite; an angular size can never be more than about 180° since you can’t see in back of your head!. (b) It’s not possible to make the magnification infinite by having $d_o = 0$. The image location and object location are related by $1/f = 1/d_o - 1/d_i$, so $1/d_i = 1/d_o - 1/f$. If $d_o$ is zero, then
$1/d_o$ is infinite, $1/d_i$ is infinite, and $d_i$ is zero as well. In other words, as $d_o$ approaches zero, so does $d_i$, and $d_i/d_o$ doesn’t blow up. Physically, the mirror’s curvature becomes irrelevant from the point of view of a tiny flea sitting on its surface: the mirror seems flat to the flea. So physically the magnification would be 1, not infinity, for very small values of $d_o$.

**Page 251, problem 9:**
The magnification is the ratio of the image’s size to the object’s size. It has nothing to do with the person’s location. The angular magnification, however, does depend on the person’s location, because things farther away subtend smaller angles. The distance to the actual object is not changed significantly, since it’s zillions of miles away in outer space, but the distance to the image does change if the observer’s point of view changes. If you can get closer to the image, the angular magnification is greater.

**Page 252, problem 11:**
The refracted ray that was bent closer to the normal in the plastic when the plastic was in air will be bent farther from the normal in the plastic when the plastic is in water. It will become a diverging lens.

**Page 252, problem 13:**
Refraction occurs only at the boundary between two substances, which in this case means the surface of the lens. Light doesn’t get bent at all inside the lens, so the thickness of the lens isn’t really what’s important. What matters is the angles of the lens’ surfaces at various points.

Ray 1 makes an angle of zero with respect to the normal as it enters the lens, so it doesn’t get bent at all, and likewise at the back.

At the edge of the lens, 2, the front and back are not parallel, so a ray that traverses the lens at the edge ends up being bent quite a bit.

Although I drew both ray 1 and ray 2 coming in along the axis of the lens, it really doesn’t matter. For instance, ray 3 bends on the way in, but bends an equal amount on the way out, so it still emerges from the lens moving in the same direction as the direction it originally had.

Summarizing and systematizing these observations, we can say that for a ray that enters the lens at the center, where the surfaces are parallel, the sum of the two deflection angles is zero. Since the total deflection is zero at the center, it must be larger away from the center.
Page 252, problem 15:
Normally, in air, your eyes do most of their focusing at the air-eye boundary. When you swim without goggles, there is almost no difference in speed at the water-eye interface, so light is not strongly refracted there (see figure), and the image is far behind the retina.

Goggles fix this problem for the following reason. The light rays cross a water-air boundary as they enter the goggles, but they’re coming in along the normal, so they don’t get bent. At the air-eye boundary, they get bent the same amount they normally would when you weren’t swimming.