to visualize correctly a great many phenomena.

As an example, let’s consider the distinctions among nonmetals, metals that are magnetic, and metals that are nonmagnetic. As shown in figure m, a metal differs from a nonmetal because its outermost electrons are free to wander rather than owing their allegiance to a particular atom. A metal that can be magnetized is one that is willing to line up the rotations of some of its electrons so that their axes are parallel. Recall that magnetic forces are forces made by moving charges; we have not yet discussed the mathematics and geometry of magnetic forces, but it is easy to see how random orientations of the atoms in the nonmagnetic substance would lead to cancellation of the forces.

Even if the planetary model does not immediately answer such questions as why one element would be a metal and another a nonmetal, these ideas would be difficult or impossible to conceptualize in the raisin cookie model.

**Discussion question**

A: In reality, charges of the same type repel one another and charges of different types are attracted. Suppose the rules were the other way around, giving repulsion between opposite charges and attraction between similar ones. What would the universe be like?

**13.4.3 Atomic number**

As alluded to in a discussion question in the previous section, scientists of this period had only a very approximate idea of how many units of charge resided in the nuclei of the various chemical elements. Although we now associate the number of units of nuclear charge with the element’s position on the periodic table, and call it the atomic number, they had no idea that such a relationship existed. Mendeleev’s table just seemed like an organizational tool, not something with any necessary physical significance. And everything Mendeleev had done seemed equally valid if you turned the table upside-down or reversed its left and right sides, so even if you wanted to number the elements sequentially with integers, there was an ambiguity as to how to do it. Mendeleev’s original table was in fact upside-down compared to the modern one.

In the period immediately following the discovery of the nucleus, physicists only had rough estimates of the charges of the various nuclei. In the case of the very lightest nuclei, they simply found the maximum number of electrons they could strip off by various methods: chemical reactions, electric sparks, ultraviolet light, and so on. For example they could easily strip off one or two electrons from helium, making He\(^+\) or He\(^{++}\), but nobody could make He\(^{+++}\), presumably because the nuclear charge of helium was only \(+2e\). Unfortunately only a few of the lightest elements could be stripped completely, because the more electrons were stripped off, the greater the
A modern periodic table, labeled with atomic numbers. Mendeleev’s original table was upside-down compared to this one.

Positive net charge remaining, and the more strongly the rest of the negatively charged electrons would be held on. The heavy elements’ atomic numbers could only be roughly extrapolated from the light elements, where the atomic number was about half the atom’s mass expressed in units of the mass of a hydrogen atom. Gold, for example, had a mass about 197 times that of hydrogen, so its atomic number was estimated to be about half that, or somewhere around 100. We now know it to be 79.

How did we finally find out? The riddle of the nuclear charges was at last successfully attacked using two different techniques, which gave consistent results. One set of experiments, involving x-rays, was performed by the young Henry Mosely, whose scientific brilliance was soon to be sacrificed in a battle between European imperialists over who would own the Dardanelles, during that pointless conflict then known as the War to End All Wars, and now referred to as World War I.

Since Mosely’s analysis requires several concepts with which you are not yet familiar, we will instead describe the technique used by James Chadwick at around the same time. An added bonus of
describing Chadwick’s experiments is that they presaged the important modern technique of studying collisions of subatomic particles. In grad school, I worked with a professor whose thesis adviser’s thesis adviser was Chadwick, and he related some interesting stories about the man. Chadwick was apparently a little nutty and a complete fanatic about science, to the extent that when he was held in a German prison camp during World War II, he managed to cajole his captors into allowing him to scrounge up parts from broken radios so that he could attempt to do physics experiments.

Chadwick’s experiment worked like this. Suppose you perform two Rutherford-type alpha scattering measurements, first one with a gold foil as a target as in Rutherford’s original experiment, and then one with a copper foil. It is possible to get large angles of deflection in both cases, but as shown in figure p, the alpha particle must be heading almost straight for the copper nucleus to get the same angle of deflection that would have occurred with an alpha that was much farther off the mark; the gold nucleus’ charge is so much greater than the copper’s that it exerts a strong force on the alpha particle even from far off. The situation is very much like that of a blindfolded person playing darts. Just as it is impossible to aim an alpha particle at an individual nucleus in the target, the blindfolded person cannot really aim the darts. Achieving a very close encounter with the copper atom would be akin to hitting an inner circle on the dartboard. It’s much more likely that one would have the luck to hit the outer circle, which covers a greater number of square inches. By analogy, if you measure the frequency with which alphas are scattered by copper at some particular angle, say between 19 and 20 degrees, and then perform the same measurement at the same angle with gold, you get a much higher percentage for gold than for copper.

In fact, the numerical ratio of the two nuclei’s charges can be derived from this same experimentally determined ratio. Using the standard notation \( Z \) for the atomic number (charge of the nucleus

\[ \text{Section 13.4 The nucleus} \]
divided by $e$), the following equation can be proved (example 4):

$$\frac{Z_{\text{gold}}^2}{Z_{\text{copper}}^2} = \frac{\text{number of alphas scattered by gold at 19-20}^\circ}{\text{number of alphas scattered by copper at 19-20}^\circ}$$

By making such measurements for targets constructed from all the elements, one can infer the ratios of all the atomic numbers, and since the atomic numbers of the light elements were already known, atomic numbers could be assigned to the entire periodic table. According to Mosely, the atomic numbers of copper, silver and platinum were 29, 47, and 78, which corresponded well with their positions on the periodic table. Chadwick’s figures for the same elements were 29.3, 46.3, and 77.4, with error bars of about 1.5 times the fundamental charge, so the two experiments were in good agreement.

The point here is absolutely not that you should be ready to plug numbers into the above equation for a homework or exam question! My overall goal in this chapter is to explain how we know what we know about atoms. An added bonus of describing Chadwick’s experiment is that the approach is very similar to that used in modern particle physics experiments, and the ideas used in the analysis are closely related to the now-ubiquitous concept of a “cross-section.” In the dartboard analogy, the cross-section would be the area of the circular ring you have to hit. The reasoning behind the invention of the term “cross-section” can be visualized as shown in figure p. In this language, Rutherford’s invention of the planetary model came from his unexpected discovery that there was a nonzero cross-section for alpha scattering from gold at large angles, and Chadwick confirmed Mosely’s determinations of the atomic numbers by measuring cross-sections for alpha scattering.

Proof of the relationship between $Z$ and scattering example 4

The equation above can be derived by the following not very rigorous proof. To deflect the alpha particle by a certain angle requires that it acquire a certain momentum component in the direction perpendicular to its original momentum. Although the nucleus’s force on the alpha particle is not constant, we can pretend that it is approximately constant during the time when the alpha is within a distance equal to, say, 150% of its distance of closest approach, and that the force is zero before and after that part of the motion. (If we chose 120% or 200%, it shouldn’t make any difference in the final result, because the final result is a ratio, and the effects on the numerator and denominator should cancel each other.) In the approximation of constant force, the change in the alpha’s perpendicular momentum component is then equal to $F\Delta t$. The Coulomb force law says the force is proportional to $Z/r^2$. Although $r$ does change somewhat during the time interval of interest, it’s good enough to treat it as a constant number, since
we’re only computing the ratio between the two experiments’ results. Since we are approximating the force as acting over the time during which the distance is not too much greater than the distance of closest approach, the time interval $\Delta t$ must be proportional to $r$, and the sideways momentum imparted to the alpha, $F \Delta t$, is proportional to $(Z/r^2)r$, or $Z/r$. If we’re comparing alphas scattered at the same angle from gold and from copper, then $\Delta p$ is the same in both cases, and the proportionality $\Delta p \propto Z/r$ tells us that the ones scattered from copper at that angle had to be headed in along a line closer to the central axis by a factor equaling $Z_{\text{gold}}/Z_{\text{copper}}$. If you imagine a “dartboard ring” that the alphas have to hit, then the ring for the gold experiment has the same proportions as the one for copper, but it is enlarged by a factor equal to $Z_{\text{gold}}/Z_{\text{copper}}$. That is, not only is the radius of the ring greater by that factor, but unlike the rings on a normal dartboard, the thickness of the outer ring is also greater in proportion to its radius. When you take a geometric shape and scale it up in size like a photographic enlargement, its area is increased in proportion to the square of the enlargement factor, so the area of the dartboard ring in the gold experiment is greater by a factor equal to $(Z_{\text{gold}}/Z_{\text{copper}})^2$. Since the alphas are aimed entirely randomly, the chances of an alpha hitting the ring are in proportion to the area of the ring, which proves the equation given above.

As an example of the modern use of scattering experiments and cross-section measurements, you may have heard of the recent experimental evidence for the existence of a particle called the top quark. Of the twelve subatomic particles currently believed to be the smallest constituents of matter, six form a family called the quarks, distinguished from the other six by the intense attractive forces that make the quarks stick to each other. (The other six consist of the electron plus five other, more exotic particles.) The only two types of quarks found in naturally occurring matter are the “up quark” and “down quark,” which are what protons and neutrons are made of, but four other types were theoretically predicted to exist, for a total of six. (The whimsical term “quark” comes from a line by James Joyce reading “Three quarks for master Mark.”) Until recently, only five types of quarks had been proven to exist via experiments, and the sixth, the top quark, was only theorized. There was no hope of ever detecting a top quark directly, since it is radioactive, and only exists for a zillionth of a second before evaporating. Instead, the researchers searching for it at the Fermi National Accelerator Laboratory near Chicago measured cross-sections for scattering of nuclei off of other nuclei. The experiment was much like those of Rutherford and Chadwick, except that the incoming nuclei had to be boosted to much higher speeds in a particle accelerator. The resulting encounter with a target nucleus was so violent that both nuclei were completely demolished, but, as Einstein proved, energy
can be converted into matter, and the energy of the collision creates a spray of exotic, radioactive particles, like the deadly shower of wood fragments produced by a cannon ball in an old naval battle. Among those particles were some top quarks. The cross-sections being measured were the cross-sections for the production of certain combinations of these secondary particles. However different the details, the principle was the same as that employed at the turn of the century: you smash things together and look at the fragments that fly off to see what was inside them. The approach has been compared to shooting a clock with a rifle and then studying the pieces that fly off to figure out how the clock worked.

Discussion questions

A The diagram, showing alpha particles being deflected by a gold nucleus, was drawn with the assumption that alpha particles came in on lines at many different distances from the nucleus. Why wouldn’t they all come in along the same line, since they all came out through the same tube?

B Why does it make sense that, as shown in the figure, the trajectories that result in 19° and 20° scattering cross each other?

C Rutherford knew the velocity of the alpha particles emitted by radium, and guessed that the positively charged part of a gold atom had a charge of about +100e (we now know it is +79e). Considering the fact that some alpha particles were deflected by 180°, how could he then use conservation of energy to derive an upper limit on the size of a gold nucleus? (For simplicity, assume the size of the alpha particle is negligible compared to that of the gold nucleus, and ignore the fact that the gold nucleus recoils a little from the collision, picking up a little kinetic energy.)

13.4.4 The structure of nuclei

The proton

The fact that the nuclear charges were all integer multiples of e suggested to many physicists that rather than being a pointlike object, the nucleus might contain smaller particles having individual charges of +e. Evidence in favor of this idea was not long in arriving. Rutherford reasoned that if he bombarded the atoms of a very light element with alpha particles, the small charge of the target nuclei would give a very weak repulsion. Perhaps those few alpha particles that happened to arrive on head-on collision courses would get so close that they would physically crash into some of the target nuclei. An alpha particle is itself a nucleus, so this would be a collision between two nuclei, and a violent one due to the high speeds involved. Rutherford hit pay dirt in an experiment with alpha particles striking a target containing nitrogen atoms. Charged particles were detected flying out of the target like parts flying off of cars in a high-speed crash. Measurements of the deflection of these particles in electric and magnetic fields showed that they had the same charge-to-mass ratio as singly-ionized hydrogen atoms. Rutherford concluded that these were the conjectured singly-charged particles.
that held the charge of the nucleus, and they were later named protons. The hydrogen nucleus consists of a single proton, and in general, an element’s atomic number gives the number of protons contained in each of its nuclei. The mass of the proton is about 1800 times greater than the mass of the electron.

The neutron

It would have been nice and simple if all the nuclei could have been built only from protons, but that couldn’t be the case. If you spend a little time looking at a periodic table, you will soon notice that although some of the atomic masses are very nearly integer multiples of hydrogen’s mass, many others are not. Even where the masses are close whole numbers, the masses of an element other than hydrogen is always greater than its atomic number, not equal to it. Helium, for instance, has two protons, but its mass is four times greater than that of hydrogen.

Chadwick cleared up the confusion by proving the existence of a new subatomic particle. Unlike the electron and proton, which are electrically charged, this particle is electrically neutral, and he named it the neutron. The method Chadwick used was to expose a sample of the light element beryllium to a stream of alpha particles from a lump of radium. Beryllium has only four protons, so an alpha that happens to be aimed directly at a beryllium nucleus can actually hit it rather than being stopped short of a collision by electrical repulsion. Neutrons were observed as a new form of radiation emerging from the collisions, and Chadwick correctly inferred that they were previously unsuspected components of the nucleus that had been knocked out. As described earlier, Chadwick also determined the mass of the neutron; it is very nearly the same as that of the proton.

To summarize, atoms are made of three types of particles:

<table>
<thead>
<tr>
<th></th>
<th>charge</th>
<th>mass in units of the proton’s mass</th>
<th>location in atom</th>
</tr>
</thead>
<tbody>
<tr>
<td>proton</td>
<td>+e</td>
<td>1</td>
<td>in nucleus</td>
</tr>
<tr>
<td>neutron</td>
<td>0</td>
<td>1.001</td>
<td>in nucleus</td>
</tr>
<tr>
<td>electron</td>
<td>−e</td>
<td>1/1836</td>
<td>orbiting nucleus</td>
</tr>
</tbody>
</table>

The existence of neutrons explained the mysterious masses of the elements. Helium, for instance, has a mass very close to four times greater than that of hydrogen. This is because it contains two neutrons in addition to its two protons. The mass of an atom is essentially determined by the total number of neutrons and protons. The total number of neutrons plus protons is therefore referred to as the atom’s mass number.
Isotopes

We now have a clear interpretation of the fact that helium is close to four times more massive than hydrogen, and similarly for all the atomic masses that are close to an integer multiple of the mass of hydrogen. But what about copper, for instance, which had an atomic mass 63.5 times that of hydrogen? It didn’t seem reasonable to think that it possessed an extra half of a neutron! The solution was found by measuring the mass-to-charge ratios of singly-ionized atoms (atoms with one electron removed). The technique is essentially that same as the one used by Thomson for cathode rays, except that whole atoms do not spontaneously leap out of the surface of an object as electrons sometimes do. Figure r shows an example of how the ions can be created and injected between the charged plates for acceleration.

Injecting a stream of copper ions into the device, we find a surprise — the beam splits into two parts! Chemists had elevated to dogma the assumption that all the atoms of a given element were identical, but we find that 69% of copper atoms have one mass, and 31% have another. Not only that, but both masses are very nearly integer multiples of the mass of hydrogen (63 and 65, respectively). Copper gets its chemical identity from the number of protons in its nucleus, 29, since chemical reactions work by electric forces. But apparently some copper atoms have $63 - 29 = 34$ neutrons while others have $65 - 29 = 36$. The atomic mass of copper, 63.5, reflects the proportions of the mixture of the mass-63 and mass-65 varieties. The different mass varieties of a given element are called isotopes of that element.

Isotopes can be named by giving the mass number as a subscript to the left of the chemical symbol, e.g., $^{65}\text{Cu}$. Examples:

<table>
<thead>
<tr>
<th></th>
<th>protons</th>
<th>neutrons</th>
<th>mass number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1\text{H}$</td>
<td>1</td>
<td>0</td>
<td>0+1 = 1</td>
</tr>
<tr>
<td>$^4\text{He}$</td>
<td>2</td>
<td>2</td>
<td>2+2 = 4</td>
</tr>
<tr>
<td>$^{12}\text{C}$</td>
<td>6</td>
<td>6</td>
<td>6+6 = 12</td>
</tr>
<tr>
<td>$^{14}\text{C}$</td>
<td>6</td>
<td>8</td>
<td>6+8 = 14</td>
</tr>
<tr>
<td>$^{262}\text{Ha}$</td>
<td>105</td>
<td>157</td>
<td>105+157 = 262</td>
</tr>
</tbody>
</table>

**self-check A**

Why are the positive and negative charges of the accelerating plates reversed in the isotope-separating apparatus compared to the Thomson apparatus?  
▷ Answer, p. 457

Chemical reactions are all about the exchange and sharing of electrons: the nuclei have to sit out this dance because the forces of electrical repulsion prevent them from ever getting close enough to make contact with each other. Although the protons do have a vitally important effect on chemical processes because of their electrical forces, the neutrons can have no effect on the atom’s chemical...
reactions. It is not possible, for instance, to separate $^{63}\text{Cu}$ from $^{65}\text{Cu}$ by chemical reactions. This is why chemists had never realized that different isotopes existed. (To be perfectly accurate, different isotopes do behave slightly differently because the more massive atoms move more sluggishly and therefore react with a tiny bit less intensity. This tiny difference is used, for instance, to separate out the isotopes of uranium needed to build a nuclear bomb. The smallness of this effect makes the separation process a slow and difficult one, which is what we have to thank for the fact that nuclear weapons have not been built by every terrorist cabal on the planet.)

Sizes and shapes of nuclei

Matter is nearly all nuclei if you count by weight, but in terms of volume nuclei don’t amount to much. The radius of an individual neutron or proton is very close to 1 fm ($1\text{ fm}=10^{-15}\text{ m}$), so even a big lead nucleus with a mass number of 208 still has a diameter of only about 13 fm, which is ten thousand times smaller than the diameter of a typical atom. Contrary to the usual imagery of the nucleus as a small sphere, it turns out that many nuclei are somewhat elongated, like an American football, and a few have exotic asymmetric shapes like pears or kiwi fruits.

Discussion questions

A Suppose the entire universe was in a (very large) cereal box, and the nutritional labeling was supposed to tell a godlike consumer what percentage of the contents was nuclei. Roughly what would the percentage be like if the labeling was according to mass? What if it was by volume?

s / A nuclear power plant at Cattenom, France. Unlike the coal and oil plants that supply most of the U.S.’s electrical power, a nuclear power plant like this one releases no pollution or greenhouse gases into the Earth’s atmosphere, and therefore doesn’t contribute to global warming. The white stuff puffing out of this plant is non-radioactive water vapor. Although nuclear power plants generate long-lived nuclear waste, this waste arguably poses much less of a threat to the biosphere than greenhouse gases would.

Section 13.4 The nucleus 309
13.4.5 **The strong nuclear force, alpha decay and fission**

Once physicists realized that nuclei consisted of positively charged protons and uncharged neutrons, they had a problem on their hands. The electrical forces among the protons are all repulsive, so the nucleus should simply fly apart! The reason all the nuclei in your body are not spontaneously exploding at this moment is that there is another force acting. This force, called the *strong nuclear force*, is always attractive, and acts between neutrons and neutrons, neutrons and protons, and protons and protons with roughly equal strength. The strong nuclear force does not have any effect on electrons, which is why it does not influence chemical reactions.

Unlike electric forces, whose strengths are given by the simple Coulomb force law, there is no simple formula for how the strong nuclear force depends on distance. Roughly speaking, it is effective over ranges of $\sim 1$ fm, but falls off extremely quickly at larger distances (much faster than $1/r^2$). Since the radius of a neutron or proton is about 1 fm, that means that when a bunch of neutrons and protons are packed together to form a nucleus, the strong nuclear force is effective only between neighbors.

Figure u illustrates how the strong nuclear force acts to keep

\[ u / 1. \text{ The forces cancel.} \quad 2. \text{ The forces don’t cancel.} \quad 3. \text{ In a heavy nucleus, the large number of electrical repulsions can add up to a force that is comparable to the strong nuclear attraction.} \quad 4. \text{ Alpha emission.} \quad 5. \text{ Fission.} \]
ordinary nuclei together, but is not able to keep very heavy nuclei from breaking apart. In u/1, a proton in the middle of a carbon nucleus feels an attractive strong nuclear force (arrows) from each of its nearest neighbors. The forces are all in different directions, and tend to cancel out. The same is true for the repulsive electrical forces (not shown). In figure u/2, a proton at the edge of the nucleus has neighbors only on one side, and therefore all the strong nuclear forces acting on it are tending to pull it back in. Although all the electrical forces from the other five protons (dark arrows) are all pushing it out of the nucleus, they are not sufficient to overcome the strong nuclear forces.

In a very heavy nucleus, u/3, a proton that finds itself near the edge has only a few neighbors close enough to attract it significantly via the strong nuclear force, but every other proton in the nucleus exerts a repulsive electrical force on it. If the nucleus is large enough, the total electrical repulsion may be sufficient to overcome the attraction of the strong force, and the nucleus may spit out a proton. Proton emission is fairly rare, however; a more common type of radioactive decay\(^1\) in heavy nuclei is alpha decay, shown in u/4. The imbalance of the forces is similar, but the chunk that is ejected is an alpha particle (two protons and two neutrons) rather than a single proton.

It is also possible for the nucleus to split into two pieces of roughly equal size, u/5, a process known as fission. Note that in addition to the two large fragments, there is a spray of individual neutrons. In a nuclear fission bomb or a nuclear fission reactor, some of these neutrons fly off and hit other nuclei, causing them to undergo fission as well. The result is a chain reaction.

When a nucleus is able to undergo one of these processes, it is said to be radioactive, and to undergo radioactive decay. Some of the naturally occurring nuclei on earth are radioactive. The term “radioactive” comes from Becquerel’s image of rays radiating out from something, not from radio waves, which are a whole different phenomenon. The term “decay” can also be a little misleading, since it implies that the nucleus turns to dust or simply disappears – actually it is splitting into two new nuclei with the same total number of neutrons and protons, so the term “radioactive transformation” would have been more appropriate. Although the original atom’s electrons are mere spectators in the process of weak radioactive decay, we often speak loosely of “radioactive atoms” rather than “radioactive nuclei.”

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\(^1\)Alpha decay is more common because an alpha particle happens to be a very stable arrangement of protons and neutrons.
Randomness in physics

How does an atom decide when to decay? We might imagine that it is like a termite-infested house that gets weaker and weaker, until finally it reaches the day on which it is destined to fall apart. Experiments, however, have not succeeded in detecting such “tick-ing clock” hidden below the surface; the evidence is that all atoms of a given isotope are absolutely identical. Why, then, would one uranium atom decay today while another lives for another million years? The answer appears to be that it is entirely random. We can make general statements about the average time required for a certain isotope to decay, or how long it will take for half the atoms in a sample to decay (its half-life), but we can never predict the behavior of a particular atom.

This is the first example we have encountered of an inescapable randomness in the laws of physics. If this kind of randomness makes you uneasy, you’re in good company. Einstein’s famous quote is “…I am convinced that He [God] does not play dice.” Einstein’s distaste for randomness, and his association of determinism with divinity, goes back to the Enlightenment conception of the universe as a gigantic piece of clockwork that only had to be set in motion initially by the Builder. Physics had to be entirely rebuilt in the 20th century to incorporate the fundamental randomness of physics, and this modern revolution is the topic of chapters 15-18. In particular, we will delay the mathematical development of the half-life concept until then.

13.4.6 The weak nuclear force; beta decay

All the nuclear processes we’ve discussed so far have involved rearrangements of neutrons and protons, with no change in the total number of neutrons or the total number of protons. Now consider the proportions of neutrons and protons in your body and in the planet earth: neutrons and protons are roughly equally numerous in your body’s carbon and oxygen nuclei, and also in the nickel and iron that make up most of the earth. The proportions are about 50-50. But the only chemical elements produced in any significant quantities by the big bang were hydrogen (about 90%) and helium (about 10%). If the early universe was almost nothing but hydrogen atoms, whose nuclei are protons, where did all those neutrons come from?

The answer is that there is another nuclear force, the weak nuclear force, that is capable of transforming neutrons into protons and vice-versa. Two possible reactions are

\[ n \rightarrow p + e^- + \bar{\nu} \quad \text{[electron decay]} \]

and

\[ p \rightarrow n + e^+ + \nu. \quad \text{[positron decay]} \]
There is also a third type called electron capture, in which a proton grabs one of the atom’s electrons and they produce a neutron and a neutrino.

Whereas alpha decay and fission are just a redivision of the previously existing particles, these reactions involve the destruction of one particle and the creation of three new particles that did not exist before.

There are three new particles here that you have never previously encountered. The symbol $e^+$ stands for an antielectron, which is a particle just like the electron in every way, except that its electric charge is positive rather than negative. Antielectrons are also known as positrons. Nobody knows why electrons are so common in the universe and antielectrons are scarce. When an antielectron encounters an electron, they annihilate each other, producing gamma rays, and this is the fate of all the antielectrons that are produced by natural radioactivity on earth. Antielectrons are an example of antimatter. A complete atom of antimatter would consist of antiprotons, antielectrons, and antineutrons.Although individual particles of antimatter occur commonly in nature due to natural radioactivity and cosmic rays, only a few complete atoms of antihydrogen have ever been produced artificially.

The notation $\nu$ stands for a particle called a neutrino, and $\bar{\nu}$ means an antineutrino. Neutrinos and antineutrinos have no electric charge (hence the name).

We can now list all four of the known fundamental forces of physics:

- gravity
- electromagnetism
- strong nuclear force
- weak nuclear force

The other forces we have learned about, such as friction and the normal force, all arise from electromagnetic interactions between atoms, and therefore are not considered to be fundamental forces of physics.

\[ \text{Decay of } ^{212}\text{Pb} \]

As an example, consider the radioactive isotope of lead $^{212}\text{Pb}$. It contains 82 protons and 130 neutrons. It decays by the process $n \rightarrow p + e^- + \bar{\nu}$. The newly created proton is held inside the nucleus by the strong nuclear force, so the new nucleus contains 83 protons and 129 neutrons. Having 83 protons makes it the element bismuth, so it will be an atom of $^{212}\text{Bi}$.
In a reaction like this one, the electron flies off at high speed (typically close to the speed of light), and the escaping electrons are the things that make large amounts of this type of radioactivity dangerous. The outgoing electron was the first thing that tipped off scientists in the early 1900s to the existence of this type of radioactivity. Since they didn’t know that the outgoing particles were electrons, they called them beta particles, and this type of radioactive decay was therefore known as beta decay. A clearer but less common terminology is to call the two processes electron decay and positron decay.

The neutrino or antineutrino emitted in such a reaction pretty much ignores all matter, because its lack of charge makes it immune to electrical forces, and it also remains aloof from strong nuclear interactions. Even if it happens to fly off going straight down, it is almost certain to make it through the entire earth without interacting with any atoms in any way. It ends up flying through outer space forever. The neutrino’s behavior makes it exceedingly difficult to detect, and when beta decay was first discovered nobody realized that neutrinos even existed. We now know that the neutrino carries off some of the energy produced in the reaction, but at the time it seemed that the total energy afterwards (not counting the unsuspected neutrino’s energy) was greater than the total energy before the reaction, violating conservation of energy. Physicists were getting ready to throw conservation of energy out the window as a basic law of physics when indirect evidence led them to the conclusion that neutrinos existed.

Discussion questions

A In the reactions \( n \rightarrow p + e^- + \bar{\nu} \) and \( p \rightarrow n + e^+ + \nu \), verify that charge is conserved. In beta decay, when one of these reactions happens to a neutron or proton within a nucleus, one or more gamma rays may also be emitted. Does this affect conservation of charge? Would it be possible for some extra electrons to be released without violating charge conservation?

B When an antielectron and an electron annihilate each other, they produce two gamma rays. Is charge conserved in this reaction?

13.4.7 Fusion

As we have seen, heavy nuclei tend to fly apart because each proton is being repelled by every other proton in the nucleus, but is only attracted by its nearest neighbors. The nucleus splits up into two parts, and as soon as those two parts are more than about 1 fm apart, the strong nuclear force no longer causes the two fragments to attract each other. The electrical repulsion then accelerates them, causing them to gain a large amount of kinetic energy. This release of kinetic energy is what powers nuclear reactors and fission bombs.
Our sun’s source of energy is nuclear fusion, so nuclear fusion is also the source of power for all life on earth, including, 2, this rain forest in Fatu-Hiva. 3. The first release of energy by nuclear fusion through human technology was the 1952 Ivy Mike test at the Enewetak Atoll. 4. This array of gamma-ray detectors is called GAMMASPHERE. During operation, the array is closed up, and a beam of ions produced by a particle accelerator strikes a target at its center, producing nuclear fusion reactions. The gamma rays can be studied for information about the structure of the fused nuclei, which are typically varieties not found in nature. 5. Nuclear fusion promises to be a clean, inexhaustible source of energy. However, the goal of commercially viable nuclear fusion power has remained elusive, due to the engineering difficulties involved in magnetically containing a plasma (ionized gas) at a sufficiently high temperature and density. This photo shows the experimental JET reactor, with the device opened up on the left, and in action on the right.

It might seem, then, that the lightest nuclei would be the most stable, but that is not the case. Let’s compare an extremely light nucleus like $^4\text{He}$ with a somewhat heavier one, $^{16}\text{O}$. A neutron or proton in $^4\text{He}$ can be attracted by the three others, but in $^{16}\text{O}$, it might have five or six neighbors attracting it. The $^{16}\text{O}$ nucleus is therefore more stable.
It turns out that the most stable nuclei of all are those around nickel and iron, having about 30 protons and 30 neutrons. Just as a nucleus that is too heavy to be stable can release energy by splitting apart into pieces that are closer to the most stable size, light nuclei can release energy if you stick them together to make bigger nuclei that are closer to the most stable size. Fusing one nucleus with another is called nuclear fusion. Nuclear fusion is what powers our sun and other stars.

13.4.8 Nuclear energy and binding energies

In the same way that chemical reactions can be classified as exothermic (releasing energy) or endothermic (requiring energy to react), so nuclear reactions may either release or use up energy. The energies involved in nuclear reactions are greater by a huge factor. Thousands of tons of coal would have to be burned to produce as much energy as would be produced in a nuclear power plant by one kg of fuel.

Although nuclear reactions that use up energy (endothermic reactions) can be initiated in accelerators, where one nucleus is rammed into another at high speed, they do not occur in nature, not even in the sun. The amount of kinetic energy required is simply not available.

To find the amount of energy consumed or released in a nuclear reaction, you need to know how much nuclear interaction energy, $U_{\text{nuc}}$, was stored or released. Experimentalists have determined the amount of nuclear energy stored in the nucleus of every stable element, as well as many unstable elements. This is the amount of mechanical work that would be required to pull the nucleus apart into its individual neutrons and protons, and is known as the nuclear binding energy.

**A reaction occurring in the sun**

The sun produces its energy through a series of nuclear fusion reactions. One of the reactions is

$$^1\text{H} + ^2\text{H} \rightarrow ^3\text{He} + \gamma$$

The excess energy is almost all carried off by the gamma ray (not by the kinetic energy of the helium-3 atom). The binding energies in units of pJ (picojoules) are:

- $^1\text{H} = 0$ pJ
- $^2\text{H} = 0.35593$ pJ
- $^3\text{He} = 1.23489$ pJ

The total initial nuclear energy is 0 pJ+0.35593 pJ, and the final nuclear energy is 1.23489 pJ, so by conservation of energy, the gamma ray must carry off 0.87896 pJ of energy. The gamma ray is then absorbed by the sun and converted to heat.

*Self-check B*
Why is the binding energy of $^1\text{H}$ exactly equal to zero?  

Answer, p. 458

Figure w is a compact way of showing the vast variety of the nuclei. Each box represents a particular number of neutrons and protons. The black boxes are nuclei that are stable, i.e., that would require an input of energy in order to change into another. The gray boxes show all the unstable nuclei that have been studied experimentally. Some of these last for billions of years on the average before decaying and are found in nature, but most have much shorter average lifetimes, and can only be created and studied in the laboratory.

The curve along which the stable nuclei lie is called the line of stability. Nuclei along this line have the most stable proportion of neutrons to protons. For light nuclei the most stable mixture is about 50-50, but we can see that stable heavy nuclei have two or three times more neutrons than protons. This is because the electrical repulsions of all the protons in a heavy nucleus add up
A map showing levels of radiation near the site of the Chernobyl nuclear accident.

13.4.9 Biological effects of ionizing radiation

Units used to measure exposure

As a science educator, I find it frustrating that nowhere in the massive amount of journalism devoted to nuclear safety does one ever find any numerical statements about the amount of radiation to which people have been exposed. Anyone capable of understanding sports statistics or weather reports ought to be able to understand such measurements, as long as something like the following explanatory text was inserted somewhere in the article:

Radiation exposure is measured in units of Sieverts (Sv). The average person is exposed to about 2000 µSv (microSieverts) each year from natural background sources.

With this context, people would be able to come to informed conclusions. For example, figure x shows a scary-looking map of the levels of radiation in the area surrounding the 1986 nuclear accident at Chernobyl, Ukraine, the most serious that has ever occurred. At the boundary of the most highly contaminated (bright red) areas, people would be exposed to about 13,000 µSv per year, or about four times the natural background level. In the pink areas, which are still densely populated, the exposure is comparable to the natural level found in a high-altitude city such as Denver.

What is a Sievert? It measures the amount of energy per kilogram deposited in the body by ionizing radiation, multiplied by a “quality factor” to account for the different health hazards posed by alphas, betas, gammas, neutrons, and other types of radiation. Only ionizing radiation is counted, since nonionizing radiation simply heats one’s body rather than killing cells or altering DNA. For instance, alpha particles are typically moving so fast that their kinetic energy is sufficient to ionize thousands of atoms, but it is possible for an alpha particle to be moving so slowly that it would not have enough kinetic energy to ionize even one atom.

Unfortunately, most people don’t know much about radiation and tend to react to it based on unscientific cultural notions. These may, as in figure y, be based on fictional tropes silly enough to require the suspension of disbelief by the audience, but they can also be more subtle. People of my kids’ generation are more familiar with the 2011 Fukushima nuclear accident than with the much more serious Chernobyl accident. The news coverage of Fukushima showed scary scenes of devastated landscapes and distraught evacuees, implying that people had been killed and displaced by the release of radiation from the reaction. In fact, there were no deaths at all due
to the radiation released at Fukushima, and no excess cancer deaths are statistically predicted in the future. The devastation and the death toll of 16,000 were caused by the earthquake and tsunami, which were also what damaged the plant.

**Effects of exposure**

Notwithstanding the pop culture images like figure z, it is not possible for a multicellular animal to become “mutated” as a whole. In most cases, a particle of ionizing radiation will not even hit the DNA, and even if it does, it will only affect the DNA of a single cell, not every cell in the animal’s body. Typically, that cell is simply killed, because the DNA becomes unable to function properly. Once in a while, however, the DNA may be altered so as to make that cell cancerous. For instance, skin cancer can be caused by UV light hitting a single skin cell in the body of a sunbather. If that cell becomes cancerous and begins reproducing uncontrollably, she will end up with a tumor twenty years later.

Other than cancer, the only other dramatic effect that can result from altering a single cell’s DNA is if that cell happens to be a sperm or ovum, which can result in nonviable or mutated offspring. Men are relatively immune to reproductive harm from radiation, because their sperm cells are replaced frequently. Women are more vulnerable because they keep the same set of ova as long as they live.

**Effects of high doses of radiation**

A whole-body exposure of 5,000,000 µSv will kill a person within a week or so. Luckily, only a small number of humans have ever been exposed to such levels: one scientist working on the Manhattan Project, some victims of the Nagasaki and Hiroshima explosions, and 31 workers at Chernobyl. Death occurs by massive killing of cells, especially in the blood-producing cells of the bone marrow.

**Effects of low doses radiation**

Lower levels, on the order of 1,000,000 µSv, were inflicted on some people at Nagasaki and Hiroshima. No acute symptoms result from this level of exposure, but certain types of cancer are significantly more common among these people. It was originally expected that the radiation would cause many mutations resulting in birth defects, but very few such inherited effects have been observed.

A great deal of time has been spent debating the effects of very low levels of ionizing radiation. The following table gives some sample figures.
A typical example of radiation hormesis: the health of mice is improved by low levels of radiation. In this study, young mice were exposed to fairly high levels of x-rays, while a control group of mice was not exposed. The mice were weighed, and their rate of growth was taken as a measure of their health. At levels below about 50,000 µSv, the radiation had a beneficial effect on the health of the mice, presumably by activating cellular damage control mechanisms. The two highest data points are statistically significant at the 99% level. The curve is a fit to a theoretical model. Redrawn from T.D. Luckey, Hormesis with Ionizing Radiation, CRC Press, 1980.

Note that the largest number, on the first line of the table, is the maximum beneficial dose. The most useful evidence comes from experiments in animals, which can intentionally be exposed to significant and well measured doses of radiation under controlled conditions. Experiments show that low levels of radiation activate cellular damage control mechanisms, increasing the health of the organism. For example, exposure to radiation up to a certain level makes mice grow faster; makes guinea pigs’ immune systems function better against diptheria; increases fertility in trout and mice; improves fetal mice’s resistance to disease; increases the life-spans of flour beetles and mice; and reduces mortality from cancer in mice. This type of effect is called radiation hormesis.

There is also some evidence that in humans, small doses of radiation increase fertility, reduce genetic abnormalities, and reduce mortality from cancer. The human data, however, tend to be very poor compared to the animal data. Due to ethical issues, one cannot do controlled experiments in humans. For example, one of the best sources of information has been from the survivors of the Hiroshima and Nagasaki bomb blasts, but these people were also exposed to high levels of carcinogenic chemicals in the smoke from their burning cities; for comparison, firefighters have a heightened risk of cancer, and there are also significant concerns about cancer from the 9/11 attacks in New York. The direct empirical evidence about radiation hormesis in humans is therefore not good enough to tell us anything unambiguous, and the most scientifically reasonable approach is to assume that the results in animals also hold for humans: small doses of radiation in humans are beneficial, rather than harmful. However, a variety of cultural and historical factors have led to a situation in which public health policy is based on the assumption, known as “linear no-threshold” (LNT), that even tiny doses of radiation are harmful, and that the risk they carry is proportional to the dose. In other words, law and policy are made based on the assumption that the effects of radiation on humans are dramatically different than its effects on mice and guinea pigs. Even with the unrealistic assumption of LNT, one can still evaluate risks by comparing with natural background radiation. For example, we can see that

<table>
<thead>
<tr>
<th>Activity</th>
<th>Radiation Dose</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum beneficial dose per day</td>
<td>∼ 10,000 µSv</td>
</tr>
<tr>
<td>CT scan</td>
<td>∼ 10,000 µSv</td>
</tr>
<tr>
<td>natural background per year</td>
<td>2,000-7,000 µSv</td>
</tr>
<tr>
<td>health guidelines for exposure to a fetus</td>
<td>1,000 µSv</td>
</tr>
<tr>
<td>flying from New York to Tokyo</td>
<td>150 µSv</td>
</tr>
<tr>
<td>chest x-ray</td>
<td>50 µSv</td>
</tr>
</tbody>
</table>

the effect of a chest x-ray is about a hundred times smaller than the effect of spending a year in Colorado, where the level of natural background radiation from cosmic rays is higher than average, due to the high altitude. Dropping the implausible LNT assumption, we can see that the impact on one’s health of spending a year in Colorado is likely to be positive, because the excess radiation is below the maximum beneficial level.

The green case for nuclear power

In the late twentieth century, antinuclear activists largely succeeded in bringing construction of new nuclear power plants to a halt in the U.S. Ironically, we now know that the burning of fossil fuels, which leads to global warming, is a far more grave threat to the environment than even the Chernobyl disaster. A team of biologists writes: “During recent visits to Chernobyl, we experienced numerous sightings of moose (Alces alces), roe deer (Capreol capreolus), Russian wild boar (Sus scrofa), foxes (Vulpes vulpes), river otter (Lutra canadensis), and rabbits (Lepus europaeus) ... Diversity of flowers and other plants in the highly radioactive regions is impressive and equals that observed in protected habitats outside the zone ... The observation that typical human activity (industrialization, farming, cattle raising, collection of firewood, hunting, etc.) is more devastating to biodiversity and abundance of local flora and fauna than is the worst nuclear power plant disaster validates the negative impact the exponential growth of human populations has on wildlife.”

Nuclear power is the only source of energy that is sufficient to replace any significant percentage of energy from fossil fuels on the rapid schedule demanded by the speed at which global warming is progressing. People worried about the downside of nuclear energy might be better off putting their energy into issues related to nuclear weapons: the poor stewardship of the former Soviet Union’s warheads; nuclear proliferation in unstable states such as Pakistan; and the poor safety and environmental history of the superpowers’ nuclear weapons programs, including the loss of several warheads in plane crashes, and the environmental disaster at the Hanford, Washington, weapons plant.

Protection from radiation

People do sometimes work with strong enough radioactivity that there is a serious health risk. Typically the scariest sources are those used in cancer treatment and in medical and biological research.
Also, a dental technician, for example, needs to take precautions to avoid accumulating a large radiation dose from giving dental x-rays to many patients. There are three general ways to reduce exposure: time, distance, and shielding. This is why a dental technician doing x-rays wears a lead apron (shielding) and steps outside of the x-ray room while running an exposure (distance). Reducing the time of exposure dictates, for example, that a person working with a hot cancer-therapy source would minimize the amount of time spent near it.

Shielding against alpha and beta particles is trivial to accomplish. (Alphas can’t even penetrate the skin.) Gammas and x-rays interact most strongly with materials that are dense and have high atomic numbers, which is why lead is so commonly used. But other materials will also work. For example, the reason that bones show up so clearly on x-ray images is that they are dense and contain plenty of calcium, which has a higher atomic number than the elements found in most other body tissues, which are mostly made of water.

Neutrons are difficult to shield against. Because they are electrically neutral, they don’t interact intensely with matter in the same way as alphas and betas. They only interact if they happen to collide head-on with a nucleus, and that doesn’t happen very often because nuclei are tiny targets. Kinematically, a collision can transfer kinetic energy most efficiently when the target is as low in mass as possible compared to the projectile. For this reason, substances that contain a lot of hydrogen make the best shielding against neutrons. Blocks of paraffin wax from the supermarket are often used for this purpose.
Problems

Key
✓ A computerized answer check is available online.
⋆ A difficult problem.

1. $^{241}$Pu decays either by electron decay or by alpha decay. (A given $^{241}$Pu nucleus may do either one; it’s random.) What are the isotopes created as products of these two modes of decay?

2. As discussed in more detail in section 13.4, a nucleus contains protons, which have positive charge, and neutrons, which have zero charge. If only the electrical force existed, a nucleus would immediately fly apart due to electrical repulsion. However, there is also another force, called the strong nuclear force, which keeps this from happening. Suppose that a proton in a lead nucleus wanders out to the surface of the nucleus, and experiences a strong nuclear force of about 8 kN from the nearby neutrons and protons pulling it back in. Compare this numerically to the repulsive electrical force from the other protons, and verify that the net force is attractive. A lead nucleus is very nearly spherical, is about 6.5 fm in radius, and contains 82 protons, each with a charge of $+e$, where $e = 1.60 \times 10^{-19}$ C.

3. The nuclear process of beta decay by electron capture is described parenthetically on page 313. The reaction is $p + e^- \rightarrow n + \nu$. (a) Show that charge is conserved in this reaction. (b) Conversion between energy and mass is discussed in sec. 3.6.1, p. 75. Based on these ideas, explain why electron capture doesn’t occur in hydrogen atoms. (If it did, matter wouldn’t exist!)

4. Potassium 40 is the strongest source of naturally occurring beta radioactivity in our environment. It decays according to

$$^{40}\text{K} \rightarrow ^{40}\text{Ca} + e^- + \bar{\nu}.$$ 

The energy released in the decay is 1.33 MeV, where 1 eV is defined as the fundamental charge $e$ multiplied by one volt. The energy is shared randomly among the products, subject to the constraint imposed by conservation of energy-momentum, which dictates that very little of the energy is carried by the recoiling calcium nucleus. Determine the maximum energy of the calcium, and compare with the typical energy of a chemical bond, which is a few eV. If the potassium is part of a molecule, do we expect the molecule to survive? Carry out the calculation first by assuming that the electron is ultrarelativistic, then without the approximation, and comment on the how good the approximation is. ⋆
Exercise 13: Nuclear decay

1. Consulting a periodic table, find the $N$, $Z$, and $A$ of the following:

<table>
<thead>
<tr>
<th></th>
<th>$N$</th>
<th>$Z$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^4\text{He}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{244}\text{Pu}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Consider the following five decay processes:

- $\alpha$ decay
- $\gamma$ decay
- $p \to n + e^+ + \nu$ ($\beta^+$ decay)
- $n \to p + e^- + \bar{\nu}$ ($\beta^-$ decay)
- $p + e^- \to n + \nu$ (electron capture)

What would be the action of each of these on the chart of the nuclei? The * represents the original nucleus.

3. (a) Suppose that $^{244}\text{Pu}$ undergoes perfectly symmetric fission, and also emits two neutrons. Find the daughter isotope.

(b) Is the daughter stable, or is it neutron-rich or -poor relative to the line of stability? (To estimate what’s stable, you can use a large chart of the nuclei, or, if you don’t have one handy, consult a periodic table and use the average atomic mass as an approximation to the stable value of $A$.)

(c) Consulting the chart of the nuclei (fig. w on p. 317), explain why it turns out this way.

(d) If the daughter is unstable, which process from question #2 would you expect it to decay by?
Chapter 14
Probability distributions
and a first glimpse of
quantum physics

14.1 Probability distributions

In ch. 7, we considered random variables such as the number of gas molecules $r$ on the right-hand side of the box in figure e, p. 170. This variable is discrete rather than continuous, so we can speak meaningfully of the probability that the integer $r$ has some particular value. On the other hand, the time $t$ at which a particular unstable nucleus decays is a continuous variable. For such a variable, there is an infinite number of possible values, and the probability of any particular value is typically zero.

How do we handle this mathematically? Let’s start finite and sneak up on the infinity.

Consider a throw of a die. If the die is honest, then we expect all six values to be equally likely. Since all six probabilities must add up to 1, then the probability of any particular value must be $1/6$. We can summarize this in a graph, a. Areas under the curve can be interpreted as total probabilities. For instance, the area under the curve from 1 to 3 is $1/6 + 1/6 + 1/6 = 1/2$, so the probability of getting a result from 1 to 3 is 1/2. The function shown on the graph is called the probability distribution.

Figure b shows the probabilities of various results obtained by rolling two dice and adding them together, as in the game of craps. The probabilities are not all the same. There is a small probability of getting a two, for example, because there is only one way to do it, by rolling a one and then another one. The probability of rolling a seven is high because there are six different ways to do it: 1+6, 2+5, etc.

If the number of possible outcomes is large but finite, for example the number of hairs on a dog, the graph would start to look like a smooth curve rather than a ziggurat.

In these examples, the probability that the result will fall within some range is proportional to the area under the bar graph. In other words, we’re talking about an integral. Passing to the case of
a continuous variable, we use this as our definition of the concept of a probability distribution. If \( x \) is a random number, the probability distribution \( D(x) \) is defined so that the probability that \( x \) lies between \( a \) and \( b \) is equal to

\[
P(a \leq x \leq b) = \int_a^b D(x) \, dx.
\]

You’ve probably heard about “the bell curve,” and seen people draw it with a pencil. When they do this, the function they’re drawing is an example of one of these probability distributions. If you’ve heard of the idea that an electron in an atom is like a probability cloud, what is being described qualitatively is actually the function \( D \) (which in this case depends on three coordinates, \( x, y, \) and \( z \)).

Suppose that \( x \) has some units such as seconds. Then \( dx \), which represents a small change in \( x \), also has units of seconds, and since \( P \) is unitless, it follows that \( D \) has units of \( s^{-1} \). That is, \( D \) represents the probability per unit of time. The same kind of thing occurs for random variables with other units: whatever units \( x \) has, \( D \) has the inverse of those units.

Recall that normalization (p. 169) is the requirement that the total probability for \( x \) to have some value must be one,

\[
\int_{-\infty}^{\infty} D(x) \, dx = 1.
\]

For a random variable that is discrete rather than continuous, we just do a sum rather than an integral, \( \sum P(x) = 1 \).

Figure c shows another example, a probability distribution for people’s height. This kind of bell-shaped curve is quite common.

**self-check A**

Compare the number of people with heights in the range of 130-135 cm to the number in the range 135-140. \( \triangleright \) Answer, p. 458

Looking for tall basketball players

\( \triangleright \) A certain country with a large population wants to find very tall people to be on its Olympic basketball team and strike a blow against western imperialism. Out of a pool of \( 10^8 \) people who are the right age and gender, how many are they likely to find who are over 225 cm (7 feet 4 inches) in height? Figure d gives a close-up of the “tail” of the distribution shown previously in figure c.

\( \triangleright \) The shaded area under the curve represents the probability that a given person is tall enough. Each rectangle represents a probability of \( 0.2 \times 10^{-7} \, \text{cm}^{-1} \times 1 \, \text{cm} = 2 \times 10^{-8} \). There are about 35 rectangles covered by the shaded area, so the probability of having a height greater than 225 cm is \( 7 \times 10^{-7} \), or just under one in a million. Using the rule for calculating averages, the average, or expected number of people this tall is \( (10^8) \times (7 \times 10^{-7}) = 70 \).
The average value of $x$ is given by

$$\langle\text{average of } x\rangle = \langle x \rangle = \bar{x} = \int_a^b xD(x)dx.$$ 

The notation $\langle \ldots \rangle$ means “the average of . . .,” and the bar in $\bar{x}$ means the same thing. We can think of the average of a probability distribution geometrically as the horizontal position at which it could be balanced if it was constructed out of cardboard, figure e. For a discrete variable, we again just switch the integral to a probability-weighted sum, $\bar{x} = \sum xP(x)$.

The average is not the only possible way to say what is a typical value for a quantity that can vary randomly; another possible definition is the median, defined as the value that is exceeded with 50% probability. When discussing incomes of people living in a certain town, the average could be very misleading, since it can be affected massively if a single resident of the town is Mark Zuckerberg.

14.2 The variance and standard deviation

If the next Martian you meet asks you, “How tall is an adult human?,” you will probably reply with a statement about the average human height, such as “Oh, about 5 feet 6 inches.” If you wanted to explain a little more, you could say, “But that’s only an average. Most people are somewhere between 5 feet and 6 feet tall.” Without bothering to draw the relevant bell curve for your new extraterrestrial acquaintance, you’ve summarized the relevant information by giving an average and a typical range of variation.

Just as an average is not the only way of defining a central value of a distribution, there are many possible ways of measuring the amount of variation about that center. But a method that is common and has nice mathematical properties is the following. We define the variance of a probability distribution as follows:

$$(\text{variance of } x) = \langle (x - \bar{x})^2 \rangle.$$ 

In other words, we consider the difference between $x$ and its average value $\bar{x}$, and we take the average of the square of that difference. If $x$ always had exactly its average value, then $x - \bar{x}$ would always be zero, and the variance would be zero. It would not make sense to define the variance without the square, because, for example, a symmetrical probability distribution would have a variance of zero — the negative values of $x - \bar{x}$ would cancel the positive ones.

The big mathematical advantage of the variance is that it is additive: the variance of $x + y$ is the same as the sum of the variances, provided that $x$ and $y$ are not correlated to one another. For instance, if Susan has two bartending jobs, bringing in two different incomes $x$ and $y$, then this week’s variation $x - \bar{x}$ in her tips at
Hipster Lounge X is probably not related to the variation $y - \bar{y}$ in what she gets from the tip jar at Y Bar and Grill. One can then plug in to the definition of the variance and show that the variances do add; this works out because the cross-term $\langle (x - \bar{x})(y - \bar{y}) \rangle$ is zero.

The only unfortunate thing about the variance is that its units aren’t the same as the units of the variable such as $x$. For example, if $x$ has units of dollars, then the variance of $x$ has units of dollars squared. For this reason, we define the standard deviation

$$(\text{standard deviation of } x) = \sigma_x = \sqrt{\langle (x - \bar{x})^2 \rangle}.$$ When people give error bars in science experiments, or ranges of error in an opinion poll, they are usually quoting a standard deviation. If someone gives you a number like $137 \pm 5$ kg, then the 5 certainly can’t be the variance, since the variance would have units of kg$^2$, not kg. The standard deviation’s name comes from its interpretation as a typical or standard amount by which $x$ deviates from $\bar{x}$. In the context of AC circuits, you have probably encountered the idea of an r.m.s. (root-mean-square) value, which is exactly a standard deviation (in the case where $\bar{x} = 0$).

As an example, consider the simplest possible case of the gas atoms in the two-sided box. Let the total number of atoms be one (figure f, top), so that the number $r$ of atoms on the right-hand side is either 0 or 1, with equal probability. By normalization, each of these probabilities is 1/2, and $\bar{r} = 1/2$ as well. A calculation (2336) shows that the standard deviation is also 1/2, which makes sense: 1/2 is not just a typical value for how much $r$ differs from $\bar{r}$, it is always the size of that deviation.

We now have an easy way to estimate the sizes of fluctuations in $r$ when the number of atoms is larger. (On p. 172 we did this by a technique that was a lot more work.) Say there are $n = 5$ atoms, as in the bottom of figure f. If this is an ideal gas, then the atoms don’t interact with each other often enough to matter, and there should be no correlation between finding one atom on the right and another atom there. Therefore the variances add. The variance in $r$ contributed by one atom is $(1/2)^2 = 1/4$. Therefore the total variance for 5 atoms is $5/4$, and the standard deviation of $r$ is $\sqrt{5}/2$.

This is the justification for our claim on p. 172 that when $n$ is large, the fluctuations in $r$ are negligible compared to $r$. For the relative size of the fluctuations, we should typically have $\sigma_r/\bar{r} = (\sqrt{n}/2)/(n/2) = 1/\sqrt{n}$. When $n = 10^{22}$, for example, the relative size of the fluctuations should be $10^{-11}$, which is much too small to measure in any experiment. This is similar to the idea of the “law of averages,” which decrees that the casino always makes a profit by the end of the month.
14.3 Errors in random counts: Poisson statistics

If you do a lab experiment as part of this course in which you count radioactive decays with a Geiger counter, the number of counts $N$ in a fixed time period will have some standard deviation $\sigma_N$. This is an example of a more generally occurring situation in statistics, which is that we have a large number of things that may happen, each with some small probability, and we count them up. The total number of them that do happen, $N$, is called a Poisson (“Pwa-SAN”) random variable. For example, the number of houses burglarized in Fullerton this year is a Poisson random variable. When you count the number of nuclear decays in a certain time interval, the result is Poisson. The helpful thing to know is that when a Poisson variable has an average value $N$, its statistical uncertainty is $\sqrt{N}$. So for example if your Geiger counter counts 100 clicks in one minute, this is $100 \pm 10$. We could anticipate based on almost the same reasoning as in section 14.2, p. 328, that the standard deviation would be proportional to $\sqrt{N}$. It just so happens that the constant of proportionality for a Poisson random variable equals one.

14.4 Exponential decay

14.4.1 Half-life

Most people know that radioactivity “lasts a certain amount of time,” but that simple statement leaves out a lot. As an example, consider the following medical procedure used to diagnose thyroid function. A very small quantity of the isotope $^{131}$I, produced in a nuclear reactor, is fed to or injected into the patient. The body’s biochemical systems treat this artificial, radioactive isotope exactly the same as $^{127}$I, which is the only naturally occurring type. (Nutritionally, iodine is a necessary trace element. Iodine taken into the body is partly excreted, but the rest becomes concentrated in the thyroid gland. Iodized salt has had iodine added to it to prevent the nutritional deficiency known as goiters, in which the iodine-starved thyroid becomes swollen.) As the $^{131}$I undergoes beta decay, it emits electrons, neutrinos, and gamma rays. The gamma rays can be measured by a detector passed over the patient’s body. As the radioactive iodine becomes concentrated in the thyroid, the amount of gamma radiation coming from the thyroid becomes greater, and that emitted by the rest of the body is reduced. The rate at which the iodine concentrates in the thyroid tells the doctor about the health of the thyroid.

If you ever undergo this procedure, someone will presumably explain a little about radioactivity to you, to allay your fears that you will turn into the Incredible Hulk, or that your next child will have an unusual number of limbs. Since iodine stays in your thyroid...
for a long time once it gets there, one thing you’ll want to know is whether your thyroid is going to become radioactive forever. They may just tell you that the radioactivity “only lasts a certain amount of time,” but we can now carry out a quantitative derivation of how the radioactivity really will die out.

Let $P(t)$ be the probability that an iodine atom will survive without decaying for a period of at least $t$. It has been experimentally measured that half all $^{131}$I atoms decay in 8 hours, so we have

$$P(8 \text{ hr}) = 0.5.$$

Now using the law of independent probabilities, the probability of surviving for 16 hours equals the probability of surviving for the first 8 hours multiplied by the probability of surviving for the second 8 hours,

$$P(16 \text{ hr}) = 0.50 \times 0.50 = 0.25.$$

Similarly we have

$$P(24 \text{ hr}) = 0.50 \times 0.5 \times 0.5 = 0.125.$$

Generalizing from this pattern, the probability of surviving for any time $t$ that is a multiple of 8 hours is

$$P(t) = 0.5^{t/8} \text{ hr}.$$

We now know how to find the probability of survival at intervals of 8 hours, but what about the points in time in between? What would be the probability of surviving for 4 hours? Well, using the law of independent probabilities again, we have

$$P(8 \text{ hr}) = P(4 \text{ hr}) \times P(4 \text{ hr}),$$

which can be rearranged to give

$$P(4 \text{ hr}) = \sqrt{P(8 \text{ hr})} = \sqrt{0.5} = 0.707.$$

This is exactly what we would have found simply by plugging in $P(t) = 0.5^{t/8} \text{ hr}$ and ignoring the restriction to multiples of 8 hours. Since 8 hours is the amount of time required for half of the atoms to decay, it is known as the half-life, written $t_{1/2}$. The general rule is then the exponential decay equation

$$P(t) = 2^{-t/t_{1/2}}. \quad \text{[probability of survival for time } t\text{]}$$
14.4.2 Calculations for exponential decay

We’ll see that all our formulas come out simpler if we state them in terms of the average lifetime \( \tau \) rather than the half-life \( t_{1/2} \). These are related by \( \tau = t_{1/2} / \ln 2 \).

Also, it’s a little awkward doing exponentials with base 2. Usually we prefer to get everything in terms of base \( e \). This has advantages, for example, when we do calculus, because it’s easy to differentiate or integrate \( e^x \), but hard to do those things with \( 2^x \). Using the identity \( 2^x = \exp(\ln 2^x) = e^{x \ln 2} \), we find that the probability of survival is given by

\[
P(t) = e^{-t/\tau}. \quad \text{[probability of survival for time } t]\]

We would like to know the probability distribution \( D(t) \) for the time at which decay occurs. Since the survival probability is \( P(t) = \int_t^{\infty} D(t') \, dt' \), the fundamental theorem of calculus gives \( D(t) = -dP/dt \), or

\[
D(t) = \frac{1}{\tau} e^{-t/\tau}. \quad \text{[prob. dist. of the time of decay } t]\]

We can see that the units of this equation make sense, since the probability distribution for a random variable with units of seconds must itself have units of inverse seconds.

If you’re fiddling around with a hunk of plutonium and want to know how badly your chromosomes are getting nuked, then you’re interested in the rate of decay, i.e., the number of decays per second, \( dN/dt \). As the cumulative number of decays goes up, the number of survivors goes down, so \( dN = N_0 D(t) \, dt \), where \( N_0 \) is the initial number of nuclei in your sample, and

\[
\frac{dN}{dt} = \frac{N_0}{\tau} e^{-t/\tau}. \quad \text{[rate of decay]}\]

Note that the three functions above are all basically the same exponential function, just with different constant factors out in front. This is because they’re integrals and derivatives of each other, and integrating or differentiating \( e^{bx} \) gives back the same function with a factor of \( b \) or \( 1/b \).

In a common heart test, the patient is injected with a molecule containing \(^{99}\)Tc (technetium-99) atoms in an excited state. This state decays by emitting a gamma ray, with a half-life of 6.01 hours. The molecules bind to red blood cells, so a gamma-ray video camera can see the flow of blood through the chambers of the heart. Once the chemical for the injection has been prepared, it has to be used fairly promptly. Suppose that usable medical results require that at least 40% of the \(^{99}\)Tc nuclei remain in their excited state. What is the shelf life of the chemical?
Once we start doing math, it’s easier to work with the mean lifetime, which in this case is \( \tau = t_{1/2}/ \ln 2 = 8.67 \text{ hr} \). We have \( P(t) = e^{-t/\tau} \), so taking logs of both sides gives \( \ln P = -t/\tau \), and \( t = -\tau(\ln 0.40) = 7.9 \text{ hr} \). This is a little more than the half-life, which makes sense, because 0.4 is a little less than 0.5.

Calibration of the \( ^{14}\text{C} \) dating method using tree rings and artifacts whose ages were known from other methods. Redrawn from Emilio Segrè, *Nuclei and Particles*, 1965.

\( ^{14}\text{C} \) Dating

Almost all the carbon on Earth is \( ^{12}\text{C} \), but not quite. The isotope \( ^{14}\text{C} \), with a half-life of 5600 years, is produced by cosmic rays in the atmosphere. It decays naturally, but is replenished at such a rate that the fraction of \( ^{14}\text{C} \) in the atmosphere remains constant, at \( 1.3 \times 10^{-12} \). Living plants and animals take in both \( ^{12}\text{C} \) and \( ^{14}\text{C} \) from the atmosphere and incorporate both into their bodies. Once the living organism dies, it no longer takes in C atoms from the atmosphere, and the proportion of \( ^{14}\text{C} \) gradually falls off as it undergoes radioactive decay. This effect can be used to find the age of dead organisms, or human artifacts made from plants or animals. Figure h on page 332 shows the exponential decay curve of \( ^{14}\text{C} \) in various objects. Similar methods, using longer-lived isotopes, provided the first firm proof that the earth was billions of
years old, not a few thousand as some had claimed on religious
grounds.

Discussion questions

A  In the medical procedure involving $^{99}$Tc, example 2, why is it the
gamma rays that are detected, not the electrons or neutrinos that are
also emitted?

B  For 1 s, Fred holds in his hands 1 kg of radioactive stuff with a
half-life of 1000 years. Ginger holds 1 kg of a different substance, with a
half-life of 1 min, for the same amount of time. Did they place themselves
in equal danger, or not?

C  Does the half-life depend on how much of the substance you have?
Does the expected time until the sample decays completely depend on
how much of the substance you have?

14.5  A first glimpse of quantum physics

Let’s step back and think for a moment about the subversive physics
assumptions behind all of our derivations about radioactive decay
curves. Back around 1900, when the nucleus and radioactivity were
first discovered, there were no clear principles underlying this sort of
thing. In observations, it seemed like nuclear decay was random, and
observations seemed to show that a nucleus’s probability of decaying
during a certain time interval was statistically independent of its
previous history. Nobody had any idea why these things were true.
One might expect that any answers to questions like these would be
very technical, and would have to await a deeper understanding of
the nucleus.

Actually these facts about nuclear decay require even less de-
tailed technical knowledge of nuclear physics than you have from
sec. 13.4. They arise from certain more basic facts of life concerning
quantum physics. Let’s preview these facts. I’ve stated these below
in nonmathematical and sometimes somewhat facetious language,
but the phrases, or variations on them, are the ones actually used
by physicists. In pop culture, there is a tendency to over-sell quan-
tum physics as if it were even more mysterious than it really is, or
as if it had more implications than it really does for philosophy, re-
ligion, consciousness, and so on. Because of these issues, I’ve added
footnotes below each statement to try to clarify where they really
come from, what physicists mean by them, and which ones have
more rigorous mathematical formulations.
1. **Totalitarian principle:** Everything not forbidden is compulsory. That is, if a process can take place without violating a conservation law, it will take place, with some probability.¹

2. **Ground state:** Every system has a lower bound on its energy. This is some number such that no state has any energy lower than that number.² In many cases, there is exactly one state, called the ground state, that has the lowest energy.

3. **State fundamentalism:** There is nothing more that can be known about a system than its state.³

The first principle is arguably the fundamental difference between quantum physics and classical physics (i.e., all the physical theories that came before). In classical physics, pigs can’t fly, particles can’t transmute themselves into other particles, and a marble locked inside a box can’t get out unless we open the lid. In quantum physics, we have to ask ourselves whether there is some conservation law that prevents these things from happening, and if there isn’t such a law, we expect that the process will happen, albeit possibly at a rate that is too low to measure. I suspect that flying pigs probably do violate conservation of energy (sorry, pigs), but particles do transmute themselves in various ways, and the marble definitely can get out of the box through a process called quantum tunneling, although we can estimate the rate, and it is very low. (See example 2, p. 398, for a crude estimate for the marble’s chances of escape.)

The second principle, a lower bound on energies, seems to be true because we see forms of matter in our universe that seem to be either relatively stable or even (apparently) completely stable. Examples include the proton (if isolated rather than inside a nucleus) and black holes. If there was always some lower-energy state to decay to, then based on the totalitarian principle, every system would have something that it could decay to without violating conservation of energy, and therefore no system would be stable.

The third principle says that, whatever these “states” are, that’s all there is. There is nothing else to know, no higher reality, no deeper insight to be gained, nothing else that can be measured or

¹The facetious wording’s first published application to the description of quantum physics was in a footnote in a 1956 paper by Murray Gell-Mann. There is no rigorous version of this principle that forms a foundational principle of quantum physics, but ever since the birth of quantum mechanics, physicists have found it to be an excellent informal guide to reasoning.

²This is a mathematically rigorous statement. Although it is not typically included in formal axiomatizations of quantum physics, any theory that violates this principle is normally discarded as unrealistic. Specialists call it the spectrum condition.

³This is a more vaguely defined philosophical statement that does not have a complete and rigorous mathematical formulation. People working on the philosophy and foundations of quantum mechanics use a variety of related phrases, such as “wavefunction fundamentalism” and “state monism.”
observed about the system. This principle is what allows us to assert that a nucleus’s probability of decay is independent of its history, as assumed in our derivation of the exponential decay equation. When we see a nucleus that’s been sitting around for a while, it’s normally in its ground state. Therefore there is nothing else to know about it besides the fact that it’s in its ground state. It can’t have cracks and strain that show it’s about to decay, nor can we see that it must be a really tough little nucleus because it’s survived for such a long time. (A more detailed description of exponential decay is given in example 12, p. 417.) Another application of this principle is discussed in sec. 17.8.1, p. 413.

Two processes involving positrons
Around 1930, Paul Dirac proposed that each of the following processes might occur:

\[ p + e^- \rightarrow \gamma + \gamma \] (1)

\[ \gamma + \gamma \rightarrow e^+ + e^- \] (2).

In process (1), a proton and an electron annihilate, creating two gamma rays. This would seem to imply that the hydrogen atom would be unstable with respect to radioactive decay, which seems like a daring prediction, although perhaps Dirac thought the rate at which the annihilation would occur would be so low that it would not yet have been noticed in laboratory experiments. He wrote, “There appears to be no reason why such processes should not actually occur somewhere in the world. They would be consistent with all the general laws of Nature.” This sounds like a use of the totalitarian principle.

But in fact, process (1) has never been observed. Because physicists tend to believe in the totalitarian principle, they need a conservation law to explain why it doesn’t occur. The process is consistent, however, with both conservation of energy-momentum (p. 148) and conservation of charge. Therefore, we invent a new conservation law. In fact, we have two conservation laws at this point that forbid this decay, of which I’ll discuss only one. This one is called conservation of lepton number. “Lepton” is a general term that refers to particles like electrons and positrons, as well as some other, similar particles that are unstable. The electron has a lepton number of +1, while the lepton numbers of the other particles in this process are zero.

Process (2) actually does occur — it is the time-reversed version of the process of electron-positron annihilation (p. 77). It was in fact this process that allowed the original experimental discovery of the positron in 1932. This process obeys conservation of energy-momentum, conservation of charge, and conservation of lepton number, because the antielectron has lepton number −1.
Notes for chapter 14

328 Standard deviation of $r$ with one atom

For a single atom in a box, the standard deviation of the number of atoms on the right is $1/2$.

We have a single atom in a box, with $r = 1$ if the atom is in the right half and $r = 0$ otherwise. Because $r$ is discrete, the variance $\langle (r - \bar{r})^2 \rangle$ can be computed as a probability-weighted sum,

$$\text{(variance of } r) = \langle \left( r - \frac{1}{2} \right)^2 \rangle$$

$$= P(r = 0) \left( 0 - \frac{1}{2} \right)^2 + P(r = 1) \left( 1 - \frac{1}{2} \right)^2$$

$$= \frac{1}{4}.$$

The standard deviation of $r$ is then $\sqrt{1/4} = 1/2$, as expected.
Problems

Key
✓ A computerized answer check is available online.
★ A difficult problem.

1. If a radioactive substance has a half-life of one year, does this mean that it will be completely decayed after two years? Explain.

2. A nuclear physicist is studying a nuclear reaction caused in an accelerator experiment, with a beam of ions from the accelerator striking a thin metal foil and causing nuclear reactions when a nucleus from one of the beam ions happens to hit one of the nuclei in the target. After the experiment has been running for a few hours, a few billion radioactive atoms have been produced, embedded in the target. She does not know what nuclei are being produced, but she suspects they are an isotope of some heavy element such as Pb, Bi, Fr or U. Following one such experiment, she takes the target foil out of the accelerator, sticks it in front of a detector, measures the activity every 5 min, and makes a graph (figure). The isotopes she thinks may have been produced are:

<table>
<thead>
<tr>
<th>isotope</th>
<th>half-life (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{211}$Pb</td>
<td>36.1</td>
</tr>
<tr>
<td>$^{214}$Pb</td>
<td>26.8</td>
</tr>
<tr>
<td>$^{214}$Bi</td>
<td>19.7</td>
</tr>
<tr>
<td>$^{223}$Fr</td>
<td>21.8</td>
</tr>
<tr>
<td>$^{239}$U</td>
<td>23.5</td>
</tr>
</tbody>
</table>

Which one is it? [Part b of this problem has been deleted.]
3 Refer to the probability distribution for people’s heights in figure c on page 326.
(a) Show that the graph is properly normalized.
(b) Estimate the fraction of the population having heights between 140 and 150 cm.

4 A blindfolded person fires a gun at a circular target of radius \( b \), and is allowed to continue firing until a shot actually hits it. Any part of the target is equally likely to get hit. We measure the random distance \( r \) from the center of the circle to where the bullet went in.
(a) Show that the probability distribution of \( r \) must be of the form \( D(r) = kr \), where \( k \) is some constant. (Of course we have \( D(r) = 0 \) for \( r > b \).)
(b) Determine \( k \) by requiring \( D \) to be properly normalized.
(c) Find the average value of \( r \).
(d) Interpreting your result from part c, how does it compare with \( b/2 \)? Does this make sense? Explain.

5 We are given some atoms of a certain radioactive isotope, with half-life \( t_{1/2} \). We pick one atom at random, and observe it for one half-life, starting at time zero. If it decays during that one-half-life period, we record the time \( t \) at which the decay occurred. If it doesn’t, we reset our clock to zero and keep trying until we get an atom that cooperates. The final result is a time \( 0 \leq t \leq t_{1/2} \), with a distribution that looks like the usual exponential decay curve, but with its tail chopped off.
(a) Find the distribution \( D(t) \), with the proper normalization.
(b) Find the average value of \( t \).
(c) Interpreting your result from part b, how does it compare with \( t_{1/2}/2 \)? Does this make sense? Explain.

6 The speed, \( v \), of an atom in an ideal gas has a probability distribution of the form \( D(v) = bve^{-cv^2} \), where \( 0 \leq v < \infty \), \( c \) relates to the temperature, and \( b \) is determined by normalization.
(a) Sketch the distribution.
(b) Find \( b \) in terms of \( c \).
(c) Find the average speed in terms of \( c \), eliminating \( b \). (Don’t try to do the indefinite integral, because it can’t be done in closed form. The relevant definite integral can be found in tables or done with computer software.)
All helium on earth is from the decay of naturally occurring heavy radioactive elements such as uranium. Each alpha particle that is emitted ends up claiming two electrons, which makes it a helium atom. If the original $^{238}\text{U}$ atom is in solid rock (as opposed to the earth’s molten regions), the He atoms are unable to diffuse out of the rock. This problem involves dating a rock using the known decay properties of uranium 238. Suppose a geologist finds a sample of hardened lava, melts it in a furnace, and finds that it contains 1230 mg of uranium and 2.3 mg of helium. $^{238}\text{U}$ decays by alpha emission, with a half-life of $4.5 \times 10^9$ years. The subsequent chain of alpha and electron (beta) decays involves much shorter half-lives, and terminates in the stable nucleus $^{206}\text{Pb}$. Almost all natural uranium is $^{238}\text{U}$, and the chemical composition of this rock indicates that there were no decay chains involved other than that of $^{238}\text{U}$.

(a) How many alphas are emitted per decay chain? [Hint: Use conservation of mass.]
(b) How many electrons are emitted per decay chain? [Hint: Use conservation of charge.]
(c) How long has it been since the lava originally hardened? √

In the year 2010, Fullerton, California, had 342 car thefts. In the following year, the number was 359. Was this a statistically significant increase?

New isotopes are continually being produced and studied. A common method is that experimenters produce a beam of nuclei in an accelerator, and the beam strikes a target such as a thin metal foil. If a beam nucleus happens to hit a target nucleus, nuclear fusion can occur. Once the fused nucleus is formed, it is common for several neutrons to boil off, and the number of neutrons lost can be random, so that more than one isotope can be produced in the same experiment.

Liza carries out such an experiment and observes beta particles being emitted afterward, meaning that she has produced an isotope that is radioactive. She counts the number of betas observed in her detector for the first three hours after the isotope has been produced, with the following results:

- first hour 770,336
- second hour 662,901
- third hour 582,813

Is this a decay curve that could be statistically consistent with a single half-life, i.e., with the production of a single isotope?
Exercise 14: Probability distributions

Questions 1-3 involve the useful concept of the *cumulative distribution* (not introduced in the text). Let \( P(x) = \int_{-\infty}^{x} D(x') \, dx' \) be the probability of finding a value for the random number that is less than or equal to \( x \). The function \( P \) is referred to as the cumulative distribution. (In the context of radioactive decay, the survival probability referred to in the text is actually \( 1 - P \) for this definition of \( P \).)

1. Using the fundamental theorem of calculus, express \( D \) in terms of \( P \). Use Leibniz (“d”) notation.

2. Suppose that \( x \) has some units such as kilograms. Use one of the relations between \( D \) and \( P \) to determine the units of both functions, and then check that it also works out according to the other relation.

3. Sketch the functions \( D \) and \( P \) for the following random variables: (a) a random real number that has a uniform probability of lying anywhere in the interval from 0 to 1; (b) the time \( t \) at which an atom of a radioactive isotope decays, shown by the exponential curve below; (c) a random number that follows the standard “bell curve,” shown below, with an average value of 0 and a standard deviation of 1; (d) the result of a die roll. You will not find that both \( D \) and \( P \) are well defined in all cases.

Pause. I’ll walk you through the following problem on the board. A certain electron wave confined to a box of length \( L \) has a probability distribution given by

\[
D(x) = \begin{cases} 
A \sin^2(2\pi x/L) & \text{if } 0 \leq x \leq L \\
0 & \text{elsewhere}
\end{cases}
\]

Sketch the function. Infer the units of \( A \). What would go wrong if the sine function wasn’t squared? Determine \( A \) from the requirement of normalization.

4. The earth is constantly exposed to neutrinos from outer space. Most neutrinos pass through the entire planet without interacting, but a small fraction of them are absorbed. These absorption events are distributed uniformly within the spherical volume of the earth, which is of radius \( b \). Let \( r \) be the distance from the center at which one of these events occurs, so that \( 0 \leq r \leq b \).

(a) Sketch the geometrical situation, showing an infinitesimally thin shell that stretches from \( r \) to \( r + dr \).

(b) Show that the probability distribution \( D(r) \) is of the form \( kr^p \), where \( k \) and \( p \) are constants, and determine \( p \). Hint: consider the volume of the shell you sketched in part a.

(c) Determine \( k \).

(d) Find the average value of \( r \).

Pause. Let \( x \) be a random variable with a known distribution \( D(x) \), and let \( y = f(x) \) be a function of \( x \). I’ll show you how to use the chain rule to determine the probability distribution \( D^*(y) \), where the star indicates that \( D^* \) is a different function from \( D \).
5. A drunk guy in a bar offers you the following bet. He holds a sharp razor blade facing up, and you toss a stick of uncooked spaghetti up in the air so that when it comes down, the blade breaks it at a random point. The spaghetti is 1 foot long, and the break point is a random variable $x$. He offers to pay you $y = 1/x$ in units of dollars. You want to determine whether this is a sucker bet and the guy is a scammer. Find the amount of money you should be willing to pay in return for the expected payout.
Chapter 15
Light as a particle

The only thing that interferes with my learning is my education. 

*Albert Einstein*

Radioactivity is random, but do the laws of physics exhibit randomness in other contexts besides radioactivity? Yes. Radioactive decay was just a good playpen to get us started with concepts of randomness, because all atoms of a given isotope are identical. By stocking the playpen with an unlimited supply of identical atom-toys, nature helped us to realize that their future behavior could be different regardless of their original identicality. We are now ready to leave the playpen, and see how randomness fits into the structure of physics at the most fundamental level.

The laws of physics describe light and matter, and the quantum revolution rewrote both descriptions. Radioactivity was a good example of matter’s behaving in a way that was inconsistent with classical physics, but if we want to get under the hood and understand how nonclassical things happen, it will be easier to focus on light rather than matter. A radioactive atom such as uranium-235 is after all an extremely complex system, consisting of 92 protons, 143 neutrons, and 92 electrons. Light, however, can be a simple sine wave.

However successful the classical wave theory of light had been — allowing the creation of radio and radar, for example — it still failed to describe many important phenomena. An example that is currently of great interest is the way the ozone layer protects us from the dangerous short-wavelength ultraviolet part of the sun’s spectrum. In the classical description, light is a wave. When a wave passes into and back out of a medium, its frequency is unchanged, and although its wavelength is altered while it is in the medium, it returns to its original value when the wave reemerges. Luckily for us, this is not at all what ultraviolet light does when it passes through the ozone layer, or the layer would offer no protection at all!

15.1 Evidence for light as a particle

For a long time, physicists tried to explain away the problems with the classical theory of light as arising from an imperfect understanding of atoms and the interaction of light with individual atoms and
molecules. The ozone paradox, for example, could have been attributed to the incorrect assumption that one could think of the ozone layer as a smooth, continuous substance, when in reality it was made of individual ozone molecules. It wasn’t until 1905 that Albert Einstein threw down the gauntlet, proposing that the problem had nothing to do with the details of light’s interaction with atoms and everything to do with the fundamental nature of light itself.

In those days the data were sketchy, the ideas vague, and the experiments difficult to interpret; it took a genius like Einstein to cut through the thicket of confusion and find a simple solution. Today, however, we can get right to the heart of the matter with a piece of ordinary consumer electronics, the digital camera. Instead of film, a digital camera has a computer chip with its surface divided up into a grid of light-sensitive squares, called “pixels.” Compared to a grain of the silver compound used to make regular photographic film, a digital camera pixel is activated by an amount of light energy orders of magnitude smaller. We can learn something new about light by using a digital camera to detect smaller and smaller amounts of light, as shown in figure a. Figure a/1 is fake, but a/2 and a/3 are real digital-camera images made by Prof. Lyman Page of Princeton University as a classroom demonstration. Figure a/1 is what we would see if we used the digital camera to take a picture of a fairly dim source of light. In figures a/2 and a/3, the intensity of the light was drastically reduced by inserting semitransparent absorbers like the tinted plastic used in sunglasses. Going from a/1 to a/2 to a/3, more and more light energy is being thrown away by the absorbers.

The results are drastically different from what we would expect based on the wave theory of light. If light was a wave and nothing but a wave, b, then the absorbers would simply cut down the wave’s amplitude across the whole wavefront. The digital camera’s entire chip would be illuminated uniformly, and weakening the wave with an absorber would just mean that every pixel would take a long time to soak up enough energy to register a signal.
But figures a/2 and a/3 show that some pixels take strong hits while others pick up no energy at all. Instead of the wave picture, the image that is naturally evoked by the data is something more like a hail of bullets from a machine gun, c. Each “bullet” of light apparently carries only a tiny amount of energy, which is why detecting them individually requires a sensitive digital camera rather than an eye or a piece of film.

Although Einstein was interpreting different observations, this is the conclusion he reached in his 1905 paper: that the pure wave theory of light is an oversimplification, and that the energy of a beam of light comes in finite chunks rather than being spread smoothly throughout a region of space.

We now think of these chunks as particles of light, and call them “photons,” although Einstein avoided the word “particle,” and the word “photon” was invented later. Regardless of words, the trouble was that waves and particles seemed like inconsistent categories. The reaction to Einstein’s paper could be kindly described as vigorously skeptical. Even twenty years later, Einstein wrote, “There are therefore now two theories of light, both indispensable, and — as one must admit today despite twenty years of tremendous effort on the part of theoretical physicists — without any logical connection.” In the remainder of this section we will learn how the seeming paradox was eventually resolved.

Discussion questions

A Suppose someone rebuts the digital camera data in figure a, claiming that the random pattern of dots occurs not because of anything fundamental about the nature of light but simply because the camera’s pixels are not all exactly the same — some are just more sensitive than others. How could we test this interpretation?

B Discuss how the correspondence principle applies to the observations and concepts discussed in this section.
15.2 How much light is one photon?

15.2.1 The photoelectric effect

We have seen evidence that light energy comes in little chunks, so the next question to be asked is naturally how much energy is in one chunk. The most straightforward experimental avenue for addressing this question is a phenomenon known as the photoelectric effect. The photoelectric effect occurs when a photon strikes the surface of a solid object and knocks out an electron. It occurs continually all around you. It is happening right now at the surface of your skin and on the paper or computer screen from which you are reading these words. It does not ordinarily lead to any observable electrical effect, however, because on the average free electrons are wandering back in just as frequently as they are being ejected. (If an object did somehow lose a significant number of electrons, its growing net positive charge would begin attracting the electrons back more and more strongly.)

Figure e shows a practical method for detecting the photoelectric effect. Two very clean parallel metal plates (the electrodes of a capacitor) are sealed inside a vacuum tube, and only one plate is exposed to light. Because there is a good vacuum between the plates, any ejected electron that happens to be headed in the right direction will almost certainly reach the other capacitor plate without colliding with any air molecules.

The illuminated (bottom) plate is left with a net positive charge, and the unilluminated (top) plate acquires a negative charge from the electrons deposited on it. There is thus an electric field between the plates, and it is because of this field that the electrons’ paths are curved, as shown in the diagram. However, since vacuum is a good insulator, any electrons that reach the top plate are prevented from responding to the electrical attraction by jumping back across the gap. Instead they are forced to make their way around the circuit, passing through an ammeter. The ammeter allows a measurement of the strength of the photoelectric effect.

15.2.2 An unexpected dependence on frequency

The photoelectric effect was discovered serendipitously by Heinrich Hertz in 1887, as he was experimenting with radio waves. He was not particularly interested in the phenomenon, but he did notice that the effect was produced strongly by ultraviolet light and more weakly by lower frequencies. Light whose frequency was lower than a certain critical value did not eject any electrons at all. (In fact this was all prior to Thomson’s discovery of the electron, so Hertz would not have described the effect in terms of electrons — we are discussing everything with the benefit of hindsight.) This dependence on frequency didn’t make any sense in terms of the classical wave theory of light. A light wave consists of electric and magnetic...
fields. The stronger the fields, i.e., the greater the wave’s amplitude, the greater the forces that would be exerted on electrons that found themselves bathed in the light. It should have been amplitude (brightness) that was relevant, not frequency. The dependence on frequency not only proves that the wave model of light needs modifying, but with the proper interpretation it allows us to determine how much energy is in one photon, and it also leads to a connection between the wave and particle models that we need in order to reconcile them.

To make any progress, we need to consider the physical process by which a photon would eject an electron from the metal electrode. A metal contains electrons that are free to move around. Ordinarily, in the interior of the metal, such an electron feels attractive forces from atoms in every direction around it. The forces cancel out. But if the electron happens to find itself at the surface of the metal, the attraction from the interior side is not balanced out by any attraction from outside. In popping out through the surface the electron therefore loses some amount of energy $E_s$, which depends on the type of metal used.

Suppose a photon strikes an electron, annihilating itself and giving up all its energy to the electron. (We now know that this is what always happens in the photoelectric effect, although it had not yet been established in 1905 whether or not the photon was completely annihilated.) The electron will (1) lose kinetic energy through collisions with other electrons as it plows through the metal on its way to the surface; (2) lose an amount of kinetic energy equal to $E_s$ as it emerges through the surface; and (3) lose more energy on its way across the gap between the plates, due to the electric field between the plates. Even if the electron happens to be right at the surface of the metal when it absorbs the photon, and even if the electric field between the plates has not yet built up very much, $E_s$ is the bare minimum amount of energy that it must receive from the photon if it is to contribute to a measurable current. The reason for using very clean electrodes is to minimize $E_s$ and make it have a definite value characteristic of the metal surface, not a mixture of values due to the various types of dirt and crud that are present in tiny amounts on all surfaces in everyday life.

We can now interpret the frequency dependence of the photoelectric effect in a simple way: apparently the amount of energy possessed by a photon is related to its frequency. A low-frequency red or infrared photon has an energy less than $E_s$, so a beam of them will not produce any current. A high-frequency blue or violet photon, on the other hand, packs enough of a punch to allow an electron to make it to the other plate. At frequencies higher than the minimum, the photoelectric current continues to increase with the frequency of the light because of effects (1) and (3).
15.2.3 Numerical relationship between energy and frequency

Figure g shows an experiment that is used sometimes in college laboratory courses to probe the relationship between the energy and frequency of a photon. The idea is simply to illuminate one plate of the vacuum tube with light of a single wavelength and monitor the voltage difference between the two plates as they charge up. Since the resistance of a voltmeter is very high (much higher than the resistance of an ammeter), we can assume to a good approximation that electrons reaching the top plate are stuck there permanently, so the voltage will keep on increasing for as long as electrons are making it across the vacuum tube.

At a moment when the voltage difference has reached a value $\Delta V$, the minimum energy required by an electron to make it out of the bottom plate and across the gap to the other plate is $E_s + e\Delta V$. As $\Delta V$ increases, we eventually reach a point at which $E_s + e\Delta V$ equals the energy of one photon. No more electrons can cross the gap, and the reading on the voltmeter stops rising. The quantity $E_s + e\Delta V$ now tells us the energy of one photon. If we determine this energy for a variety of wavelengths, $h$, we find the following simple relationship between the energy of a photon and the frequency of the light:

$$E = hf,$$

where $h$ is a constant with the value $6.63 \times 10^{-34}$ J · s. Note how the equation brings the wave and particle models of light under the same roof: the left side is the energy of one particle of light, while the right side is the frequency of the same light, interpreted as a wave. The constant $h$ is known as Planck’s constant, for historical reasons explained in the footnote beginning on the preceding page.

**Self-check A**

How would you extract $h$ from the graph in figure h? What if you didn’t even know $E_s$ in advance, and could only graph $e\Delta V$ versus $f$?

Answer, p. 458

Since the energy of a photon is $hf$, a beam of light can only have energies of $hf$, $2hf$, $3hf$, etc. Its energy is quantized — there is no such thing as a fraction of a photon. Quantum physics gets its name from the fact that it quantizes quantities like energy, momentum, and angular momentum that had previously been thought to be smooth, continuous and infinitely divisible.
Photons from a lightbulb example 1

Roughly how many photons are emitted by a 100 watt lightbulb in 1 second?

People tend to remember wavelengths rather than frequencies for visible light. The bulb emits photons with a range of frequencies and wavelengths, but let’s take 600 nm as a typical wavelength for purposes of estimation. The energy of a single photon is

\[ E_{\text{photon}} = hf = hc/\lambda \]

A power of 100 W means 100 joules per second, so the number of photons is

\[ (100 \text{ J})/E_{\text{photon}} = (100 \text{ J})/(hc/\lambda) \approx 3 \times 10^{20} \]

This hugeness of this number is consistent with the correspondence principle. The experiments that established the classical theory of optics weren’t wrong. They were right, within their domain of applicability, in which the number of photons was so large as to be indistinguishable from a continuous beam.

Measuring the wave example 2

When surfers are out on the water waiting for their chance to catch a wave, they’re interested in both the height of the waves and when the waves are going to arrive. In other words, they observe both the amplitude and phase of the waves, and it doesn’t matter to them that the water is granular at the molecular level. The correspondence principle requires that we be able to do the same thing for electromagnetic waves, since the classical theory of electricity and magnetism was all stated and verified experimentally in terms of the fields \( E \) and \( B \), which are the amplitude of an electromagnetic wave. The phase is also necessary, since the induction effects predicted by Maxwell’s equation would flip their signs depending on whether an oscillating field is on its way up or on its way back down.

This is a more demanding application of the correspondence principle than the one in example 1, since amplitudes and phases constitute more detailed information than the over-all intensity of a beam of light. Eyeball measurements can’t detect this type of information, since the eye is much bigger than a wavelength, but for example an AM radio receiver can do it with radio waves, since the wavelength for a station at 1000 kHz is about 300 meters, which is much larger than the antenna. The correspondence principle demands that we be able to explain this in terms of the photon theory, and this requires not just that we have a large
number of photons emitted by the transmitter per second, as in example 1, but that even by the time they spread out and reach the receiving antenna, there should be many photons overlapping each other within a space of one cubic wavelength. Problem 9 on p. 362 verifies that the number is in fact extremely large.

**Momentum of a photon example 3**

According to the theory of relativity, the momentum of a beam of light is given by \( p = \frac{E}{c} \). Apply this to find the momentum of a single photon in terms of its frequency, and in terms of its wavelength.

Combining the equations \( p = \frac{E}{c} \) and \( E = hf \), we find

\[
p = \frac{E}{c} = \frac{hf}{c}
\]

To reexpress this in terms of wavelength, we use \( c = f\lambda \):

\[
p = \frac{h}{c} \cdot \frac{c}{\lambda} = \frac{h}{\lambda}
\]

The second form turns out to be simpler.

**Discussion questions**

A The photoelectric effect only ever ejects a very tiny percentage of the electrons available near the surface of an object. How well does this agree with the wave model of light, and how well with the particle model? Consider the two different distance scales involved: the wavelength of the light, and the size of an atom, which is on the order of \( 10^{-10} \) or \( 10^{-9} \) m.

B What is the significance of the fact that Planck’s constant is numerically very small? How would our everyday experience of light be different if it was not so small?

C How would the experiments described above be affected if a single electron was likely to get hit by more than one photon?

D Draw some representative trajectories of electrons for \( \Delta V = 0 \), \( \Delta V \) less than the maximum value, and \( \Delta V \) greater than the maximum value.

E Explain based on the photon theory of light why ultraviolet light would be more likely than visible or infrared light to cause cancer by damaging DNA molecules. How does this relate to discussion question C?

F Does \( E = hf \) imply that a photon changes its energy when it passes from one transparent material into another substance with a different index of refraction?