The vibrations of this electric bass string are converted to electrical vibrations, then to sound vibrations, and finally to vibrations of our eardrums.

Chapter 16
Vibrations

Dandelion. Cello. Read those two words, and your brain instantly conjures a stream of associations, the most prominent of which have to do with vibrations. Our mental category of “dandelion-ness” is strongly linked to the color of light waves that vibrate about half a million billion times a second: yellow. The velvety throb of a cello has as its most obvious characteristic a relatively low musical pitch — the note you are spontaneously imagining right now might be one whose sound vibrations repeat at a rate of a hundred times a second.

Evolution has designed our two most important senses around the assumption that not only will our environment be drenched with information-bearing vibrations, but in addition those vibrations will often be repetitive, so that we can judge colors and pitches by the rate of repetition. Granting that we do sometimes encounter non-repeating waves such as the consonant “sh,” which has no recognizable pitch, why was Nature’s assumption of repetition nevertheless so right in general?

Repeating phenomena occur throughout nature, from the orbits of electrons in atoms to the reappearance of Halley’s Comet every 75 years. Ancient cultures tended to attribute repetitious phenomena
a. If we try to draw a non-repeating orbit for Halley’s Comet, it will inevitably end up crossing itself.

b. A spring has an equilibrium length, 1, and can be stretched, 2, or compressed, 3. A mass attached to the spring can be set into motion initially, 4, and will then vibrate, 4-13.

Like the seasons to the cyclical nature of time itself, but we now have a less mystical explanation. Suppose that instead of Halley’s Comet’s true, repeating elliptical orbit that closes seamlessly upon itself with each revolution, we decide to take a pen and draw a whimsical alternative path that never repeats. We will not be able to draw for very long without having the path cross itself. But at such a crossing point, the comet has returned to a place it visited once before, and since its potential energy is the same as it was on the last visit, conservation of energy proves that it must again have the same kinetic energy and therefore the same speed. Not only that, but the comet’s direction of motion cannot be randomly chosen, because angular momentum must be conserved as well. Although this falls short of being an ironclad proof that the comet’s orbit must repeat, it no longer seems surprising that it does.

Conservation laws, then, provide us with a good reason why repetitive motion is so prevalent in the universe. But it goes deeper than that. Up to this point in your study of physics, I have been indoctrinating you with a mechanistic vision of the universe as a giant piece of clockwork. Breaking the clockwork down into smaller and smaller bits, we end up at the atomic level, where the electrons circling the nucleus resemble — well, little clocks! From this point of view, particles of matter are the fundamental building blocks of everything, and vibrations and waves are just a couple of the tricks that groups of particles can do. But at the beginning of the 20th century, the tables were turned. A chain of discoveries initiated by Albert Einstein led to the realization that the so-called subatomic “particles” were in fact waves. In this new worldview, it is vibrations and waves that are fundamental, and the formation of matter is just one of the tricks that waves can do.

16.1 Period, frequency, and amplitude

Figure b shows our most basic example of a vibration. With no forces on it, the spring assumes its equilibrium length, b/1. It can be stretched, 2, or compressed, 3. We attach the spring to a wall on the left and to a mass on the right. If we now hit the mass with a hammer, 4, it oscillates as shown in the series of snapshots, 4-13. If we assume that the mass slides back and forth without friction and that the motion is one-dimensional, then conservation of energy proves that the motion must be repetitive. When the block comes back to its initial position again, 7, its potential energy is the same again, so it must have the same kinetic energy again. The motion is in the opposite direction, however. Finally, at 10, it returns to its initial position with the same kinetic energy and the same direction of motion. The motion has gone through one complete cycle, and will now repeat forever in the absence of friction.

The usual physics terminology for motion that repeats itself over
and over is periodic motion, and the time required for one repetition is called the period, \(T\). (The symbol \(P\) is not used because of the possible confusion with momentum.) One complete repetition of the motion is called a cycle.

We are used to referring to short-period sound vibrations as “high” in pitch, and it sounds odd to have to say that high pitches have low periods. It is therefore more common to discuss the rapidity of a vibration in terms of the number of vibrations per second, a quantity called the frequency, \(f\). Since the period is the number of seconds per cycle and the frequency is the number of cycles per second, they are reciprocals of each other,

\[
f = \frac{1}{T}.
\]

The forms of various equations turn out to be simpler when they are expressed not in terms of \(f\) but in terms of \(\omega = 2\pi f\). It’s not a coincidence that this relationship looks the same as the one relating angular velocity and frequency in circular motion. In machines, mechanical linkages are used to convert back and forth between vibrational motion and circular motion. For example, a car engine’s pistons oscillate in their cylinders at a frequency \(f\), driving the crankshaft at the same frequency \(f\). Either of these motions can be described using \(\omega\) instead of \(f\), even though only in the case of the crankshaft’s rotational motion does it make sense to interpret \(\omega\) as the number of radians per second. When the motion is not rotational, we usually refer to \(\omega\) as the angular frequency, and we often use the word “frequency” to mean either \(f\) or \(\omega\), relying on context to make the meaning clear.

**A carnival game example 1**

In the carnival game shown in figure e, the rube is supposed to push the bowling ball on the track just hard enough so that it goes over the hump and into the valley, but does not come back out again. If the only types of energy involved are kinetic and potential, this is impossible. Suppose you expect the ball to come back to a point such as the one shown with the dashed outline, then stop and turn around. It would already have passed through this point once before, going to the left on its way into the valley. It was moving then, so conservation of energy tells us that it cannot be at rest when it comes back to the same point. The motion that the customer hopes for is physically impossible. There is a physically possible periodic motion in which the ball rolls back and forth, staying confined within the valley, but there is no way to get the ball into that motion beginning from the place where we start. There is a way to beat the game, though. If you put enough spin on the ball, you can create enough kinetic friction so that a significant amount of heat is generated. Conservation of energy then allows the ball to be at rest when it comes back to a point.
The amplitude of the vibrations of the mass on a spring could be defined in two different ways. It would have units of distance. The amplitude of a swinging pendulum would more naturally be defined as an angle.

\[ f / 1 \]

\[ f / 2 \]

Period and frequency of a fly's wing-beats example 2
A Victorian parlor trick was to listen to the pitch of a fly's buzz, reproduce the musical note on the piano, and announce how many times the fly's wings had flapped in one second. If the fly's wings flap, say, 200 times in one second, then the frequency of their motion is \( f = \frac{200}{1} \text{ s} = 200 \text{ s}^{-1} \). The period is one 200th of a second, \( T = \frac{1}{f} = (1/200) \text{ s} = 0.005 \text{ s} \).

Units of inverse second, \( \text{s}^{-1} \), are awkward in speech, so an abbreviation has been created. One Hertz, named in honor of a pioneer of radio technology, is one cycle per second. In abbreviated form, \( 1 \text{ Hz} = 1 \text{ s}^{-1} \). This is the familiar unit used for the frequencies on the radio dial.

Frequency of a radio station example 3
KKJZ's frequency is 88.1 MHz. What does this mean, and what period does this correspond to?

The metric prefix M- is mega-, i.e., millions. The radio waves emitted by KKJZ's transmitting antenna vibrate 88.1 million times per second. This corresponds to a period of

\[ T = \frac{1}{f} = 1.14 \times 10^{-8} \text{ s} \]

This example shows a second reason why we normally speak in terms of frequency rather than period: it would be painful to have to refer to such small time intervals routinely. I could abbreviate by telling people that KKJZ's period was 11.4 nanoseconds, but most people are more familiar with the big metric prefixes than with the small ones.

Units of frequency are also commonly used to specify the speeds of computers. The idea is that all the little circuits on a computer chip are synchronized by the very fast ticks of an electronic clock, so that the circuits can all cooperate on a task without getting ahead or behind. Adding two numbers might require, say, 30 clock cycles. Microcomputers these days operate at clock frequencies of about a gigahertz.

We have discussed how to measure how fast something vibrates, but not how big the vibrations are. The general term for this is amplitude, \( A \). The definition of amplitude depends on the system being discussed, and two people discussing the same system may not even use the same definition. In the example of the block on the end of the spring, \( f / 1 \), the amplitude will be measured in distance units such as cm. One could work in terms of the distance traveled by the block from the extreme left to the extreme right, but it would be somewhat more common in physics to use the distance from the center to one extreme. The former is usually referred to as
the peak-to-peak amplitude, since the extremes of the motion look like mountain peaks or upside-down mountain peaks on a graph of position versus time.

In other situations we would not even use the same units for amplitude. The amplitude of a child on a swing, or a pendulum, \( f/2 \), would most conveniently be measured as an angle, not a distance, since her feet will move a greater distance than her head. The electrical vibrations in a radio receiver would be measured in electrical units such as volts or amperes.

**16.2 Simple harmonic motion**

**Why are sine-wave vibrations so common?**

If we actually construct the mass-on-a-spring system discussed in the previous section and measure its motion accurately, we will find that its \( x - t \) graph is nearly a perfect sine-wave shape, as shown in figure g/1. (We call it a “sine wave” or “sinusoidal” even if it is a cosine, or a sine or cosine shifted by some arbitrary horizontal amount.) It may not be surprising that it is a wiggle of this general sort, but why is it a specific mathematically perfect shape? Why is it not a sawtooth shape like 2 or some other shape like 3? The mystery deepens as we find that a vast number of apparently unrelated vibrating systems show the same mathematical feature. A tuning fork, a sapling pulled to one side and released, a car bouncing on its shock absorbers, all these systems will exhibit sine-wave motion under one condition: the amplitude of the motion must be small.

It is not hard to see intuitively why extremes of amplitude would act differently. For example, a car that is bouncing lightly on its shock absorbers may behave smoothly, but if we try to double the amplitude of the vibrations the bottom of the car may begin hitting the ground, g/4. (Although we are assuming for simplicity in this chapter that energy is never dissipated, this is clearly not a very realistic assumption in this example. Each time the car hits the ground it will convert quite a bit of its potential and kinetic energy into heat and sound, so the vibrations would actually die out quite quickly, rather than repeating for many cycles as shown in the figure.)

The key to understanding how an object vibrates is to know how the force on the object depends on the object’s position. If an object is vibrating to the right and left, then it must have a leftward force on it when it is on the right side, and a rightward force when it is on the left side. In one dimension, we can represent the direction of the force using a positive or negative sign, and since the force changes from positive to negative there must be a point in the middle where the force is zero. This is the equilibrium point, where the object would stay at rest if it was released at rest. For convenience of
The force exerted by an ideal spring, which behaves exactly according to Hooke’s law.

The simplest example is the mass on a spring, for which the force on the mass is given by Hooke’s law,
\[ F = -kx. \]

We can visualize the behavior of this force using a graph of \( F \) versus \( x \), as shown in figure h. The graph is a line, and the spring constant, \( k \), is equal to minus its slope. A stiffer spring has a larger value of \( k \) and a steeper slope. Hooke’s law is only an approximation, but it works very well for most springs in real life, as long as the spring isn’t compressed or stretched so much that it is permanently bent or damaged.

The following important theorem relates the motion graph to the force graph.

**Theorem:** A linear force graph makes a sinusoidal motion graph.

If the total force on a vibrating object depends only on the object’s position, and is related to the object’s displacement from equilibrium by an equation of the form \( F = -kx \), then the object’s motion displays a sinusoidal graph with frequency \( \omega = \sqrt{k/m} \).

**Proof:** By Newton’s second law, \( -kx = ma \), so we need a function \( x(t) \) that satisfies the equation \( \frac{d^2 x}{dt^2} = -cx \), where for convenience we write \( c \) for \( k/m \). This type of equation is called a differential equation, because it relates a function to its own derivative (in this case the second derivative).

Just to make things easier to think about, suppose that we happen to have an oscillator with \( c = 1 \). Then our goal is to find a function whose second derivative is equal to minus the original function. We know of two such functions, the sine and the cosine. These two solutions can be combined to make anything of the form \( P \sin t + Q \cos t \), where \( P \) and \( Q \) are constants, and the result will still be a solution. Using trig identities, such an expression can always be rewritten as \( A \cos(\omega t + \delta) \).

Now what about the more general case where \( c \) need not equal 1? The role of \( c \) in \( \frac{d^2 x}{dt^2} = -cx \) is to set the time scale. For example, suppose we produce a fake video of an object oscillating according to \( A \cos(t + \delta) \), which violates Newton’s second law because \( c \) doesn’t equal 1, so the acceleration is too small. We can always make the video physically accurate by speeding it up. This suggests generalizing the solution to \( A \cos(\omega t + \delta) \). Plugging in to the differential equation, we find that \( \omega = \sqrt{k/m} \), and \( T = 2\pi/\omega \) brings us to the claimed result.

We’ve proved that anything of this form is a solution, but we
Because simple harmonic motion involves sinusoidal functions, it is equivalent to circular motion that has been projected into one dimension. This figure shows a simulated view of Jupiter and its four largest moons at intervals of three hours. Seen from the side from within the plane of the solar system, the circular orbits appear linear. In coordinates with the origin at Jupiter, a moon has coordinates $x = r \cos \theta$ and $y = r \sin \theta$, where $\theta = \omega t$. If the view is along the $y$ axis, then we see only the $x$ motion, which is of the form $A \cos(\omega t)$.

haven’t shown that any solution must be of this form. Physically, this must be true because the motion is fully determined by the oscillator’s initial position and initial velocity, which can always be matched by choosing $A$ and $\delta$ appropriately. Mathematically, the uniqueness result is a standard one about second-order differential equations.

This may seem like only an obscure theorem about the mass-on-a-spring system, but figure j shows it to be far more general than that. Figure j/1 depicts a force curve that is not a straight line. A system with this $F - x$ curve would have large-amplitude vibrations that were complex and not sinusoidal. But the same system would exhibit sinusoidal small-amplitude vibrations. This is because any curve looks linear from very close up. If we magnify the $F - x$ graph as shown in figure j/2, it becomes very difficult to tell that the graph is not a straight line. If the vibrations were confined to the region shown in j/2, they would be very nearly sinusoidal. This is the reason why sinusoidal vibrations are a universal feature of all vibrating systems, if we restrict ourselves to small amplitudes. The theorem is therefore of great general significance. It applies throughout the universe, to objects ranging from vibrating stars to vibrating nuclei. A sinusoidal vibration is known as simple harmonic motion.

This relates to the fundamental idea behind differential calculus, which is that up close, any smooth function looks linear. To characterize small oscillations about the equilibrium at $x = 0$ in figure h, all we need to know is the derivative $dF/dx|_0$, which equals $-k$. That is, a force function $F(x)$ has no “individuality” except as defined by $k$. 

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Three functions with the same curvature at $x = 0$.

Example 5. The rod pivots on the hinge at the bottom.

The same idea about lack of individuality can be expressed in terms of energy.

On a graph of $PE$ versus $x$, an equilibrium is a local minimum. We can imagine an oscillation about this equilibrium point as if a marble was rolling back and forth in the depression of the graph. Let's choose a coordinate system in which $x = 0$ is the equilibrium, and since the potential energy is only well defined up to an additive constant, we'll simply define it to be zero at equilibrium:

$$PE(0) = 0$$

Since $x = 0$ is a local minimum,

$$\frac{dPE}{dx}(0) = 0.$$  

There are still infinitely many functions that could satisfy these criteria, including the three shown in figure k, which are $x^2/2$, $x^2/2(1+x^2)$, and $(e^{3x}+e^{-3x} - 2)/18$. Note, however, how all three functions are virtually identical right near the minimum. That's because they all have the same curvature. More specifically, each function has its second derivative equal to 1 at $x = 0$, and the second derivative is a measure of curvature. Since the $F = -dPE/dx$ and $k = -dPE/dx$, $k$ equals the second derivative of the PE,

$$\frac{d^2PE}{dx^2}(0) = k.$$  

As shown in figure k, any two functions that have $PE(0) = 0$, $dPE/dx = 0$, and $d^2PE/dx^2 = k$, with the same value of $k$, are virtually indistinguishable for small values of $x$, so if we want to analyze small oscillations, it doesn’t even matter which function we assume. For simplicity, we can always use $PE(x) = (1/2)kx^2$, which is the form that gives a constant second derivative.

A spring and a lever example 5

What is the period of small oscillations of the system shown in the figure? Neglect the mass of the lever and the spring. Assume that the spring is so stiff that gravity is not an important effect. The spring is relaxed when the lever is vertical.

This is a little tricky, because the spring constant $k$, although it is relevant, is not the $k$ we should be putting into the equation $\omega = \sqrt{k/m}$. I find this easier to understand by working with energy rather than force. (Another method would be to use torque, as in problem 15.) The $k$ that goes into $\sqrt{k/m}$ has to be the second derivative of $PE$ with respect to the position, $x$, of the mass that’s moving. The energy $PE$ stored in the spring depends on how far the tip of the lever is from the center. This distance equals $(L/b)x,$
so the energy in the spring is

\[ PE = \frac{1}{2} k \left( \frac{L}{b} \right)^2 \]

\[ = \frac{kL^2}{2b^2} \times \] 2

and the \( k \) we have to put in \( T = 2\pi \sqrt{\frac{m}{k}} \) is

\[ \frac{d^2 PE}{dx^2} = \frac{kL^2}{b^2} \]

The result is

\[ \omega = \sqrt{\frac{kL^2}{mb^2}} \]

\[ = \frac{L}{b} \sqrt{\frac{k}{m}} \]

The leverage of the lever makes it as if the spring was stronger, decreasing the period of the oscillations by a factor of \( \frac{b}{L} \).

Water in a U-shaped tube example 6

The U-shaped tube in figure m has cross-sectional area \( A \), and the density of the water inside is \( \rho \). Find the gravitational potential energy as a function of the quantity \( y \) shown in the figure, show that there is an equilibrium at \( y = 0 \), and find the frequency of oscillation of the water.

Potential energy is only well defined up to an additive constant. To fix this constant, let's define \( PE \) to be zero when \( y = 0 \). The difference between \( PE(y) \) and \( PE(0) \) is the energy that would be required to lift a water column of height \( y \) out of the right side, and place it above the dashed line, on the left side, raising it through a height \( y \). This water column has height \( y \) and cross-sectional area \( A \), so its volume is \( Ay \), its mass is \( \rho Ay \), and the energy required is \( mgy = (\rho Ay)gy = \rho gAy^2 \). We then have \( PE(y) = PE(0) + \rho gAy^2 = \rho gAy^2 \).

The “spring constant” is

\[ k = \frac{d^2 PE}{dy^2} \]

\[ = 2\rho gA. \]

This is an interesting example, because \( k \) can be calculated without any approximations, but the kinetic energy requires an approximation, because we don’t know the details of the pattern of
flow of the water. It could be very complicated. There will be a tendency for the water near the walls to flow more slowly due to friction, and there may also be swirling, turbulent motion. However, if we make the approximation that all the water moves with the same velocity as the surface, $d y / d t$, then the mass-on-a-spring analysis applies. Letting $L$ be the total length of the filled part of the tube, the mass is $\rho L A$, and we have

\[
\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2 \rho g A}{\rho L A}} = \sqrt{\frac{2g}{L}}.
\]

**Period is approximately independent of amplitude, if the amplitude is small.**

Until now we have not even mentioned the most counterintuitive aspect of the equation $\omega = \sqrt{k/m}$: it does not depend on amplitude at all. Intuitively, most people would expect the mass-on-a-spring system to take longer to complete a cycle if the amplitude was larger. (We are comparing amplitudes that are different from each other, but both small enough that the theorem applies.) In fact the larger-amplitude vibrations take the same amount of time as the small-amplitude ones. This is because at large amplitudes, the force is greater, and therefore accelerates the object to higher speeds.

Legend has it that this fact was first noticed by Galileo during what was apparently a less than enthralling church service. A gust of wind would now and then start one of the chandeliers in the cathedral swaying back and forth, and he noticed that regardless of the amplitude of the vibrations, the period of oscillation seemed to be the same. Up until that time, he had been carrying out his physics experiments with such crude time-measuring techniques as feeling his own pulse or singing a tune to keep a musical beat. But after going home and testing a pendulum, he convinced himself that he had found a superior method of measuring time. Even without a fancy system of pulleys to keep the pendulum's vibrations from dying down, he could get very accurate time measurements, because the gradual decrease in amplitude due to friction would have no effect on the pendulum's period. (Galileo never produced a modern-style pendulum clock with pulleys, a minute hand, and a second hand, but within a generation the device had taken on the form that persisted for hundreds of years after.)
The pendulum

Compare the frequencies of pendula having bobs with different masses.

From the equation \( \omega = \sqrt{k/m} \), we might expect that a larger mass would lead to a lower frequency. However, increasing the mass also increases the forces that act on the pendulum: gravity and the tension in the string. This increases \( k \) as well as \( m \), so the frequency of a pendulum is independent of \( m \).

Discussion questions

A Suppose that a pendulum has a rigid arm mounted on a bearing, rather than a string tied at its top with a knot. The bob can then oscillate with center-to-side amplitudes greater than 90°. For the maximum amplitude of 180°, what can you say about the period?

B In the language of calculus, Newton’s second law for a simple harmonic oscillator can be written in the form \( \frac{d^2 x}{dt^2} = -(\ldots) x \), where “\( \ldots \)” refers to a constant, and the minus sign says that if we pull the object away from equilibrium, a restoring force tries to bring it back to equilibrium, which is the opposite direction. This is why we get motion that looks like a sine or cosine function: these are functions that, when differentiated twice, give back the original function but with an opposite sign. Now consider the example described in discussion question A, where a pendulum is upright or nearly upright. How does the analysis play out differently?
Summary

Selected vocabulary

- periodic motion . motion that repeats itself over and over
- period . . . . the time required for one cycle of a periodic motion
- frequency . . . the number of cycles per second, the inverse of the period
- amplitude . . . the amount of vibration, often measured from the center to one side; may have different units depending on the nature of the vibration
- simple harmonic motion . . . motion whose $x - t$ graph is a sine wave

Notation

- $T$ . . . . . . . period
- $f$ . . . . . . . frequency
- $A$ . . . . . . . amplitude
- $k$ . . . . . . . the slope of the graph of $F$ versus $x$, where $F$ is the total force acting on an object and $x$ is the object’s position; for a spring, this is known as the spring constant.
- $\omega$ (Greek letter “omega”) . . . .

Other terminology and notation

- $\nu$ . . . . . . . The Greek letter $\nu$, nu, is used in many books for frequency.

Summary

Periodic motion is common in the world around us because of conservation laws. An important example is one-dimensional motion in which the only two forms of energy involved are potential and kinetic; in such a situation, conservation of energy requires that an object repeat its motion, because otherwise when it came back to the same point, it would have to have a different kinetic energy and therefore a different total energy.

Not only are periodic vibrations very common, but small-amplitude vibrations are always sinusoidal as well. That is, the $x - t$ graph is a sine wave. This is because the graph of force versus position will always look like a straight line on a sufficiently small scale. This type of vibration is called simple harmonic motion. In simple harmonic motion, the frequency is independent of the amplitude, and is given by

$$\omega = \sqrt{\frac{k}{m}}.$$
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.
1 This problem has been deleted.

2 Many single-celled organisms propel themselves through water with long tails, which they wiggle back and forth. (The most obvious example is the sperm cell.) The frequency of the tail’s vibration is typically about 10-15 Hz. To what range of periods does this range of frequencies correspond? √

3 The figure shows the oscillation of a microphone in response to the author whistling the musical note “A.” The horizontal axis, representing time, has a scale of 1.0 ms per square. Find the period T, the frequency f, and the angular frequency ω. √

4 (a) Pendulum 2 has a string twice as long as pendulum 1. If we define x as the distance traveled by the bob along a circle away from the bottom, how does the k of pendulum 2 compare with the k of pendulum 1? Give a numerical ratio. [Hint: the total force on the bob is the same if the angles away from the bottom are the same, but equal angles do not correspond to equal values of x.]

(b) Based on your answer from part (a), how does the period of pendulum 2 compare with the period of pendulum 1? Give a numerical ratio.

5 A pneumatic spring consists of a piston riding on top of the air in a cylinder. The upward force of the air on the piston is given by \( F_{\text{air}} = ax^{-\beta} \), where \( \beta = 1.4 \) and \( a \) is a constant with funny units of N·m\(^{1.4}\). For simplicity, assume the air only supports the weight \( mg \) of the piston itself, although in practice this device is used to support some other object. The equilibrium position, \( x_0 \), is where \( mg \) equals \( -F_{\text{air}} \). (Note that in the main text I have assumed the equilibrium position to be at \( x = 0 \), but that is not the natural choice here.) Assume friction is negligible, and consider a case where the amplitude of the vibrations is very small. Find the angular frequency of oscillation. √
Verify that energy is conserved in simple harmonic motion.

Consider the same pneumatic piston described in problem 5, but now imagine that the oscillations are not small. Sketch a graph of the total force on the piston as it would appear over this wider range of motion. For a wider range of motion, explain why the vibration of the piston about equilibrium is not simple harmonic motion, and sketch a graph of \( x \) vs \( t \), showing roughly how the curve is different from a sine wave. [Hint: Acceleration corresponds to the curvature of the \( x - t \) graph, so if the force is greater, the graph should curve around more quickly.]

Archimedes' principle states that an object partly or wholly immersed in fluid experiences a buoyant force equal to the weight of the fluid it displaces. For instance, if a boat is floating in water, the upward pressure of the water (vector sum of all the forces of the water pressing inward and upward on every square inch of its hull) must be equal to the weight of the water displaced, because if the boat was instantly removed and the hole in the water filled back in, the force of the surrounding water would be just the right amount to hold up this new “chunk” of water. (a) Show that a cube of mass \( m \) with edges of length \( b \) floating upright (not tilted) in a fluid of density \( \rho \) will have a draft (depth to which it sinks below the waterline) \( h \) given at equilibrium by \( h_0 = \frac{m}{b^2 \rho} \). (b) Find the total force on the cube when its draft is \( h \), and verify that plugging in \( h - h_0 \) gives a total force of zero. (c) Find the cube’s period of oscillation as it bobs up and down in the water, and show that can be expressed in terms of \( \rho \) and \( g \) only.

The figure shows a see-saw with two springs at Codornices Park in Berkeley, California. Each spring has spring constant \( k \), and a kid of mass \( m \) sits on each seat. (a) Find the period of vibration in terms of the variables \( k \), \( m \), \( a \), and \( b \). (b) Discuss the special case where \( a = b \), rather than \( a > b \) as in the real see-saw. (c) Show that your answer to part a also makes sense in the case of \( b = 0 \).

Show that the equation \( \omega = \sqrt{\frac{k}{m}} \) has units that make sense.
11 A hot scientific question of the 18th century was the shape of the earth: whether its radius was greater at the equator than at the poles, or the other way around. One method used to attack this question was to measure gravity accurately in different locations on the earth using pendula. If the highest and lowest latitudes accessible to explorers were 0 and 70 degrees, then the the strength of gravity would in reality be observed to vary over a range from about 9.780 to 9.826 m/s². This change, amounting to 0.046 m/s², is greater than the 0.022 m/s² effect to be expected if the earth had been spherical. The greater effect occurs because the equator feels a reduction due not just to the acceleration of the spinning earth out from under it, but also to the greater radius of the earth at the equator. What is the accuracy with which the period of a one-second pendulum would have to be measured in order to prove that the earth was not a sphere, and that it bulged at the equator?

12 A certain mass, when hung from a certain spring, causes the spring to stretch by an amount \( h \) compared to its equilibrium length. If the mass is displaced vertically from this equilibrium, it will oscillate up and down with a period \( T_{osc} \). Give a numerical comparison between \( T_{osc} \) and \( T_{fall} \), the time required for the mass to fall from rest through a height \( h \), when it isn’t attached to the spring.

13 Find the period of vertical oscillations of the mass \( m \). The spring, pulley, and ropes have negligible mass.

14 The equilibrium length of each spring in the figure is \( b \), so when the mass \( m \) is at the center, neither spring exerts any force on it. When the mass is displaced to the side, the springs stretch; their spring constants are both \( k \).
(a) Find the energy, \( U \), stored in the springs, as a function of \( y \), the distance of the mass up or down from the center.
(b) Show that the period of small up-down oscillations is infinite.
For a one-dimensional harmonic oscillator, the solution to the energy conservation equation,

\[ U + K = \frac{1}{2} kx^2 + \frac{1}{2} mv^2 = \text{constant}, \]

is an oscillation with frequency \( \omega = \sqrt{\frac{k}{m}} \).

Now consider an analogous system consisting of a bar magnet hung from a thread, which acts like a magnetic compass. A normal compass is full of water, so its oscillations are strongly damped, but the magnet-on-a-thread compass has very little friction, and will oscillate repeatedly around its equilibrium direction. The magnetic energy of the bar magnet is

\[ U = -Bm \cos \theta, \]

where \( B \) is a constant that measures the strength of the earth’s magnetic field, \( m \) is a constant that parametrizes the strength of the magnet, and \( \theta \) is the angle, measured in radians, between the bar magnet and magnetic north. The equilibrium occurs at \( \theta = 0 \), which is the minimum of \( U \).

(a) Problem 26 on p. 490 gave some examples of how to construct analogies between rotational and linear motion. Using the same technique, translate the equation defining the linear quantity \( k \) to one that defines an analogous angular one \( \kappa \) (Greek letter kappa). Applying this to the present example, find an expression for \( \kappa \). (Assume the thread is so thin that its stiffness does not have any significant effect compared to earth’s magnetic field.) √

(b) Find the frequency of the compass’s vibrations. √

A mass \( m \) on a spring oscillates around an equilibrium at \( x = 0 \). Any function \( F(x) \) with an equilibrium at \( x = 0, F(0) = 0 \), can be approximated as \( F(x) = -kx \), and if the spring’s behavior is symmetric with respect to positive and negative values of \( x \), so that \( F(-x) = -F(x) \), then the next level of improvement in such an approximation would be \( F(x) = -kx - bx^3 \). The general idea here is that any smooth function can be approximated locally by a polynomial, and if you want a better approximation, you can use a polynomial with more terms in it. When you ask your calculator to calculate a function like \( \sin \) or \( e^x \), it’s using a polynomial approximation with 10 or 12 terms. Physically, a spring with a positive value of \( b \) gets stiffer when stretched strongly than an “ideal” spring with \( b = 0 \). A spring with a negative \( b \) is like a person who cracks under stress — when you stretch it too much, it becomes more elastic than an ideal spring would. We should not expect any spring to give totally ideal behavior no matter how much it is stretched. For example, there has to be some point at which it breaks.

Do a numerical simulation of the oscillation of a mass on a spring whose force has a nonvanishing \( b \). Is the period still independent of
amplitude? Is the amplitude-independent equation for the period still approximately valid for small enough amplitudes? Does the addition of an $x^3$ term with $b > 0$ tend to increase the period, or decrease it? Include a printout of your program and its output with your homework paper.

17 An idealized pendulum consists of a pointlike mass $m$ on the end of a massless, rigid rod of length $L$. Its amplitude, $\theta$, is the angle the rod makes with the vertical when the pendulum is at the end of its swing. Write a numerical simulation to determine the period of the pendulum for any combination of $m$, $L$, and $\theta$. Examine the effect of changing each variable while manipulating the others. ⋆
Exercise 16: Vibrations

Equipment:

- air track and carts of two different masses
- springs
- spring scales

Place the cart on the air track and attach springs so that it can vibrate.

1. Test whether the period of vibration depends on amplitude. Try at least one moderate amplitude, for which the springs do not go slack, at least one amplitude that is large enough so that they do go slack, and one amplitude that’s the very smallest you can possibly observe.

2. Try a cart with a different mass. Does the period change by the expected factor, based on the equation $\omega = \sqrt{\frac{k}{m}}$?

3. Use a spring scale to pull the cart away from equilibrium, and make a graph of force versus position. Is it linear? If so, what is its slope?

4. Test the equation $\omega = \sqrt{\frac{k}{m}}$ numerically.
Chapter 17

Resonance

Soon after the mile-long Tacoma Narrows Bridge opened in July 1940, motorists began to notice its tendency to vibrate frighteningly in even a moderate wind. Nicknamed “Galloping Gertie,” the bridge collapsed in a steady 42-mile-per-hour wind on November 7 of the same year. The following is an eyewitness report from a newspaper editor who found himself on the bridge as the vibrations approached the breaking point.

“Just as I drove past the towers, the bridge began to sway violently from side to side. Before I realized it, the tilt became so violent that I lost control of the car... I jammed on the brakes and
got out, only to be thrown onto my face against the curb.

“Around me I could hear concrete cracking. I started to get my
dog Tubby, but was thrown again before I could reach the car. The
car itself began to slide from side to side of the roadway.

“On hands and knees most of the time, I crawled 500 yards or
more to the towers... My breath was coming in gasps; my knees
were raw and bleeding, my hands bruised and swollen from gripping
the concrete curb... Toward the last, I risked rising to my feet and
running a few yards at a time... Safely back at the toll plaza, I
saw the bridge in its final collapse and saw my car plunge into the
Narrows.”

The ruins of the bridge formed an artificial reef, one of the
world’s largest. It was not replaced for ten years. The reason for its
collapse was not substandard materials or construction, nor was the
bridge under-designed: the piers were hundred-foot blocks of con-
crete, the girders massive and made of carbon steel. The bridge was
destroyed because the bridge absorbed energy efficiently from the
wind, but didn’t dissipate it efficiently into heat. The replacement
bridge, which has lasted half a century so far, was built smarter, not
stronger. The engineers learned their lesson and simply included
some slight modifications to avoid the phenomenon that spelled the
doom of the first one.

17.1 Energy in vibrations

One way of describing the collapse of the bridge is that the bridge
kept taking energy from the steadily blowing wind and building up
more and more energetic vibrations. In this section, we discuss the
energy contained in a vibration, and in the subsequent sections we
will move on to the loss of energy and the adding of energy to a
vibrating system, all with the goal of understanding the important
phenomenon of resonance.

Going back to our standard example of a mass on a spring,
we find that there are two forms of energy involved: the potential
energy stored in the spring and the kinetic energy of the moving
mass. We may start the system in motion either by hitting the
mass to put in kinetic energy or by pulling it to one side to put in
potential energy. Either way, the subsequent behavior of the system
is identical. It trades energy back and forth between kinetic and
potential energy. (We are still assuming there is no friction, so that
no energy is converted to heat, and the system never runs down.)

The most important thing to understand about the energy con-
tent of vibrations is that the total energy is proportional to the
square of the amplitude. Although the total energy is constant, it
is instructive to consider two specific moments in the motion of the mass on a spring as examples. When the mass is all the way to one side, at rest and ready to reverse directions, all its energy is potential. We have already seen that the potential energy stored in a spring equals \((1/2)kx^2\), so the energy is proportional to the square of the amplitude. Now consider the moment when the mass is passing through the equilibrium point at \(x = 0\). At this point it has no potential energy, but it does have kinetic energy. The velocity is proportional to the amplitude of the motion, and the kinetic energy, \((1/2)mv^2\), is proportional to the square of the velocity, so again we find that the energy is proportional to the square of the amplitude. The reason for singling out these two points is merely instructive; proving that energy is proportional to \(A^2\) at any point would suffice to prove that energy is proportional to \(A^2\) in general, since the energy is constant.

Are these conclusions restricted to the mass-on-a-spring example? No. We have already seen that \(F = -kx\) is a valid approximation for any vibrating object, as long as the amplitude is small. We are thus left with a very general conclusion: the energy of any vibration is approximately proportional to the square of the amplitude, provided that the amplitude is small.

---

1. **Water in a U-tube**  
   If water is poured into a U-shaped tube as shown in the figure, it can undergo vibrations about equilibrium. The energy of such a vibration is most easily calculated by considering the “turnaround point” when the water has stopped and is about to reverse directions. At this point, it has only potential energy and no kinetic energy, so by calculating its potential energy we can find the energy of the vibration. This potential energy is the same as the work that would have to be done to take the water out of the right-hand side down to a depth \(A\) below the equilibrium level, raise it through a height \(A\), and place it in the left-hand side. The weight of this chunk of water is proportional to \(A\), and so is the height through which it must be lifted, so the energy is proportional to \(A^2\).

---

1. **The range of energies of sound waves**  
   The amplitude of vibration of your eardrum at the threshold of pain is about \(10^6\) times greater than the amplitude with which it vibrates in response to the softest sound you can hear. How many times greater is the energy with which your ear has to cope for the painfully loud sound, compared to the soft sound?

   - The amplitude is \(10^6\) times greater, and energy is proportional to the square of the amplitude, so the energy is greater by a factor of \(10^{12}\). This is a phenomenally large factor!
We are only studying vibrations right now, not waves, so we are not yet concerned with how a sound wave works, or how the energy gets to us through the air. Note that because of the huge range of energies that our ear can sense, it would not be reasonable to have a sense of loudness that was additive. Consider, for instance, the following three levels of sound:

- barely audible wind
- quiet conversation . . . . \(10^5\) times more energy than the wind
- heavy metal concert . . \(10^{12}\) times more energy than the wind

In terms of addition and subtraction, the difference between the wind and the quiet conversation is nothing compared to the difference between the quiet conversation and the heavy metal concert. Evolution wanted our sense of hearing to be able to encompass all these sounds without collapsing the bottom of the scale so that anything softer than the crack of doom would sound the same. So rather than making our sense of loudness additive, mother nature made it multiplicative. We sense the difference between the wind and the quiet conversation as spanning a range of about \(5/12\) as much as the whole range from the wind to the heavy metal concert. Although a detailed discussion of the decibel scale is not relevant here, the basic point to note about the decibel scale is that it is logarithmic. The zero of the decibel scale is close to the lower limit of human hearing, and adding 1 unit to the decibel measurement corresponds to multiplying the energy level (or actually the power per unit area) by a certain factor.

### 17.2 Energy lost from vibrations

**Numerical treatment**

An oscillator that has friction is referred to as damped. Let’s use numerical techniques to find the motion of a damped oscillator that is released away from equilibrium, but experiences no driving force after that. We can expect that the motion will consist of oscillations that gradually die out.

Friction is in general a very complicated phenomenon, and for example video games with racing cars in them include extremely sophisticated models of the friction between the tires and the road. On p. 172 I presented a very simple model of friction that dates back to the French physicist Coulomb, who worked in the era of the French revolution. There is nothing sacred about this model. For example, it doesn’t work well for lubricated surfaces. Most of the ideas we’re going to learn about damped vibrations are at least qualitatively correct regardless of what technical assumptions are made about friction, but the math turns out to be much simpler if
we choose, instead of the Coulomb model, one in which the force of friction on an object is given by $F = -bv$, where $v$ is the object’s speed and $b$ is a constant. This is in contrast to the Coulomb model, in which the force is independent of speed.

Newton’s second law, $a = F/m$, gives $a = (-kx - bv)/m$. This becomes a little prettier if we rewrite it in the form

$$ma + bv + kx = 0,$$

which gives symmetric treatment to three terms involving $x$ and its first and second derivatives, $v$ and $a$.

```python
import math

k = 39.4784  # chosen to give a period of 1 second
m = 1.

b = 0.211    # chosen to make the results simple
x = 1.

v = 0.
t = 0.
dt = 0.01
n = 1000

for j in range(n):
x = x + v * dt
a = (-k * x - b * v) / m
if (v > 0) and (v + a * dt < 0):
    print("turnaround at t=", t, ", x=", x)
v = v + a * dt
t = t + dt

turnaround at t= 0.99 , x= 0.899919262445
turnaround at t= 1.99 , x= 0.809844934046
turnaround at t= 2.99 , x= 0.728777519477
turnaround at t= 3.99 , x= 0.655817260033
turnaround at t= 4.99 , x= 0.590154191135
turnaround at t= 5.99 , x= 0.531059189965
turnaround at t= 6.99 , x= 0.477875914756
turnaround at t= 7.99 , x= 0.430013546991
turnaround at t= 8.99 , x= 0.386940256644
turnaround at t= 9.99 , x= 0.348177318484
```

The spring constant, $k = 4\pi = 39.4784$ N/m, is designed so that if the undamped equation $f = (1/2\pi)\sqrt{k/m}$ was still true, the frequency would be 1 Hz. We start by noting that the addition of a small amount of damping doesn’t seem to have changed the period at all, or at least not to within the accuracy of the calculation. You can check for yourself, however, that a large value of $b$, say 5 N·s/m, does change the period significantly.
We release the mass from $x = 1$ m, and after one cycle, it only comes back to about $x = 0.9$ m. I chose $b = 0.211$ N·s/m by fiddling around until I got this result, since a decrease of exactly 10% is easy to discuss. Notice how the amplitude after two cycles is about 0.81 m, i.e., 1 m times 0.9²: the amplitude has again dropped by exactly 10%. This pattern continues for as long as the simulation runs, e.g., for the last two cycles, we have $0.34818/0.38694=0.89982$, or almost exactly 0.9 again. It might have seemed capricious when I chose to use the unrealistic equation $F = -bv$, but this is the payoff. Only with $-bv$ friction do we get this kind of mathematically simple exponential decay.

Because the decay is exponential, it never dies out completely; this is different from the behavior we would have had with Coulomb friction, which does make objects grind completely to a stop at some point. With friction that acts like $F = -bv$, $v$ gets smaller as the oscillations get smaller. The smaller and smaller force then causes them to die out at a rate that is slower and slower.

**Analytic treatment**

Taking advantage of this unexpectedly simple result, let’s find an analytic solution for the motion. The numerical output suggests that we assume a solution of the form

$$x = Ae^{-ct}\sin(\omega_ft + \delta),$$

where the unknown constants $\omega_f$ and $c$ will presumably be related to $m$, $b$, and $k$. The constant $c$ indicates how quickly the oscillations die out. The constant $\omega_f$ is, as before, defined as $2\pi$ times the frequency, with the subscript $f$ to indicate a free (undriven) solution. All our equations will come out much simpler if we use $\omega$s everywhere instead of $f$s from now on, and, as physicists often do, I’ll generally use the word “frequency” to refer to $\omega$ when the context makes it clear what I’m talking about. The phase angle $\delta$ has no real physical significance, since we can define $t = 0$ to be any moment in time we like.

**self-check A**

In figure c, which graph has the greater value of $c$?  ▶ Answer, p. 571

The factor $A$ for the initial amplitude can also be omitted without loss of generality, since the equation we’re trying to solve, $ma + bv + kx = 0$, is linear. That is, $v$ and $a$ are the first and second derivatives of $x$, and the derivative of $Ax$ is simply $A$ times the derivative of $x$. Thus, if $x(t)$ is a solution of the equation, then multiplying it by a constant gives an equally valid solution. This is another place where we see that a damping force proportional to $v$ is the easiest to handle mathematically. For a damping force proportional to $v^2$, for example, we would have had to solve the equation $ma + bv^2 + kx = 0$, which is nonlinear.

For the purpose of determining $\omega_f$ and $c$, the most general form
we need to consider is therefore \( x = e^{-ct} \sin \omega_f t \), whose first and second derivatives are \( v = e^{-ct} \left( -c \sin \omega_f t + \omega \cos \omega_f t \right) \) and \( a = e^{-ct} \left( c^2 \sin \omega_f t - 2 \omega_f c \cos \omega_f t - \omega^2 \sin \omega_f t \right) \). Plugging these into the equation \( ma + bv + kx = 0 \) and setting the sine and cosine parts equal to zero gives, after some tedious algebra,

\[
c = \frac{b}{2m}
\]

and

\[
\omega_f = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.
\]

Intuitively, we expect friction to “slow down” the motion, as when we ride a bike into a big patch of mud. “Slow down,” however, could have more than one meaning here. It could mean that the oscillator would take more time to complete each cycle, or it could mean that as time went on, the oscillations would die out, thus giving smaller velocities.

Our mathematical results show that both of these things happen. The first equation says that \( c \), which indicates how quickly the oscillations damp out, is directly related to \( b \), the strength of the damping.

The second equation, for the frequency, can be compared with the result from page 507 of \( \sqrt{k/m} \) for the undamped system. Let’s refer to this now as \( \omega_0 \), to distinguish it from the actual frequency \( \omega_f \) of the free oscillations when damping is present. The result for \( \omega_f \) will be less than \( \omega_0 \), due to the presence of the \( b^2/4m^2 \) term. This tells us that the addition of friction to the system does increase the time required for each cycle. However, it is very common for the \( b^2/4m^2 \) term to be negligible, so that \( \omega_f \approx \omega_0 \).

Figure d shows an example. The damping here is quite strong: after only one cycle of oscillation, the amplitude has already been reduced by a factor of 2, corresponding to a factor of 4 in energy. However, the frequency of the damped oscillator is only about 1% lower than that of the undamped one; after five periods, the accumulated lag is just barely visible in the offsetting of the arrows. We can see that extremely strong damping — even stronger than this — would have been necessary in order to make \( \omega_f \approx \omega_0 \) a poor approximation.

It is customary to describe the amount of damping with a quantity called the quality factor, \( Q \), defined as the number of cycles required for the energy to fall off by a factor of 535. (The origin of this obscure numerical factor is \( e^{2\pi} \), where \( e = 2.71828 \ldots \) is the base of natural logarithms. Choosing this particular number causes some of our later equations to come out nice and simple.) The terminology arises from the fact that friction is often considered a bad
thing, so a mechanical device that can vibrate for many oscillations before it loses a significant fraction of its energy would be considered a high-quality device.

17.3 Putting energy into vibrations

When pushing a child on a swing, you cannot just apply a constant force. A constant force will move the swing out to a certain angle, but will not allow the swing to start swinging. Nor can you give short pushes at randomly chosen times. That type of random pushing would increase the child’s kinetic energy whenever you happened to be pushing in the same direction as her motion, but it would reduce her energy when your pushing happened to be in the opposite direction compared to her motion. To make her build up her energy, you need to make your pushes rhythmic, pushing at the same point in each cycle. In other words, your force needs to form a repeating pattern with the same frequency as the normal frequency of vibration of the swing. Graph e/1 shows what the child’s $x - t$ graph would look like as you gradually put more and more energy into her vibrations. A graph of your force versus time would probably look something like graph 2. It turns out, however, that it is much simpler mathematically to consider a vibration with energy being pumped into it by a driving force that is itself a sine-wave, 3. A good example of this is your eardrum being driven by the force of a sound wave.

Now we know realistically that the child on the swing will not keep increasing her energy forever, nor does your eardrum end up exploding because a continuing sound wave keeps pumping more and more energy into it. In any realistic system, there is energy going out as well as in. As the vibrations increase in amplitude, there is an increase in the amount of energy taken away by damping with each cycle. This occurs for two reasons. Work equals force times distance (or, more accurately, the area under the force-distance curve). As the amplitude of the vibrations increases, the damping force is being applied over a longer distance. Furthermore, the damping force usually increases with velocity (we usually assume for simplicity that it is proportional to velocity), and this also serves to increase the rate at which damping forces remove energy as the amplitude increases. Eventually (and small children and our eardrums are thankful for this!), the amplitude approaches a maximum value, f, at which energy is removed by the damping force just as quickly as it is being put in by the driving force.

This process of approaching a maximum amplitude happens extremely quickly in many cases, e.g., the ear or a radio receiver, and we don’t even notice that it took a millisecond or a microsecond for the vibrations to “build up steam.” We are therefore mainly interested in predicting the behavior of the system once it has had
enough time to reach essentially its maximum amplitude. This is known as the steady-state behavior of a vibrating system.

Now comes the interesting part: what happens if the frequency of the driving force is mismatched to the frequency at which the system would naturally vibrate on its own? We all know that a radio station doesn’t have to be tuned in exactly, although there is only a small range over which a given station can be received. The designers of the radio had to make the range fairly small to make it possible to eliminate unwanted stations that happened to be nearby in frequency, but it couldn’t be too small or you wouldn’t be able to adjust the knob accurately enough. (Even a digital radio can be tuned to 88.0 MHz and still bring in a station at 88.1 MHz.) The ear also has some natural frequency of vibration, but in this case the range of frequencies to which it can respond is quite broad. Evolution has made the ear’s frequency response as broad as possible because it was to our ancestors’ advantage to be able to hear everything from a low roar to a high-pitched shriek.

The remainder of this section develops four important facts about the response of a system to a driving force whose frequency is not necessarily the same as the system’s natural frequency of vibration. The style is approximate and intuitive, but proofs are given in section 17.4.

First, although we know the ear has a frequency — about 4000 Hz — at which it would vibrate naturally, it does not vibrate at 4000 Hz in response to a low-pitched 200 Hz tone. It always responds at the frequency at which it is driven. Otherwise all pitches would sound like 4000 Hz to us. This is a general fact about driven vibrations:

1. The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system’s own natural frequency of vibration.

Now let’s think about the amplitude of the steady-state response. Imagine that a child on a swing has a natural frequency of vibration of 1 Hz, but we are going to try to make her swing back and forth at 3 Hz. We intuitively realize that quite a large force would be needed to achieve an amplitude of even 30 cm, i.e., the amplitude is less in proportion to the force. When we push at the natural frequency of 1 Hz, we are essentially just pumping energy back into the system to compensate for the loss of energy due to the damping (friction) force. At 3 Hz, however, we are not just counteracting friction. We are also providing an extra force to make the child’s momentum reverse itself more rapidly than it would if gravity and the tension in the chain were the only forces acting. It is as if we are artificially increasing the $k$ of the swing, but this is wasted effort because we
spend just as much time decelerating the child (taking energy out of the system) as accelerating her (putting energy in).

Now imagine the case in which we drive the child at a very low frequency, say 0.02 Hz or about one vibration per minute. We are essentially just holding the child in position while very slowly walking back and forth. Again we intuitively recognize that the amplitude will be very small in proportion to our driving force. Imagine how hard it would be to hold the child at our own head-level when she is at the end of her swing! As in the too-fast 3 Hz case, we are spending most of our effort in artificially changing the $k$ of the swing, but now rather than reinforcing the gravity and tension forces we are working against them, effectively reducing $k$. Only a very small part of our force goes into counteracting friction, and the rest is used in repetitively putting potential energy in on the upswing and taking it back out on the downswing, without any long-term gain.

We can now generalize to make the following statement, which is true for all driven vibrations:

\[ (2) \text{ A vibrating system resonates at its own natural frequency.} \]

That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

An opera singer breaking a wine glass example 3
In order to break a wineglass by singing, an opera singer must first tap the glass to find its natural frequency of vibration, and then sing the same note back.

Collapse of the Nimitz Freeway in an earthquake example 4
I led off the chapter with the dramatic collapse of the Tacoma Narrows Bridge, mainly because it was well documented by a local physics professor, and an unknown person made a movie of the collapse. The collapse of a section of the Nimitz Freeway in Oakland, CA, during a 1989 earthquake is however a simpler example to analyze.

An earthquake consists of many low-frequency vibrations that occur simultaneously, which is why it sounds like a rumble of indeterminate pitch rather than a low hum. The frequencies that we can hear are not even the strongest ones; most of the energy is in the form of vibrations in the range of frequencies from about 1 Hz to 10 Hz.

Now all the structures we build are resting on geological layers of dirt, mud, sand, or rock. When an earthquake wave comes along, the topmost layer acts like a system with a certain natural frequency of vibration, sort of like a cube of jello on a plate being
shaken from side to side. The resonant frequency of the layer depends on how stiff it is and also on how deep it is. The ill-fated section of the Nimitz freeway was built on a layer of mud, and analysis by geologist Susan E. Hough of the U.S. Geological Survey shows that the mud layer’s resonance was centered on about 2.5 Hz, and had a width covering a range from about 1 Hz to 4 Hz.

When the earthquake wave came along with its mixture of frequencies, the mud responded strongly to those that were close to its own natural 2.5 Hz frequency. Unfortunately, an engineering analysis after the quake showed that the overpass itself had a resonant frequency of 2.5 Hz as well! The mud responded strongly to the earthquake waves with frequencies close to 2.5 Hz, and the bridge responded strongly to the 2.5 Hz vibrations of the mud, causing sections of it to collapse.

'Collapse of the Tacoma Narrows Bridge example 5

Let’s now examine the more conceptually difficult case of the Tacoma Narrows Bridge. The surprise here is that the wind was steady. If the wind was blowing at constant velocity, why did it shake the bridge back and forth? The answer is a little complicated. Based on film footage and after-the-fact wind tunnel experiments, it appears that two different mechanisms were involved.

The first mechanism was the one responsible for the initial, relatively weak vibrations, and it involved resonance. As the wind moved over the bridge, it began acting like a kite or an airplane wing. As shown in the figure, it established swirling patterns of air flow around itself, of the kind that you can see in a moving cloud of smoke. As one of these swirls moved off of the bridge, there was an abrupt change in air pressure, which resulted in an up or down force on the bridge. We see something similar when a flag flaps in the wind, except that the flag’s surface is usually vertical. This back-and-forth sequence of forces is exactly the kind of periodic driving force that would excite a resonance. The faster the wind, the more quickly the swirls would get across the bridge, and the higher the frequency of the driving force would be. At just the right velocity, the frequency would be the right one to excite the resonance. The wind-tunnel models, however, show that the pattern of vibration of the bridge excited by this mechanism would have been a different one than the one that finally destroyed the bridge.

The bridge was probably destroyed by a different mechanism, in which its vibrations at its own natural frequency of 0.2 Hz set up an alternating pattern of wind gusts in the air immediately around it, which then increased the amplitude of the bridge’s vibrations. This vicious cycle fed upon itself, increasing the amplitude of the vibrations until the bridge finally collapsed.
As long as we’re on the subject of collapsing bridges, it is worth bringing up the reports of bridges falling down when soldiers marching over them happened to step in rhythm with the bridge’s natural frequency of oscillation. This is supposed to have happened in 1831 in Manchester, England, and again in 1849 in Anjou, France. Many modern engineers and scientists, however, are suspicious of the analysis of these reports. It is possible that the collapses had more to do with poor construction and overloading than with resonance. The Nimitz Freeway and Tacoma Narrows Bridge are far better documented, and occurred in an era when engineers’ abilities to analyze the vibrations of a complex structure were much more advanced.

\[ Emission \text{ and absorption of light waves by atoms ~ example 6} \]

In a very thin gas, the atoms are sufficiently far apart that they can act as individual vibrating systems. Although the vibrations are of a very strange and abstract type described by the theory of quantum mechanics, they nevertheless obey the same basic rules as ordinary mechanical vibrations. When a thin gas made of a certain element is heated, it emits light waves with certain specific frequencies, which are like a fingerprint of that element. As with all other vibrations, these atomic vibrations respond most strongly to a driving force that matches their own natural frequency. Thus if we have a relatively cold gas with light waves of various frequencies passing through it, the gas will absorb light at precisely those frequencies at which it would emit light if heated.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to \( Q \).

This is fairly intuitive. The steady-state behavior is an equilibrium between energy input from the driving force and energy loss due to damping. A low-\( Q \) oscillator, i.e., one with strong damping, dumps its energy faster, resulting in lower-amplitude steady-state motion.

\[ \text{self-check B} \]

If an opera singer is shopping for a wine glass that she can impress her friends by breaking, what should she look for? \( \triangleright \) Answer, p. 571

\[ \text{Piano strings ringing in sympathy with a sung note ~ example 7} \]

\( \triangleright \) A sufficiently loud musical note sung near a piano with the lid raised can cause the corresponding strings in the piano to vibrate. (A piano has a set of three strings for each note, all struck by the same hammer.) Why would this trick be unlikely to work with a violin?

\( \triangleright \) If you have heard the sound of a violin being plucked (the pizzicato effect), you know that the note dies away very quickly. In other words, a violin’s \( Q \) is much lower than a piano’s. This means
that its resonances are much weaker in amplitude.

Our fourth and final fact about resonance is perhaps the most surprising. It gives us a way to determine numerically how wide a range of driving frequencies will produce a strong response. As shown in the graph, resonances do not suddenly fall off to zero outside a certain frequency range. It is usual to describe the width of a resonance by its full width at half-maximum (FWHM) as illustrated in figure h.

(4) The FWHM of a resonance is related to its $Q$ and its resonant frequency $f_{res}$ by the equation

$$\text{FWHM} = \frac{f_{res}}{Q}.$$  

(This equation is only a good approximation when $Q$ is large.)

Why? It is not immediately obvious that there should be any logical relationship between $Q$ and the FWHM. Here’s the idea. As we have seen already, the reason why the response of an oscillator is smaller away from resonance is that much of the driving force is being used to make the system act as if it had a different $k$. Roughly speaking, the half-maximum points on the graph correspond to the places where the amount of the driving force being wasted in this way is the same as the amount of driving force being used productively to replace the energy being dumped out by the damping force. If the damping force is strong, then a large amount of force is needed to counteract it, and we can waste quite a bit of driving force on changing $k$ before it becomes comparable to the damping force. If, on the other hand, the damping force is weak, then even a small amount of force being wasted on changing $k$ will become significant in proportion, and we cannot get very far from the resonant frequency before the two are comparable.

The response is in general out of phase with the driving force by an angle $\delta$.

Changing the pitch of a wind instrument example 8

A saxophone player normally selects which note to play by choosing a certain fingering, which gives the saxophone a certain resonant frequency. The musician can also, however, change the pitch significantly by altering the tightness of her lips. This corresponds to driving the horn slightly off of resonance. If the pitch can be altered by about 5% up or down (about one musical half-step) without too much effort, roughly what is the $Q$ of a saxophone?

Five percent is the width on one side of the resonance, so the
Dependence of the amplitude and phase angle on the driving frequency. The undamped case is $Q = \infty$, and the other curves represent $Q=1$, 3, and 10. $F_m$, $m$, and $\omega_0$ are all set to 1.

If a typical saxophone setup has a $Q$ of about 10, how long will it take for a 100-Hz tone played on a baritone saxophone to die down by a factor of 535 in energy, after the player suddenly stops blowing?

A $Q$ of 10 means that it takes 10 cycles for the vibrations to die down in energy by a factor of 535. Ten cycles at a frequency of 100 Hz would correspond to a time of 0.1 seconds, which is not
very long. This is why a saxophone note doesn’t “ring” like a note played on a piano or an electric guitar.

\[ Q \text{ of a radio receiver} \quad \text{example 10} \]

A radio receiver used in the FM band needs to be tuned in to within about 0.1 MHz for signals at about 100 MHz. What is its \( Q \)?

\[ Q = \frac{f_{res}}{\text{FWHM}} = 1000. \] This is an extremely high \( Q \) compared to most mechanical systems.

\[ Q \text{ of a stereo speaker} \quad \text{example 11} \]

We have already given one reason why a stereo speaker should have a low \( Q \): otherwise it would continue ringing after the end of the musical note on the recording. The second reason is that we want it to be able to respond to a large range of frequencies.

\[ \text{Nuclear magnetic resonance} \quad \text{example 12} \]

If you have ever played with a magnetic compass, you have undoubtedly noticed that if you shake it, it takes some time to settle down, \( j/1 \). As it settles down, it acts like a damped oscillator of the type we have been discussing. The compass needle is simply a small magnet, and the planet earth is a big magnet. The magnetic forces between them tend to bring the needle to an equilibrium position in which it lines up with the planet-earth-magnet.

Essentially the same physics lies behind the technique called Nuclear Magnetic Resonance (NMR). NMR is a technique used to deduce the molecular structure of unknown chemical substances, and it is also used for making medical images of the inside of people’s bodies. If you ever have an NMR scan, they will actually tell you you are undergoing “magnetic resonance imaging” or “MRI,” because people are scared of the word “nuclear.” In fact, the nuclei being referred to are simply the non-radioactive nuclei of atoms found naturally in your body.

Here’s how NMR works. Your body contains large numbers of hydrogen atoms, each consisting of a small, lightweight electron orbiting around a large, heavy proton. That is, the nucleus of a hydrogen atom is just one proton. A proton is always spinning on its own axis, and the combination of its spin and its electrical charge causes it to behave like a tiny magnet. The principle is identical to that of an electromagnet, which consists of a coil of wire through which electrical charges pass; the circling motion of the charges in the coil of wire makes it magnetic, and in the same way, the circling motion of the proton’s charge makes it magnetic.

Now a proton in one of your body’s hydrogen atoms finds itself surrounded by many other whirling, electrically charged particles: its own electron, plus the electrons and nuclei of the other nearby atoms. These neighbors act like magnets, and exert magnetic forces on the proton, \( j/2 \). The \( k \) of the vibrating proton is simply a
measure of the total strength of these magnetic forces. Depending on the structure of the molecule in which the hydrogen atom finds itself, there will be a particular set of magnetic forces acting on the proton and a particular value of \( k \). The NMR apparatus bombards the sample with radio waves, and if the frequency of the radio waves matches the resonant frequency of the proton, the proton will absorb radio-wave energy strongly and oscillate wildly. Its vibrations are damped not by friction, because there is no friction inside an atom, but by the reemission of radio waves.

By working backward through this chain of reasoning, one can determine the geometric arrangement of the hydrogen atom’s neighboring atoms. It is also possible to locate atoms in space, allowing medical images to be made.

Finally, it should be noted that the behavior of the proton cannot be described entirely correctly by Newtonian physics. Its vibrations are of the strange and spooky kind described by the laws of quantum mechanics. It is impressive, however, that the few simple ideas we have learned about resonance can still be applied successfully to describe many aspects of this exotic system.

**Discussion question**

A Nikola Tesla, one of the inventors of radio and an archetypical mad scientist, told a credulous reporter in 1912 the following story about an application of resonance. He built an electric vibrator that fit in his pocket, and attached it to one of the steel beams of a building that was under construction in New York. Although the article in which he was quoted didn’t say so, he presumably claimed to have tuned it to the resonant frequency of the building. “In a few minutes, I could feel the beam trembling. Gradually the trembling increased in intensity and extended throughout the whole great mass of steel. Finally, the structure began to creak and weave, and the steelworkers came to the ground panic-stricken, believing that there had been an earthquake. ... [If] I had kept on ten minutes more, I could have laid that building flat in the street.” Is this physically plausible?

17.4 *Proofs*

Our first goal is to predict the amplitude of the steady-state vibrations as a function of the frequency of the driving force and the amplitude of the driving force. With that equation in hand, we will then prove statements 2, 3, and 4 from section 17.3.

We have an external driving force \( F = F_m \sin \omega t \), where the constant \( F_m \) indicates the maximum strength of the force in either direction. The equation of motion is

\[
ma + bv + kx = F_m \sin \omega t.
\]

For the steady-state motion, we’re going to look for a solution of the form

\[
x = A \sin(\omega t + \delta).
\]
The left-hand side of the equation of motion will clearly be a sinusoidal function with frequency $\omega$, so it can only equal the right-hand side if, as we have already implicitly assumed, the frequency of the motion matches the frequency of the driving force. This proves statement (1).

In contrast to the undriven case, here it’s not possible to sweep $A$ and $\delta$ under the rug. The amplitude of the steady-state motion, $A$, is actually the most interesting thing to know about the steady-state motion, and it’s not true that we still have a solution no matter how we fiddle with $A$; if we have a solution for a certain value of $A$, then multiplying $A$ by some constant would break the equality between the two sides of the equation of motion. It’s also no longer true that we can get rid of $\delta$ simply by redefining when we start the clock; here $\delta$ represents a difference in time between the start of one cycle of the driving force and the start of the corresponding cycle of the motion.

The velocity and acceleration are $v = \omega A \cos(\omega t + \delta)$ and $a = -\omega^2 A \sin(\omega t + \delta)$, and if we plug these into the equation of motion, [1], and simplify a little, we find

$$\begin{align*}
(k - m\omega^2) \sin(\omega t + \delta) + \omega b \cos(\omega t + \delta) &= \frac{F_m}{A} \sin \omega t.
\end{align*}$$

The sum of any two sinusoidal functions with the same frequency is also a sinusoidal, so the whole left side adds up to a sinusoidal. By fiddling with $A$ and $\delta$ we can make the amplitudes and phases of the two sides of the equation match up.

Using the trig identities for the sine of a sum and cosine of a sum, we can change equation [2] into the form

$$\begin{align*}
&\left[ (-m\omega^2 + k) \cos \delta - b \omega \sin \delta - \frac{F_m}{A} \right] \sin \omega t \\
&+ \left[ (-m\omega^2 + k) \sin \delta + b \omega \cos \delta \right] \cos \omega t = 0.
\end{align*}$$

Both the quantities in square brackets must equal zero, which gives us two equations we can use to determine the unknowns $A$ and $\delta$. The results are

$$\begin{align*}
\delta &= \tan^{-1} \frac{\omega \omega_0}{Q(\omega_0^2 - \omega^2)} \\
A &= \frac{F_m}{m \sqrt{(\omega^2 - \omega_0^2)^2 + \omega_0^2 \omega^2 Q^{-2}}}.
\end{align*}$$

**Statement 2: maximum amplitude at resonance**

Equation [4] makes it plausible that the amplitude is maximized when the system is driven at close to its resonant frequency. At
\( f = f_0 \), the first term inside the square root vanishes, and this makes the denominator as small as possible, causing the amplitude to be as big as possible. (Actually this is only approximately true, because it is possible to make \( A \) a little bigger by decreasing \( f \) a little below \( f_0 \), which makes the second term smaller. This technical issue is addressed in homework problem 3 on page 541.)

**Statement 3: amplitude at resonance proportional to \( Q \)**

Equation [4] shows that the amplitude at resonance is proportional to \( 1/b \), and the \( Q \) of the system is inversely proportional to \( b \), so the amplitude at resonance is proportional to \( Q \).

**Statement 4: FWHM related to \( Q \)**

We will satisfy ourselves by proving only the proportionality \( \text{FWHM} \propto f_0/Q \), not the actual equation \( \text{FWHM} = f_0/Q \). The energy is proportional to \( A^2 \), i.e., to the inverse of the quantity inside the square root in equation [4]. At resonance, the first term inside the square root vanishes, and the half-maximum points occur at frequencies for which the whole quantity inside the square root is double its value at resonance, i.e., when the two terms are equal. At the half-maximum points, we have

\[
f^2 - f_0^2 = \left( f_0 \pm \frac{\text{FWHM}}{2} \right)^2 - f_0^2\]

\[
= \pm f_0 \cdot \text{FWHM} + \frac{1}{4} \text{FWHM}^2
\]

If we assume that the width of the resonance is small compared to the resonant frequency, then the \( \text{FWHM}^2 \) term is negligible compared to the \( f_0 \cdot \text{FWHM} \) term, and setting the terms in equation 4 equal to each other gives

\[
4\pi^2 m^2 (f_0 \cdot \text{FWHM})^2 = b^2 f^2.
\]

We are assuming that the width of the resonance is small compared to the resonant frequency, so \( f \) and \( f_0 \) can be taken as synonyms. Thus,

\[
\text{FWHM} = \frac{b}{2\pi m}.
\]

We wish to connect this to \( Q \), which can be interpreted as the energy of the free (undriven) vibrations divided by the work done by damping in one cycle. The former equals \( kA^2/2 \), and the latter is proportional to the force, \( bv \propto bAf_0 \), multiplied by the distance traveled, \( A \). (This is only a proportionality, not an equation, since the force is not constant.) We therefore find that \( Q \) is proportional to \( k/bf_0 \). The equation for the FWHM can then be restated as a proportionality \( \text{FWHM} \propto k/Qf_0m \propto f_0/Q \).
Summary

Selected vocabulary

damping . . . . . the dissipation of a vibration’s energy into heat energy, or the frictional force that causes the loss of energy

quality factor . . the number of oscillations required for a system’s energy to fall off by a factor of 535 due to damping

driving force . . . an external force that pumps energy into a vibrating system

resonance . . . . . the tendency of a vibrating system to respond most strongly to a driving force whose frequency is close to its own natural frequency of vibration

steady state . . . . the behavior of a vibrating system after it has had plenty of time to settle into a steady response to a driving force

Notation

$Q$ . . . . . . . the quality factor
$f_o$ . . . . . . . the natural (resonant) frequency of a vibrating system, i.e., the frequency at which it would vibrate if it was simply kicked and left alone

$f$ . . . . . . . . the frequency at which the system actually vibrates, which in the case of a driven system is equal to the frequency of the driving force, not the natural frequency

Summary

The energy of a vibration is always proportional to the square of the amplitude, assuming the amplitude is small. Energy is lost from a vibrating system for various reasons such as the conversion to heat via friction or the emission of sound. This effect, called damping, will cause the vibrations to decay exponentially unless energy is pumped into the system to replace the loss. A driving force that pumps energy into the system may drive the system at its own natural frequency or at some other frequency. When a vibrating system is driven by an external force, we are usually interested in its steady-state behavior, i.e., its behavior after it has had time to settle into a steady response to a driving force. In the steady state, the same amount of energy is pumped into the system during each cycle as is lost to damping during the same period.

The following are four important facts about a vibrating system being driven by an external force:

(1) The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system’s own natural frequency of vibration.
(2) A vibrating system resonates at its own natural frequency. That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to $Q$.

(4) The FWHM of a resonance is related to its $Q$ and its resonant frequency $f_o$ by the equation

$$\text{FWHM} = \frac{f_o}{Q}.$$ 

(This equation is only a good approximation when $Q$ is large.)
Problems

Key
✓ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1  If one stereo system is capable of producing 20 watts of sound power and another can put out 50 watts, how many times greater is the amplitude of the sound wave that can be created by the more powerful system? (Assume they are playing the same music.)

2  Many fish have an organ known as a swim bladder, an air-filled cavity whose main purpose is to control the fish’s buoyancy and allow it to keep from rising or sinking without having to use its muscles. In some fish, however, the swim bladder (or a small extension of it) is linked to the ear and serves the additional purpose of amplifying sound waves. For a typical fish having such an anatomy, the bladder has a resonant frequency of 300 Hz, the bladder’s $Q$ is 3, and the maximum amplification is about a factor of 100 in energy. Over what range of frequencies would the amplification be at least a factor of 50?

✓

3  As noted in section 17.4, it is only approximately true that the amplitude has its maximum at the natural frequency $(1/2\pi)\sqrt{k/m}$. Being more careful, we should actually define two different symbols, $f_0 = (1/2\pi)\sqrt{k/m}$ and $f_{res}$ for the slightly different frequency at which the amplitude is a maximum, i.e., the actual resonant frequency. In this notation, the amplitude as a function of frequency is

$$A = \frac{F}{2\pi \sqrt{4\pi^2 m^2 (f^2 - f_0^2)^2 + b^2 f^2}}.$$ 

Show that the maximum occurs not at $f_0$ but rather at

$$f_{res} = \sqrt{f_0^2 - \frac{b^2}{8\pi^2 m^2}} = \sqrt{f_0^2 - \frac{1}{2} \text{FWHM}^2}$$

Hint: Finding the frequency that minimizes the quantity inside the square root is equivalent to, but much easier than, finding the frequency that maximizes the amplitude.
4. (a) Let $W$ be the amount of work done by friction in the first cycle of oscillation, i.e., the amount of energy lost to heat. Find the fraction of the original energy $E$ that remains in the oscillations after $n$ cycles of motion.

(b) From this, prove the equation

$$\left(1 - \frac{W}{E}\right)^Q = e^{-2\pi}$$

(recalling that the number 535 in the definition of $Q$ is $e^{2\pi}$).

(c) Use this to prove the approximation $1/Q \approx (1/2\pi)W/E$. (Hint: Use the approximation $\ln(1 + x) \approx x$, which is valid for small values of $x$, as shown on p. ??.)

5. (a) We observe that the amplitude of a certain free oscillation decreases from $A_0$ to $A_0/Z$ after $n$ oscillations. Find its $Q$. 

(b) The figure is from *Shape memory in Spider draglines*, Emile, Le Floch, and Vollrath, *Nature* 440:621 (2006). Panel 1 shows an electron microscope’s image of a thread of spider silk. In 2, a spider is hanging from such a thread. From an evolutionary point of view, it’s probably a bad thing for the spider if it twists back and forth while hanging like this. (We’re referring to a back-and-forth rotation about the axis of the thread, not a swinging motion like a pendulum.) The authors speculate that such a vibration could make the spider easier for predators to see, and it also seems to me that it would be a bad thing just because the spider wouldn’t be able to control its orientation and do what it was trying to do. Panel 3 shows a graph of such an oscillation, which the authors measured using a video camera and a computer, with a 0.1 g mass hung from it.
in place of a spider. Compared to human-made fibers such as kevlar or copper wire, the spider thread has an unusual set of properties:

1. It has a low $Q$, so the vibrations damp out quickly.

2. It doesn’t become brittle with repeated twisting as a copper wire would.

3. When twisted, it tends to settle in to a new equilibrium angle, rather than insisting on returning to its original angle. You can see this in panel 2, because although the experimenters initially twisted the wire by 35 degrees, the thread only performed oscillations with an amplitude much smaller than $\pm35$ degrees, settling down to a new equilibrium at 27 degrees.

4. Over much longer time scales (hours), the thread eventually resets itself to its original equilibrium angle (shown as zero degrees on the graph). (The graph reproduced here only shows the motion over a much shorter time scale.) Some human-made materials have this “memory” property as well, but they typically need to be heated in order to make them go back to their original shapes.

Focusing on property number 1, estimate the $Q$ of spider silk from the graph.

6. An oscillator with sufficiently strong damping has its maximum response at $\omega = 0$. Using equation [4] on p. 537, find the value of $Q$ at which this behavior sets in.

   $\triangleright$ Hint, p. 551  $\triangleright$ Answer, p. 572
The goal of this problem is to refine the proportionality FWHM \( \propto \frac{f_{res}}{Q} \) into the equation FWHM = \( \frac{f_{res}}{Q} \), i.e., to prove that the constant of proportionality equals 1.

(a) Show that the work done by a damping force \( F = -bv \) over one cycle of steady-state motion equals \( W_{damp} = -2\pi^2 bf A^2 \). Hint: It is less confusing to calculate the work done over half a cycle, from \( x = -A \) to \( x = +A \), and then double it.

(b) Show that the fraction of the undriven oscillator’s energy lost to damping over one cycle is \( \frac{|W_{damp}|}{E} = 4\pi^2 bf/k \).

(c) Use the previous result, combined with the result of problem 4, to prove that \( Q \) equals \( k/2\pi bf \).

(d) Combine the preceding result for \( Q \) with the equation FWHM = \( b/2\pi m \) from section 17.4 to prove the equation FWHM = \( \frac{f_{res}}{Q} \).

8 An oscillator has \( Q=6.00 \), and, for convenience, let’s assume \( F_m = 1.00 \), \( \omega_0 = 1.00 \), and \( m = 1.00 \). The usual approximations would give

\[
\omega_{res} = \omega_0, \\
A_{res} = 6.00, \quad \text{and} \\
\Delta \omega = 1/6.00.
\]

Determine these three quantities numerically using equation [4] on p. 537, and compare with the approximations.
Exercise 17: Resonance

1. Compare the oscillator's energies at A, B, C, and D.

![Graph showing energies at A, B, C, and D.]

2. Compare the Q values of the two oscillators.

![Graphs showing x-t and amplitude-frequency relationships.]

3. Match the x-t graphs in #2 with the amplitude-frequency graphs below.

![Amplitude-frequency graphs showing response.]

Exercise 17: Resonance
Three essential mathematical skills

More often than not when a search-and-rescue team finds a hiker dead in the wilderness, it turns out that the person was not carrying some item from a short list of essentials, such as water and a map. There are three mathematical essentials in this course.

1. Converting units

Basic technique: section 0.9, p. 25; conversion of area, volume, etc.: section 1.1, p. 37

Examples:

\[ 0.7 \text{ kg} \times \frac{10^3 \text{ g}}{1 \text{ kg}} = 700 \text{ g}. \]

To check that we have the conversion factor the right way up (\(10^3\) rather than \(1/10^3\)), we note that the smaller unit of grams has been compensated for by making the number larger.

For units like \(\text{m}^2\), \(\text{kg}/\text{m}^3\), etc., we have to raise the conversion factor to the appropriate power:

\[ 4 \text{ m}^3 \times \left( \frac{10^3 \text{ mm}}{1 \text{ m}} \right)^3 = 4 \times 10^9 \text{ m}^3 \times \frac{\text{mm}^3}{\text{m}^3} = 4 \times 10^9 \text{ mm}^3 \]

Examples with solutions — p. 32, #1; p. 52, #2

Problems you can check at lightandmatter.com/area1checker.html — p. 32, #3; p. 32, #2; p. 32, #4; p. 52, #5; p. 52, #1

2. Reasoning about ratios and proportionalities

The technique is introduced in section 1.2, p. 39, in the context of area and volume, but it applies more generally to any relationship in which one variable depends on another raised to some power.

Example: When a car or truck travels over a road, there is wear and tear on the road surface, which incurs a cost. Studies show that the cost per kilometer of travel \(C\) is given by

\[ C = kw^4, \]

where \(w\) is the weight per axle and \(k\) is a constant. The weight per axle is about 13 times higher for a semi-trailer than for my Honda Fit. How many times greater is the cost imposed on the federal government when the semi travels a given distance on an interstate freeway?

▷ First we convert the equation into a proportionality by throwing out \(k\), which is the same for both vehicles:

\[ C \propto w^4 \]

Next we convert this proportionality to a statement about ratios:

\[ \frac{C_1}{C_2} = \left( \frac{w_1}{w_2} \right)^4 \approx 29,000 \]

Since the gas taxes paid by the trucker are nowhere near 29,000 times more than those I pay to drive my Fit the same distance, the federal government is effectively awarding a massive subsidy to the trucking company. Plus my Fit is cuter.
3. Vector addition

Example: The $\Delta r$ vector from San Diego to Los Angeles has magnitude 190 km and direction 129° counterclockwise from east. The one from LA to Las Vegas is 370 km at 38° counterclockwise from east. Find the distance and direction from San Diego to Las Vegas.

Graphical addition is discussed on p. 222. Here we concentrate on analytic addition, which involves adding the $x$ components to find the total $x$ component, and similarly for $y$. The trig needed in order to find the components of the second leg (LA to Vegas) is laid out in figure e on p. 219 and explained in detail in example 3 on p. 219:

\[
\begin{align*}
\Delta x_2 &= (370 \text{ km}) \cos 38^\circ = 292 \text{ km} \\
\Delta y_2 &= (370 \text{ km}) \sin 38^\circ = 228 \text{ km}
\end{align*}
\]

(Since these are intermediate results, we keep an extra sig fig to avoid accumulating too much rounding error.) Once we understand the trig for one example, we don’t need to reinvent the wheel every time. The pattern is completely universal, provided that we first make sure to get the angle expressed according to the usual trig convention, counterclockwise from the $x$ axis. Applying the pattern to the first leg, we have:

\[
\begin{align*}
\Delta x_1 &= (190 \text{ km}) \cos 129^\circ = -120 \text{ km} \\
\Delta y_1 &= (190 \text{ km}) \sin 129^\circ = 148 \text{ km}
\end{align*}
\]

For the vector directly from San Diego to Las Vegas, we have

\[
\begin{align*}
\Delta x &= \Delta x_1 + \Delta x_2 = 172 \text{ km} \\
\Delta y &= \Delta y_1 + \Delta y_2 = 376 \text{ km}.
\end{align*}
\]

The distance from San Diego to Las Vegas is found using the Pythagorean theorem,

\[
\sqrt{(172 \text{ km})^2 + (376 \text{ km})^2} = 410 \text{ km}
\]

(rounded to two sig figs because it’s one of our final results). The direction is one of the two possible values of the inverse tangent

\[
\tan^{-1}(\Delta y/\Delta x) = \{65^\circ, 245^\circ\}.
\]

Consulting a sketch shows that the first of these values is the correct one.

Examples with solutions — p. 245, #3; p. 247, #9; p. 427, #8

Problems you can check at lightandmatter.com/arealchecker.html — p. 229, #3; p. 229, #4; p. 245, #4; p. 245, #5; p. 248, #16; p. 300, #13; p. 300, #14; p. 427, #9
Programming with python

The purpose of this tutorial is to help you get familiar with a computer programming language called Python, which I’ve chosen because (a) it’s free, and (b) it’s easy to use interactively. I won’t assume you have any previous experience with computer programming; you won’t need to learn very much Python, and what little you do need to learn I’ll explain explicitly. If you really want to learn Python more thoroughly, there are a couple of excellent books that you can download for free on the Web:

How to Think Like a Computer Scientist (Python Version), Allen B. Downey, Jeffrey Elkner, Moshe Zadka, http://www.ibiblio.org/obp/

Dive Into Python, Mark Pilgrim, http://diveintopython.net/

The first book is meant for people who have never programmed before, while the second is a more complete introduction aimed at veteran programmers who know a different language already.

Using Python as a calculator

The easiest way to get Python going is to go to the web site ideone.com. Under “choose a language,” select Python. Inside the window where it says “paste your source code or insert template or sample,” type print(2+2). Click on the “submit” button. The result, 4, is shown under “output.” In other words, you can use Python just like a calculator.

For compactness, I’ll show examples in the following style:

>>> print(2+2)
4

Here the >>> is not something you would type yourself; it’s just a marker to distinguish your input from the program’s output. (In some other versions of Python, the computer will actually print out >>> as a prompt to tell you it’s ready to type something.)

There are only a couple of things to watch out for. First, Python distinguishes between integers and real numbers, so the following gives an unexpected result:

>>> print(2/3)
0

To get it to treat these values as real numbers, you have to use decimal points:

>>> print(2./3.)
0.6666666666666666666663

Multiplication is represented by “*”:

>>> print(2.*3.)
6.0

Also, Python doesn’t know about its own library of math functions unless you tell it explicitly to load them in:

>>> print (sqrt(2.))
Traceback (most recent call last):
  File ‘<stdin>’, line 1, in ?
NameError: There is no variable named ‘sqrt’

Here are the steps you have to go through to calculate the square root of 2 successfully:

>>> import math
>>> print(math.sqrt(2.))
1.4142135623730951

The first line is just something you can make a habit of doing every time you start up Python. In the second line, the name of the square root function had to be prefixed with “math.” to tell Python where you wanted to get this “sqrt” function from. (All of this may seem like a nuisance if you’re just using Python as a
calculator, but it’s a good way to design a programming language so that names of functions never conflict.)

Try it. Experiment and figure out whether Python’s trig functions assume radians or degrees.

Variables
Python lets you define variables and assign values to them using an equals sign:

```python
>>> dwarfs=7
>>> print(dwarfs)
7
>>> print(dwarfs+3)
10
```

Note that a variable in computer programming isn’t quite like a variable in algebra. In algebra, if \( a=7 \) then \( a=7 \) always, throughout a particular calculation. But in a programming language, the variable name really represents a place in memory where a number can be stored, so you can change its value:

```python
>>> dwarfs=7
>>> dwarfs=37
>>> print(dwarfs)
37
```

You can even do stuff like this,

```python
>>> dwarfs=37
>>> dwarfs=dwarfs+1
>>> print(dwarfs)
38
```

In algebra it would be nonsense to have a variable equal to itself plus one, but in a computer program, it’s not an assertion that the two things are equal, its a command to calculate the value of the expression on the right side of the equals, and then put that number into the memory location referred to by the variable name on the left.

Try it. What happens if you do `dwarfs+1 = dwarfs`? Do you understand why?

Functions
Somebody had to teach Python how to do functions like `sqrt`, and it’s handy to be able to define your own functions in the same way. Here’s how to do it:

```python
>>> def double(x):
...     return 2.*x
>>> print(double(5.))
10.0
```

Note that the indentation is mandatory. The first and second lines define a function called `double`. The final line evaluates that function with an input of 5.

Loops
Suppose we want to add up all the numbers from 0 to 99. Automating this kind of thing is exactly what computers are best at, and Python provides a mechanism for this called a loop:

```python
>>> sum=0
>>> for j in range(100):
...     sum=sum+j
>>> print(sum)
4950
```

The stuff that gets repeated — the inside of the loop — has to be indented, just like in a function definition. Python always counts loops starting from 0, so for `j in range(100)` actually causes `j` to range from 0 to 99, not from 1 to 100.

Exercise 17: Resonance
Hints

Hints for chapter 8
Page 249, problem 18:
The easiest way to do this problem is to use two different coordinate systems: one that’s tilted to coincide with the upper slope, and one that’s tilted to coincide with the lower one.

Page 249, problem 19:
Consider a section of the rope subtending a very small angle, and find an approximate equation relating the normal force to the tension. Apply small-angle approximations to any trig functions occurring in your result. Eliminate all variables except for the tension and the angle, and separate these variables.

Hints for chapter 10
Page 302, problem 22:
If you try to calculate the two forces and subtract, your calculator will probably give a result of zero due to rounding. Instead, reason about the fractional amount by which the quantity $1/r^2$ will change. As a warm-up, you may wish to observe the percentage change in $1/r^2$ that results from changing $r$ from 1 to 1.01.

Hints for chapter 13
Page 389, problem 13:
What does the total energy have to be if the projectile’s velocity is exactly escape velocity? Write down conservation of energy, change $v$ to $dr/dt$, separate the variables, and integrate.

Page 391, problem 20:
You can use the geometric interpretation of the dot product.

Page 392, problem 25:
The analytic approach is a little cumbersome, although it can be done by using approximations like $1/\sqrt{1+\epsilon} \approx 1 - (1/2)\epsilon$. A more straightforward, brute-force method is simply to write a computer program that calculates $U/m$ for a given point in spherical coordinates. By trial and error, you can fairly rapidly find the $r$ that gives a desired value of $U/m$.

Hints for chapter 15
Page 490, problem 28:
The choice of axis theorem only applies to a closed system, or to a system acted on by a total force of zero. Even if the box is not going to rotate, its center of mass is going to accelerate, and this can still cause a change in its angular momentum, unless the right axis is chosen. For example, if the axis is chosen at the bottom right corner, then the box will start accumulating clockwise angular momentum, even if it is just accelerating to the right without rotating. Only by choosing the axis at the center of mass (or at some other point on the same horizontal line) do we get a constant, zero angular momentum.

Page 492, problem 41:
You’ll need the result of problem 26 in order to relate the energy and angular momentum of a rigidly rotating body. Since this relationship involves a variable raised to a power, you can’t just graph the data and get the moment of inertia directly. One way to get around this is to manipulate one of the variables to make the graph linear. Here is an example of this technique from another context. Suppose you were given a table of the masses, $m$, of cubical pieces of