by that definition the wall’s force on the pole is applied at \( r = 0 \) and thus makes no torque on the pole. This is good, because we don’t know what the wall’s force on the pole is, and we are not trying to find it.

With this choice of axis, there are two nonzero torques on the pole, a counterclockwise torque from the cable and a clockwise torque from gravity. Choosing to represent counterclockwise torques as positive numbers, and using the equation \( |\tau| = r|F|\sin \theta \), we have

\[
r_{\text{cable}}|F_{\text{cable}}| \sin \theta_{\text{cable}} - r_{\text{grav}}|F_{\text{grav}}| \sin \theta_{\text{grav}} = 0.
\]

A little geometry gives \( \theta_{\text{cable}} = 90^\circ - \alpha \) and \( \theta_{\text{grav}} = \alpha \), so

\[
r_{\text{cable}}|F_{\text{cable}}| \sin(90^\circ - \alpha) - r_{\text{grav}}|F_{\text{grav}}| \sin \alpha = 0.
\]

The gravitational force can be considered as acting at the pole’s center of mass, i.e., at its geometrical center, so \( r_{\text{cable}} \) is twice \( r_{\text{grav}} \), and we can simplify the equation to read

\[
2|F_{\text{cable}}| \sin(90^\circ - \alpha) - |F_{\text{grav}}| \sin \alpha = 0.
\]

These are all quantities we were given, except for \( \alpha \), which is the angle we want to find. To solve for \( \alpha \) we need to use the trig identity \( \sin(90^\circ - x) = \cos x \),

\[
2|F_{\text{cable}}| \cos \alpha - |F_{\text{grav}}| \sin \alpha = 0,
\]

which allows us to find

\[
\tan \alpha = 2 \frac{|F_{\text{cable}}|}{|F_{\text{grav}}|} = \tan^{-1} \left( 2 \frac{|F_{\text{cable}}|}{|F_{\text{grav}}|} \right) = \tan^{-1} \left( 2 \times \frac{70 \text{ N}}{98 \text{ N}} \right) = 55^\circ.
\]
Stable and unstable equilibria

A pencil balanced upright on its tip could theoretically be in equilibrium, but even if it was initially perfectly balanced, it would topple in response to the first air current or vibration from a passing truck. The pencil can be put in equilibrium, but not in stable equilibrium. The things around us that we really do see staying still are all in stable equilibrium.

Why is one equilibrium stable and another unstable? Try pushing your own nose to the left or the right. If you push it a millimeter to the left, your head responds with a gentle force to the right, which keeps your nose from flying off of your face. If you push your nose a centimeter to the left, your face’s force on your nose becomes much stronger. The defining characteristic of a stable equilibrium is that the farther the object is moved away from equilibrium, the stronger the force is that tries to bring it back.

The opposite is true for an unstable equilibrium. In the top figure, the ball resting on the round hill theoretically has zero total force on it when it is exactly at the top. But in reality the total force will not be exactly zero, and the ball will begin to move off to one side. Once it has moved, the net force on the ball is greater than it was, and it accelerates more rapidly. In an unstable equilibrium, the farther the object gets from equilibrium, the stronger the force that pushes it farther from equilibrium.

This idea can be rephrased in terms of energy. The difference between the stable and unstable equilibria shown in figure y is that in the stable equilibrium, the potential energy is at a minimum, and moving to either side of equilibrium will increase it, whereas the unstable equilibrium represents a maximum.

Note that we are using the term “stable” in a weaker sense than in ordinary speech. A domino standing upright is stable in the sense we are using, since it will not spontaneously fall over in response to a sneeze from across the room or the vibration from a passing truck. We would only call it unstable in the technical sense if it could be toppled by any force, no matter how small. In everyday usage, of course, it would be considered unstable, since the force required to topple it is so small.

An application of calculus example 11

Nancy Neutron is living in a uranium nucleus that is undergoing fission. Nancy’s potential energy as a function of position can be approximated by \( PE = x^4 - x^2 \), where all the units and numerical constants have been suppressed for simplicity. Use calculus to locate the equilibrium points, and determine whether they are stable or unstable.

The equilibrium points occur where the PE is at a minimum or maximum, and minima and maxima occur where the derivative
(which equals minus the force on Nancy) is zero. This derivative is 
\[ \frac{dP_E}{dx} = 4x^3 - 2x \], and setting it equal to zero, we have 
\[ x = 0, \pm 1/\sqrt{2} \]. Minima occur where the second derivative is 
positive, and maxima where it is negative. The second derivative 
is \[ 12x^2 - 2 \], which is negative at \( x = 0 \) (unstable) and positive at 
\( x = \pm 1/\sqrt{2} \) (stable). Interpretation: the graph of the PE is shaped 
like a rounded letter ‘W,’ with the two troughs representing the two 
halves of the splitting nucleus. Nancy is going to have to decide 
which half she wants to go with.

### 15.6 Simple Machines: the lever

Although we have discussed some simple machines such as the pulley, without the concept of torque we were not yet ready to address the lever, which is the machine nature used in designing living things, almost to the exclusion of all others. (We can speculate what life on our planet might have been like if living things had evolved wheels, gears, pulleys, and screws.) The figures show two examples of levers within your arm. Different muscles are used to flex and extend the arm, because muscles work only by contraction.

Analyzing example ab physically, there are two forces that do work. When we lift a load with our biceps muscle, the muscle does positive work, because it brings the bone in the forearm in the direction it is moving. The load’s force on the arm does negative work, because the arm moves in the direction opposite to the load’s force. This makes sense, because we expect our arm to do positive work on the load, so the load must do an equal amount of negative work on the arm. (If the biceps was lowering a load, the signs of the works would be reversed. Any muscle is capable of doing either positive or negative work.)

There is also a third force on the forearm: the force of the upper arm’s bone exerted on the forearm at the elbow joint (not shown with an arrow in the figure). This force does no work, because the elbow joint is not moving.

Because the elbow joint is motionless, it is natural to define our torques using the joint as the axis. The situation now becomes quite simple, because the upper arm bone’s force exerted at the elbow neither does work nor creates a torque. We can ignore it completely. In any lever there is such a point, called the fulcrum.

If we restrict ourselves to the case in which the forearm rotates with constant angular momentum, then we know that the total torque on the forearm is zero,

\[ \tau_{\text{muscle}} + \tau_{\text{load}} = 0. \]

If we choose to represent counterclockwise torques as positive, then the muscle’s torque is positive, and the load’s is negative. In terms
of their absolute values,

$$|\tau_{\text{muscle}}| = |\tau_{\text{load}}|.$$ 

Assuming for simplicity that both forces act at angles of 90° with respect to the lines connecting the axis to the points at which they act, the absolute values of the torques are

$$r_{\text{muscle}}F_{\text{muscle}} = r_{\text{load}}F_{\text{arm}},$$

where \(r_{\text{muscle}}\), the distance from the elbow joint to the biceps’ point of insertion on the forearm, is only a few cm, while \(r_{\text{load}}\) might be 30 cm or so. The force exerted by the muscle must therefore be about ten times the force exerted by the load. We thus see that this lever is a force reducer. In general, a lever may be used either to increase or to reduce a force.

Why did our arms evolve so as to reduce force? In general, your body is built for compactness and maximum speed of motion rather than maximum force. This is the main anatomical difference between us and the Neanderthals (their brains covered the same range of sizes as those of modern humans), and it seems to have worked for us.

As with all machines, the lever is incapable of changing the amount of mechanical work we can do. A lever that increases force will always reduce motion, and vice versa, leaving the amount of work unchanged.

It is worth noting how simple and yet how powerful this analysis was. It was simple because we were well prepared with the concepts of torque and mechanical work. In anatomy textbooks, whose readers are assumed not to know physics, there is usually a long and complicated discussion of the different types of levers. For example, the biceps lever, ab, would be classified as a class III lever, since it has the fulcrum and load on the ends and the muscle’s force acting in the middle. The triceps, ac, is called a class I lever, because the load and muscle’s force are on the ends and the fulcrum is in the middle. How tiresome! With a firm grasp of the concept of torque, we realize that all such examples can be analyzed in much the same way. Physics is at its best when it lets us understand many apparently complicated phenomena in terms of a few simple yet powerful concepts.
15.7 Rigid-body rotation

Kinematics

When a rigid object rotates, every part of it (every atom) moves in a circle, covering the same angle in the same amount of time, ad. Every atom has a different velocity vector, ae. Since all the velocities are different, we can’t measure the speed of rotation of the top by giving a single velocity. We can, however, specify its speed of rotation consistently in terms of angle per unit time. Let the position of some reference point on the top be denoted by its angle \( \theta \), measured in a circle around the axis. For reasons that will become more apparent shortly, we measure all our angles in radians. Then the change in the angular position of any point on the top can be written as \( d\theta \), and all parts of the top have the same value of \( d\theta \) over a certain time interval \( dt \). We define the angular velocity, \( \omega \) (Greek omega),

\[
\omega = \frac{d\theta}{dt}.
\]

[definition of angular velocity; \( \theta \) in units of radians]

The relationship between \( \omega \) and \( t \) is exactly analogous to that between \( x \) and \( t \) for the motion of a particle through space.

**self-check B**

If two different people chose two different reference points on the top in order to define \( \theta = 0 \), how would their \( \theta \)-\( t \) graphs differ? What effect would this have on the angular velocities? \( \triangleright \) Answer, p. 570

The angular velocity has units of radians per second, rad/s. However, radians are not really units at all. The radian measure of an angle is defined, as the length of the circular arc it makes, divided by the radius of the circle. Dividing one length by another gives a unitless quantity, so anything with units of radians is really unitless. We can therefore simplify the units of angular velocity, and call them inverse seconds, s\(^{-1}\).

\( \triangleright \) A 78-rpm record example 12

If we measure angles in units of revolutions and time in units of minutes, then 78 rpm is the angular velocity of such a disk?

\( \triangleright \) If we measure angles in units of revolutions and time in units of minutes, then 78 rpm is the angular velocity. Using standard physics units of radians/second, however, we have

\[
\frac{78 \text{ revolutions}}{1 \text{ minute}} \times \frac{2\pi \text{ radians}}{1 \text{ revolution}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} = 8.2 \text{ s}^{-1}.
\]
In the absence of any torque, a rigid body will rotate indefinitely with the same angular velocity. If the angular velocity is changing because of a torque, we define an angular acceleration,

\[ \alpha = \frac{d\omega}{dt}, \]

[definition of angular acceleration]

The symbol is the Greek letter alpha. The units of this quantity are \( \text{rad/s}^2 \) or simply \( \text{s}^{-2} \).

The mathematical relationship between \( \omega \) and \( \theta \) is the same as the one between \( v \) and \( x \), and similarly for \( \alpha \) and \( a \). We can thus make a system of analogies, af, and recycle all the familiar kinematic equations for constant-acceleration motion.

\[ \text{The synodic period} \]

Mars takes nearly twice as long as the Earth to complete an orbit. If the two planets are alongside one another on a certain day, then one year later, Earth will be back at the same place, but Mars will have moved on, and it will take more time for Earth to finish catching up. Angular velocities add and subtract, just as velocity vectors do. If the two planets’ angular velocities are \( \omega_1 \) and \( \omega_2 \), then the angular velocity of one relative to the other is \( \omega_1 - \omega_2 \). The corresponding period, \( 1/(1/T_1 - 1/T_2) \), is known as the synodic period.

\[ \text{A neutron star} \]

A neutron star is initially observed to be rotating with an angular velocity of \( 2.0 \text{ s}^{-1} \), determined via the radio pulses it emits. If its angular acceleration is a constant \( -1.0 \times 10^{-8} \text{ s}^{-2} \), how many rotations will it complete before it stops? (In reality, the angular acceleration is not always constant; sudden changes often occur, and are referred to as “starquakes!”)

\[ \text{The equation} \quad v_f^2 - v_i^2 = 2a\Delta x \quad \text{can be translated into} \quad \omega_f^2 - \omega_i^2 = 2\alpha\Delta \theta, \]

\[ \text{giving} \]

\[ \Delta \theta = \frac{(\omega_f^2 - \omega_i^2)}{2\alpha} \]
\[ = 2.0 \times 10^8 \text{ radians} \]
\[ = 3.2 \times 10^7 \text{ rotations}. \]

\[ \text{Relations between angular quantities and motion of a point} \]

It is often necessary to be able to relate the angular quantities to the motion of a particular point on the rotating object. As we develop these, we will encounter the first example where the advantages of radians over degrees become apparent.

The speed at which a point on the object moves depends on both the object’s angular velocity \( \omega \) and the point’s distance \( r \) from the axis. We adopt a coordinate system, ag, with an inward (radial)
axis and a tangential axis. The length of the infinitesimal circular arc $ds$ traveled by the point in a time interval $dt$ is related to $d\theta$ by the definition of radian measure, $d\theta = ds/r$, where positive and negative values of $ds$ represent the two possible directions of motion along the tangential axis. We then have $v_t = ds/dt = r \cdot d\theta/ dt = \omega r$, or

$$v_t = \omega r.$$  

[tangential velocity of a point at a
distance $r$ from the axis of rotation]

The radial component is zero, since the point is not moving inward or outward,

$$v_r = 0.$$  

[radial velocity of a point at a
distance $r$ from the axis of rotation]

Note that we had to use the definition of radian measure in this derivation. Suppose instead we had used units of degrees for our angles and degrees per second for angular velocities. The relationship between $d\theta_{\text{degrees}}$ and $ds$ is $d\theta_{\text{degrees}} = (360/2\pi)s/r$, where the extra conversion factor of $(360/2\pi)$ comes from the fact that there are 360 degrees in a full circle, which is equivalent to $2\pi$ radians. The equation for $v_t$ would then have been $v_t = (2\pi/360)(\omega_{\text{degrees per second}})(r)$, which would have been much messier. Simplicity, then, is the reason for using radians rather than degrees; by using radians we avoid infecting all our equations with annoying conversion factors.

Since the velocity of a point on the object is directly proportional to the angular velocity, you might expect that its acceleration would be directly proportional to the angular acceleration. This is not true, however. Even if the angular acceleration is zero, i.e., if the object is rotating at constant angular velocity, every point on it will have an acceleration vector directed toward the axis, $ah$. As derived on page 264, the magnitude of this acceleration is

$$a_r = \omega^2 r.$$  

[radial acceleration of a point
at a distance $r$ from the axis]

For the tangential component, any change in the angular velocity $d\omega$ will lead to a change $d\omega \cdot r$ in the tangential velocity, so it is easily shown that

$$a_t = \alpha r.$$  

[tangential acceleration of a point
at a distance $r$ from the axis]

**self-check C**

Positive and negative signs of $\omega$ represent rotation in opposite directions. Why does it therefore make sense physically that $\omega$ is raised to the first power in the equation for $v_t$ and to the second power in the one for $a_r$?  

▷ Answer, p. 570
What is your radial acceleration due to the rotation of the Earth if you are at the equator?

At the equator, your distance from the Earth’s rotation axis is the same as the radius of the spherical Earth, $6.4 \times 10^6$ m. Your angular velocity is

$$\omega = \frac{2\pi \text{ radians}}{1 \text{ day}} = 7.3 \times 10^{-5} \text{ s}^{-1},$$

which gives an acceleration of

$$a_r = \omega^2 r = 0.034 \text{ m/s}^2.$$
Analogies between rotational and linear quantities.

Example 16

\[ I = \sum m_i r_i^2, \]  
[definition of the moment of inertia;  
for rigid-body rotation in a plane; \( r \) is the distance  
from the axis, measured perpendicular to the axis]

The angular momentum of a rigidly rotating body is then

\[ L = I \omega. \]  
[angular momentum of  
rigid-body rotation in a plane]

Since torque is defined as \( dL / dt \), and a rigid body has a constant  
moment of inertia, we have \( \tau = dL / dt = I d\omega / dt = I \alpha, \)

\[ \tau = I \alpha, \]  
[relationship between torque and  
angular acceleration for rigid-body rotation in a plane]

which is analogous to \( F = ma. \)

The complete system of analogies between linear motion and  
rigid-body rotation is given in figure ai.

\[ ^A \text{barbell} \quad \text{example 16} \]

\[ x \leftrightarrow \theta \]
\[ v \leftrightarrow \omega \]
\[ a \leftrightarrow \alpha \]
\[ m \leftrightarrow I \]
\[ p \leftrightarrow L \]
\[ F \leftrightarrow \tau \]

ai / Analogies between rotational and linear quantities.

\[ \begin{align*}
\text{Example 16} & \\
\text{The barbell shown in figure aj consists of two small, dense, massive balls at the ends of a very light rod. The balls have masses of 2.0 kg and 1.0 kg, and the length of the rod is 3.0 m. Find the moment of inertia of the rod (1) for rotation about its center of mass, and (2) for rotation about the center of the more massive ball.} \\
\text{(1) The ball's center of mass lies 1/3 of the way from the greater mass to the lesser mass, i.e., 1.0 m from one and 2.0 m from the other. Since the balls are small, we approximate them as if they were two pointlike particles. The moment of inertia is} \\
I &= (2.0 \text{ kg})(1.0 \text{ m})^2 + (1.0 \text{ kg})(2.0 \text{ m})^2 \\
&= 2.0 \text{ kg} \cdot \text{m}^2 + 4.0 \text{ kg} \cdot \text{m}^2 \\
&= 6.0 \text{ kg} \cdot \text{m}^2 \\
\text{Perhaps counterintuitively, the less massive ball contributes far more to the moment of inertia.} \\
\text{(2) The big ball theoretically contributes a little bit to the moment of inertia, since essentially none of its atoms are exactly at \( r=0 \). However, since the balls are said to be small and dense, we assume all the big ball's atoms are so close to the axis that we can ignore their small contributions to the total moment of inertia:} \\
I &= (1.0 \text{ kg})(3.0 \text{ m})^2 \\
&= 9.0 \text{ kg} \cdot \text{m}^2 \\
\text{This example shows that the moment of inertia depends on the choice of axis. For example, it is easier to wiggle a pen about its center than about one end.} 
\end{align*} \]
The parallel axis theorem example 17

Generalizing the previous example, suppose we pick any axis parallel to axis 1, but offset from it by a distance \( h \). Part (2) of the previous example then corresponds to the special case of \( h = -1.0 \text{ m} \) (negative being to the left). What is the moment of inertia about this new axis?

The big ball’s distance from the new axis is \((1.0 \text{ m}) + h\), and the small one’s is \((2.0 \text{ m}) - h\). The new moment of inertia is

\[
I = (2.0 \text{ kg})[(1.0 \text{ m}) + h]^2 + (1.0 \text{ kg})[2.0 \text{ m}]^2 - h^2
\]

\[
= 6.0 \text{ kg} \cdot \text{m}^2 + (4.0 \text{ kg} \cdot \text{m})h - (4.0 \text{ kg} \cdot \text{m})h + (3.0 \text{ kg})h^2.
\]

The constant term is the same as the moment of inertia about the center-of-mass axis, the first-order terms cancel out, and the third term is just the total mass multiplied by \( h^2 \). The interested reader will have no difficulty in generalizing this to any set of particles (problem 27, p. 490), resulting in the parallel axis theorem: If an object of total mass \( M \) rotates about a line at a distance \( h \) from its center of mass, then its moment of inertia equals \( I_{cm} + Mh^2 \), where \( I_{cm} \) is the moment of inertia for rotation about a parallel line through the center of mass.

Scaling of the moment of inertia example 18

(1) Suppose two objects have the same mass and the same shape, but one is less dense, and larger by a factor \( k \). How do their moments of inertia compare?
(2) What if the densities are equal rather than the masses?

(1) This is like increasing all the distances between atoms by a factor \( k \). All the \( r \)'s become greater by this factor, so the moment of inertia is increased by a factor of \( k^2 \).
(2) This introduces an increase in mass by a factor of \( k^3 \), so the moment of inertia of the bigger object is greater by a factor of \( k^5 \).

Iterated integrals

In various places in this book, starting with subsection 15.7.5, we’ll come across integrals stuck inside other integrals. These are known as iterated integrals, or double integrals, triple integrals, etc. Similar concepts crop up all the time even when you’re not doing calculus, so let’s start by imagining such an example. Suppose you want to count how many squares there are on a chess board, and you don’t know how to multiply eight times eight. You could start from the upper left, count eight squares across, then continue with the second row, and so on, until you how counted every square, giving the result of 64. In slightly more formal mathematical language, we could write the following recipe: for each row, \( r \), from 1 to 8, consider the columns, \( c \), from 1 to 8, and add one to the count for
each one of them. Using the sigma notation, this becomes

\[ \sum_{r=1}^{8} \sum_{c=1}^{8} 1. \]

If you’re familiar with computer programming, then you can think of this as a sum that could be calculated using a loop nested inside another loop. To evaluate the result (again, assuming we don’t know how to multiply, so we have to use brute force), we can first evaluate the inside sum, which equals 8, giving

\[ \sum_{r=1}^{8} 8. \]

Notice how the “dummy” variable \( c \) has disappeared. Finally we do the outside sum, over \( r \), and find the result of 64.

Now imagine doing the same thing with the pixels on a TV screen. The electron beam sweeps across the screen, painting the pixels in each row, one at a time. This is really no different than the example of the chess board, but because the pixels are so small, you normally think of the image on a TV screen as continuous rather than discrete. This is the idea of an integral in calculus. Suppose we want to find the area of a rectangle of width \( a \) and height \( b \), and we don’t know that we can just multiply to get the area \( ab \). The brute force way to do this is to break up the rectangle into a grid of infinitesimally small squares, each having width \( dx \) and height \( dy \), and therefore the infinitesimal area \( dA = dx \, dy \). For convenience, we’ll imagine that the rectangle’s lower left corner is at the origin. Then the area is given by this integral:

\[
\begin{align*}
\text{area} &= \int_{y=0}^{b} \int_{x=0}^{a} dA \\
&= \int_{y=0}^{b} \int_{x=0}^{a} dx \, dy
\end{align*}
\]

Notice how the leftmost integral sign, over \( y \), and the rightmost differential, \( dy \), act like bookends, or the pieces of bread on a sandwich. Inside them, we have the integral sign that runs over \( x \), and the differential \( dx \) that matches it on the right. Finally, on the innermost layer, we’d normally have the thing we’re integrating, but here’s it’s 1, so I’ve omitted it. Writing the lower limits of the integrals with \( x = a \) and \( y = 0 \) helps to keep it straight which integral goes with which.
The result is

\[
\text{area} = \int_{y=0}^{b} \int_{x=0}^{a} \text{d}A = \int_{y=0}^{b} \int_{x=0}^{a} \text{d}x \text{ d}y = \int_{y=0}^{b} \left( \int_{x=0}^{a} \text{d}x \right) \text{d}y = \int_{y=0}^{b} a \text{d}y = a \int_{y=0}^{b} \text{d}y = ab.
\]

**Area of a triangle**

Find the area of a 45-45-90 right triangle having legs \(a\).

Let the triangle’s hypotenuse run from the origin to the point \((a, a)\), and let its legs run from the origin to \((0, a)\), and then to \((a, a)\). In other words, the triangle sits on top of its hypotenuse.

Then the integral can be set up the same way as the one before, but for a particular value of \(y\), values of \(x\) only run from 0 (on the \(y\) axis) to \(y\) (on the hypotenuse). We then have

\[
\text{area} = \int_{y=0}^{a} \int_{x=0}^{y} \text{d}A = \int_{y=0}^{a} \int_{x=0}^{y} \text{d}x \text{ d}y = \int_{y=0}^{a} \left( \int_{x=0}^{y} \text{d}x \right) \text{d}y = \int_{y=0}^{a} y \text{d}y = \frac{1}{2} a^2
\]

Note that in this example, because the upper end of the \(x\) values depends on the value of \(y\), it makes a difference which order we do the integrals in. The \(x\) integral has to be on the inside, and we have to do it first.

**Volume of a cube**

Find the volume of a cube with sides of length \(a\).

This is a three-dimensional example, so we’ll have integrals nested three deep, and the thing we’re integrating is the volume \(\text{d}V = \text{d}x \text{ d}y \text{ d}z\).
volume = \int_a^z \int_a^y \int_a^x \, dx \, dy \, dz
= \int_a^z \int_a^y \, ady \, dz
= a \int_a^z \int_a^y \, dy \, dz
= a \int_a^z \, adz
= a^3

\textit{Area of a circle} example 21

Find the area of a circle.

To make it easy, let’s find the area of a semicircle and then double it. Let the circle’s radius be \( r \), and let it be centered on the origin and bounded below by the \( x \) axis. Then the curved edge is given by the equation \( r^2 = x^2 + y^2 \), or \( y = \sqrt{r^2 - x^2} \). Since the \( y \) integral’s limit depends on \( x \), the \( x \) integral has to be on the outside. The area is

\[
\text{area} = \int_{x=-r}^r \int_{y=0}^{\sqrt{r^2-x^2}} \, dy \, dx
= \int_{x=-r}^r \sqrt{r^2 - x^2} \, dx
= r \int_{x=-r}^r \sqrt{1 - (x/r)^2} \, dx.
\]

Substituting \( u = x/r \),

\[
\text{area} = r^2 \int_{u=-1}^1 \sqrt{1 - u^2} \, du
\]

The definite integral equals \( \pi \), as you can find using a trig substitution or simply by looking it up in a table, and the result is, as expected, \( \pi r^2 / 2 \) for the area of the semicircle.

\textbf{Finding moments of inertia by integration}

When calculating the moment of inertia of an ordinary-sized object with perhaps \( 10^{26} \) atoms, it would be impossible to do an actual sum over atoms, even with the world’s fastest supercomputer. Calculus, however, offers a tool, the integral, for breaking a sum down to infinitely many small parts. If we don’t worry about the existence of atoms, then we can use an integral to compute a moment
of inertia as if the object was smooth and continuous throughout, rather than granular at the atomic level. Of course this granularity typically has a negligible effect on the result unless the object is itself an individual molecule. This subsection consists of three examples of how to do such a computation, at three distinct levels of mathematical complication.

**Moment of inertia of a thin rod**

What is the moment of inertia of a thin rod of mass $M$ and length $L$ about a line perpendicular to the rod and passing through its center? We generalize the discrete sum

$$I = \sum m_i r_i^2$$

to a continuous one,

$$I = \int r^2 \, dm,$$

$$= \int_{-L/2}^{L/2} x^2 \frac{M}{L} \, dx \quad [r = |x|, \text{ so } r^2 = x^2]$$

$$= \frac{1}{12} ML^2$$

In this example the object was one-dimensional, which made the math simple. The next example shows a strategy that can be used to simplify the math for objects that are three-dimensional, but possess some kind of symmetry.

**Moment of inertia of a disk**

What is the moment of inertia of a disk of radius $b$, thickness $t$, and mass $M$, for rotation about its central axis?

We break the disk down into concentric circular rings of thickness $d\, r$. Since all the mass in a given circular slice has essentially the same value of $r$ (ranging only from $r$ to $r + dr$), the slice’s contribution to the total moment of inertia is simply $r^2 \, dm$. We then have

$$I = \int r^2 \, dm$$

$$= \int r^2 \rho \, dV,$$

where $V = \pi b^2 t$ is the total volume, $\rho = M/V = M/\pi b^2 t$ is the density, and the volume of one slice can be calculated as the volume enclosed by its outer surface minus the volume enclosed by its inner surface, $dV = \pi (r + dr)^2 t - \pi r^2 t = 2\pi tr \, dr$.

$$I = \int_0^b r^2 \frac{M}{\pi b^2 t} 2\pi t r \, dr$$

$$= \frac{1}{2} Mb^2.$$
In the most general case where there is no symmetry about the rotation axis, we must use iterated integrals, as discussed in subsection 15.7.4. The example of the disk possessed two types of symmetry with respect to the rotation axis: (1) the disk is the same when rotated through any angle about the axis, and (2) all slices perpendicular to the axis are the same. These two symmetries reduced the number of layers of integrals from three to one. The following example possesses only one symmetry, of type (2), and we simply set it up as a triple integral. You may not have seen multiple integrals yet in a math course. If so, just skim this example.

**Moment of inertia of a cube**

What is the moment of inertia of a cube of side $b$, for rotation about an axis that passes through its center and is parallel to four of its faces? Let the origin be at the center of the cube, and let $x$ be the rotation axis.

\[
I = \int r^2 \, dm
\]
\[
= \rho \int r^2 \, dV
\]
\[
= \rho \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} (y^2 + z^2) \, dx \, dy \, dz
\]
\[
= \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} (y^2 + z^2) \, dy \, dz
\]

The fact that the last step is a trivial integral results from the symmetry of the problem. The integrand of the remaining double integral breaks down into two terms, each of which depends on only one of the variables, so we break it into two integrals,

\[
I = \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} y^2 \, dy \, dz + \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} z^2 \, dy \, dz
\]

which we know have identical results. We therefore only need to evaluate one of them and double the result:

\[
I = 2 \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} z^2 \, dy \, dz
\]
\[
= 2 \rho b^2 \int_{-b/2}^{b/2} z^2 \, dz
\]
\[
= \frac{1}{6} \rho b^5
\]
\[
= \frac{1}{6} Mb^2
\]

Figure ak shows the moments of inertia of some shapes, which were evaluated with techniques like these.
Moments of inertia of some geometric shapes.

In the men's Olympic hammer throw, a steel ball of radius 6.1 cm is swung on the end of a wire of length 1.22 m. What fraction of the ball's angular momentum comes from its rotation, as opposed to its motion through space?

It's always important to solve problems symbolically first, and plug in numbers only at the end, so let the radius of the ball be \( b \), and the length of the wire \( \ell \). If the time the ball takes to go once around the circle is \( T \), then this is also the time it takes to revolve once around its own axis. Its speed is \( v = \frac{2\pi\ell}{T} \), so its angular momentum due to its motion through space is \( mv\ell = \frac{2\pi m\ell^2}{T} \). Its angular momentum due to its rotation around its own center is \( \left(\frac{4\pi}{5}\right)mb^2/T \). The ratio of these two angular momenta is \( \left(\frac{2}{5}\right)(b/\ell)^2 = 1.0 \times 10^{-3} \). The angular momentum due to the ball's spin is extremely small.

A rod of length \( b \) and mass \( m \) stands upright. We want to strike the rod at the bottom, causing it to fall and land flat. Find the momentum, \( p \), that should be delivered, in terms of \( m \), \( b \), and \( g \). Can this really be done without having the rod scrape on the floor?

This is a nice example of a question that can very nearly be answered based only on units. Since the three variables, \( m \), \( b \), and \( g \), all have different units, they can't be added or subtracted. The only way to combine them mathematically is by multiplication or division. Multiplying one of them by itself is exponentiation, so in general we expect that the answer must be of the form

\[
p = An^j b^k g^l,
\]

where \( A \), \( j \), \( k \), and \( l \) are unitless constants. The result has to have units of kg·m/s. To get kilograms to the first power, we need

\[
j = 1,
\]
meters to the first power requires

\[ k + l = 1, \]

and seconds to the power \(-1\) implies

\[ l = 1/2. \]

We find \( j = 1, k = 1/2, \) and \( l = 1/2, \) so the solution must be of the form

\[ p = Am\sqrt{bg}. \]

Note that no physics was required!

Consideration of units, however, won’t help us to find the unitless constant \( A. \) Let \( t \) be the time the rod takes to fall, so that \((1/2)gt^2 = b/2.\) If the rod is going to land exactly on its side, then the number of revolutions it completes while in the air must be \(1/4,\) or \(3/4,\) or \(5/4,\) \ldots, but all the possibilities greater than \(1/4\) would cause the head of the rod to collide with the floor prematurely. The rod must therefore rotate at a rate that would cause it to complete a full rotation in a time \(T = 4t,\) and it has angular momentum

\[ L = \frac{\pi}{6}mb^2/T. \]

The momentum lost by the object striking the rod is \( p,\) and by conservation of momentum, this is the amount of momentum, in the horizontal direction, that the rod acquires. In other words, the rod will fly forward a little. However, this has no effect on the solution to the problem. More importantly, the object striking the rod loses angular momentum \( bp/2,\) which is also transferred to the rod. Equating this to the expression above for \( L,\) we find

\[ p = \frac{\pi}{12}m\sqrt{bg}. \]

Finally, we need to know whether this can really be done without having the foot of the rod scrape on the floor. The figure shows that the answer is no for this rod of finite width, but it appears that the answer would be yes for a sufficiently thin rod. This is analyzed further in homework problem 46 on page 493.
15.8 Angular momentum in three dimensions

Conservation of angular momentum produces some surprising phenomena when extended to three dimensions. Try the following experiment, for example. Take off your shoe, and toss it in to the air, making it spin along its long (toe-to-heel) axis. You should observe a nice steady pattern of rotation. The same happens when you spin the shoe about its shortest (top-to-bottom) axis. But something unexpected happens when you spin it about its third (left-to-right) axis, which is intermediate in length between the other two. Instead of a steady pattern of rotation, you will observe something more complicated, with the shoe changing its orientation with respect to the rotation axis.

Rigid-body kinematics in three dimensions

How do we generalize rigid-body kinematics to three dimensions? When we wanted to generalize the kinematics of a moving particle to three dimensions, we made the numbers $r$, $v$, and $a$ into vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$. This worked because these quantities all obeyed the same laws of vector addition. For instance, one of the laws of vector addition is that, just like addition of numbers, vector addition gives the same result regardless of the order of the two quantities being added. Thus you can step sideways 1 m to the right and then step forward 1 m, and the end result is the same as if you stepped forward first and then to the side. In other words, it didn’t matter whether you took $\Delta \mathbf{r}_1 + \Delta \mathbf{r}_2$ or $\Delta \mathbf{r}_2 + \Delta \mathbf{r}_1$. In math this is called the commutative property of addition.

Angular motion, unfortunately doesn’t have this property, as shown in figure am. Doing a rotation about the $x$ axis and then
about y gives one result, while doing them in the opposite order gives a different result. These operations don’t “commute,” i.e., it makes a difference what order you do them in.

This means that there is in general no possible way to construct a $\Delta \theta$ vector. However, if you try doing the operations shown in figure am using small rotation, say about 10 degrees instead of 90, you’ll find that the result is nearly the same regardless of what order you use; small rotations are very nearly commutative. Not only that, but the result of the two 10-degree rotations is about the same as a single, somewhat larger, rotation about an axis that lies symmetrically at between the x and y axes at 45 degree angles to each one. This is exactly what we would expect if the two small rotations did act like vectors whose directions were along the axis of rotation. We therefore define a $d\theta$ vector whose magnitude is the amount of rotation in units of radians, and whose direction is along the axis of rotation. Actually this definition is ambiguous, because there it could point in either direction along the axis. We therefore use a right-hand rule as shown in figure an to define the direction of the $d\theta$ vector, and the $\omega$ vector, $\omega = d\theta / dt$, based on it. Aliens on planet Tammyfaye may decide to define it using their left hands rather than their right, but as long as they keep their scientific literature separate from ours, there is no problem. When entering a physics exam, always be sure to write a large warning note on your left hand in magic marker so that you won’t be tempted to use it for the right-hand rule while keeping your pen in your right.

**self-check D**

Use the right-hand rule to determine the directions of the $\omega$ vectors in each rotation shown in figures am/1 through am/5. Answer, p. 571

Because the vector relationships among $d\theta$, $\omega$, and $\alpha$ are strictly analogous to the ones involving $dr$, $v$, and $a$ (with the proviso that we avoid describing large rotations using $\Delta \theta$ vectors), any operation in $r$-$v$-$a$ vector kinematics has an exact analog in $\theta$-$\omega$-$\alpha$ kinematics.

1. **Result of successive 10-degree rotations** example 24

   $\Rightarrow$ What is the result of two successive (positive) 10-degree rotations about the x and y axes? That is, what single rotation about a single axis would be equivalent to executing these in succession?

   $\Rightarrow$ The result is only going to be approximate, since 10 degrees is not an infinitesimally small angle, and we are not told in what order the rotations occur. To some approximation, however, we can add the $\Delta \theta$ vectors in exactly the same way we would add $\Delta r$ vectors, so we have

   $$
   \Delta \theta \approx \Delta \theta_1 + \Delta \theta_2 \\
   \approx (10 \text{ degrees})\hat{x} + (10 \text{ degrees})\hat{y}.
   $$

   This is a vector with a magnitude of $\sqrt{(10 \text{ deg})^2 + (10 \text{ deg})^2}$ =

Section 15.8  Angular momentum in three dimensions 469
14 deg, and it points along an axis midway between the $x$ and $y$ axes.

**Angular momentum in three dimensions**

*The vector cross product*

In order to expand our system of three-dimensional kinematics to include dynamics, we will have to generalize equations like $v_t = \omega r$, $\tau = rF \sin \theta_{DF}$, and $L = rp \sin \theta_{rp}$, each of which involves three quantities that we have either already defined as vectors or that we want to redefine as vectors. Although the first one appears to differ from the others in its form, it could just as well be rewritten as $v_t = \omega r \sin \theta_{\omega r}$, since $\theta_{\omega r} = 90^\circ$, and $\sin 90^\circ = 1$.

It thus appears that we have discovered something general about the physically useful way to relate three vectors in a multiplicative way: the magnitude of the result always seems to be proportional to the product of the magnitudes of the two vectors being “multiplied,” and also to the sine of the angle between them.

Is this pattern just an accident? Actually the sine factor has a very important physical property: it goes to zero when the two vectors are parallel. This is a Good Thing. The generalization of angular momentum into a three-dimensional vector, for example, is presumably going to describe not just the clockwise or counterclockwise nature of the motion but also from which direction we would have to view the motion so that it was clockwise or counterclockwise. (A clock’s hands go counterclockwise as seen from behind the clock, and don’t rotate at all as seen from above or to the side.) Now suppose a particle is moving directly away from the origin, so that its $r$ and $p$ vectors are parallel. It is not going around the origin from any point of view, so its angular momentum vector had better be zero.

Thinking in a slightly more abstract way, we would expect the angular momentum vector to point perpendicular to the plane of motion, just as the angular velocity vector points perpendicular to the plane of motion. The plane of motion is the plane containing both $r$ and $p$, if we place the two vectors tail-to-tail. But if $r$ and $p$ are parallel and are placed tail-to-tail, then there are infinitely many planes containing them both. To pick one of these planes in preference to the others would violate the symmetry of space, since they should all be equally good. Thus the zero-if-parallel property is a necessary consequence of the underlying symmetry of the laws of physics.

The following definition of a kind of vector multiplication is consistent with everything we’ve seen so far, and on p. 480 we’ll prove that the definition is unique, i.e., if we believe in the symmetry of space, it is essentially the only way of defining the multiplication of
two vectors to produce a third vector:

**Definition of the vector cross product:**

The cross product \( \mathbf{A} \times \mathbf{B} \) of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is defined as follows:

1. Its magnitude is defined by \( |\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin \theta_{AB} \), where \( \theta_{AB} \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \) when they are placed tail-to-tail.
2. Its direction is along the line perpendicular to both \( \mathbf{A} \) and \( \mathbf{B} \). Of the two such directions, it is the one that obeys the right-hand rule shown in figure ao.

The name “cross product” refers to the symbol, and distinguishes it from the dot product, which acts on two vectors but produces a scalar.

Although the vector cross-product has nearly all the properties of numerical multiplication, e.g., \( \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \), it lacks the usual property of commutativity. Try applying the right-hand rule to find the direction of the vector cross product \( \mathbf{B} \times \mathbf{A} \) using the two vectors shown in the figure. This requires starting with a flattened hand with the four fingers pointing along \( \mathbf{B} \), and then curling the hand so that the fingers point along \( \mathbf{A} \). The only possible way to do this is to point your thumb toward the floor, in the opposite direction. Thus for the vector cross product we have

\[
\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A},
\]

a property known as anticommutativity. The vector cross product is also not associative, i.e., \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \) is usually not the same as \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \).

A geometric interpretation of the cross product, ap, is that if both \( \mathbf{A} \) and \( \mathbf{B} \) are vectors with units of distance, then the magnitude of their cross product can be interpreted as the area of the parallelogram they form when placed tail-to-tail.

A useful expression for the components of the vector cross product in terms of the components of the two vectors being multiplied is as follows:

\[
\begin{align*}
(\mathbf{A} \times \mathbf{B})_x &= A_y B_z - B_y A_z \\
(\mathbf{A} \times \mathbf{B})_y &= A_z B_x - B_z A_x \\
(\mathbf{A} \times \mathbf{B})_z &= A_x B_y - B_x A_y
\end{align*}
\]

I’ll prove later that these expressions are equivalent to the previous definition of the cross product. Although they may appear formidable, they have a simple structure: the subscripts on the right are the other two besides the one on the left, and each equation is related to the preceding one by a cyclic change in the subscripts,
Angular momentum in three dimensions

In terms of the vector cross product, we have:

\[ \mathbf{v} = \omega \times \mathbf{r} \]
\[ \mathbf{L} = \mathbf{r} \times \mathbf{p} \]
\[ \mathbf{\tau} = \mathbf{r} \times \mathbf{F} \]

But wait, how do we know these equations are even correct? For instance, how do we know that the quantity defined by \( \mathbf{r} \times \mathbf{p} \) is in fact conserved? Well, just as we saw on page 378 that the dot product is unique (i.e., can only be defined in one way while observing rotational invariance), the cross product is also unique, as proved on page 480. If \( \mathbf{r} \times \mathbf{p} \) was not conserved, then there could not be any generally conserved quantity that would reduce to our old definition of angular momentum in the special case of plane rotation. This doesn’t prove conservation of angular momentum — only experiments can prove that — but it does prove that if angular momentum is conserved in three dimensions, there is only one possible way to generalize from two dimensions to three.

Angular momentum of a spinning top example 25

As an illustration, we consider the angular momentum of a spinning top. Figures ar and as show the use of the vector cross product to determine the contribution of a representative atom to the total angular momentum. Since every other atom’s angular momentum vector will be in the same direction, this will also be the direction of the total angular momentum of the top. This happens to be rigid-body rotation, and perhaps not surprisingly, the angular momentum vector is along the same direction as the angular velocity vector.

Three important points are illustrated by this example: (1) When we do the full three-dimensional treatment of angular momentum, the “axis” from which we measure the position vectors is just an arbitrarily chosen point. If this had not been rigid-body rotation, we would not even have been able to identify a single line about which every atom circled. (2) Starting from figure ar, we had to rearrange the vectors to get them tail-to-tail before applying the right-hand rule. If we had attempted to apply the right-hand rule to figure ar, the direction of the result would have been exactly the opposite of the correct answer. (3) The equation \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \) cannot
be applied all at once to an entire system of particles. The total momentum of the top is zero, which would give an erroneous result of zero angular momentum (never mind the fact that $r$ is not well defined for the top as a whole).

Doing the right-hand rule like this requires some practice. I urge you to make models like as out of rolled up pieces of paper and to practice with the model in various orientations until it becomes natural.

---

**Precession example 26**

Figure at shows a counterintuitive example of the concepts we’ve been discussing. One expects the torque due to gravity to cause the top to flop down. Instead, the top remains spinning in the horizontal plane, but its axis of rotation starts moving in the direction shown by the shaded arrow. This phenomenon is called precession. Figure au shows that the torque due to gravity is out of the page. (Actually we should add up all the torques on all the atoms in the top, but the qualitative result is the same.) Since torque is the rate of change of angular momentum, $\tau = dL/dt$, the $\Delta L$ vector must be in the same direction as the torque (division by a positive scalar doesn’t change the direction of the vector). As shown in av, this causes the angular momentum vector to twist in space without changing its magnitude.

For similar reasons, the Earth’s axis precesses once every 26,000 years (although not through a great circle, since the angle between the axis and the force isn’t 90 degrees as in figure at). This precession is due to a torque exerted by the moon. If the Earth was a perfect sphere, there could be no precession effect due to symmetry. However, the Earth’s own rotation causes it to be slightly flattened (oblate) relative to a perfect sphere, giving it “love handles” on which the moon’s gravity can act. The moon’s gravity on the nearer side of the equatorial bulge is stronger, so the torques do not cancel out perfectly. Presently the earth’s axis very nearly lines up with the star Polaris, but in 12,000 years, the pole star will be Vega instead.

---

**The frisbee example 27**

The flow of the air over a flying frisbee generates lift, and the lift at the front and back of the frisbee isn’t necessarily balanced. If you throw a frisbee without rotating it, as if you were shooting a basketball with two hands, you’ll find that it pitches, i.e., its nose goes either up or down. When I do this with my frisbee, it goes nose down, which apparently means that the lift at the back of the disc is greater than the lift at the front. The two torques are unbalanced, resulting in a total torque that points to the left.

The way you actually throw a frisbee is with one hand, putting a lot of spin on it. If you throw backhand, which is how most people first learn to do it, the angular momentum vector points down...
(assuming you’re right-handed). On my frisbee, the aerodynamic torque to the left would therefore tend to make the angular momentum vector precess in the clockwise direction as seen by the thrower. This would cause the disc to roll to the right, and therefore follow a curved trajectory. Some specialized discs, used in the sport of disc golf, are actually designed intentionally to show this behavior; they’re known as “understable” discs. However, the typical frisbee that most people play with is designed to be stable: as the disc rolls to one side, the airflow around it is altered in a way that tends to bring the disc back into level flight. Such a disc will therefore tend to fly in a straight line, provided that it is thrown with enough angular momentum.

Finding a cross product by components example 28

What is the torque produced by a force given by \( \hat{x} + 2\hat{y} + 3\hat{z} \) (in units of Newtons) acting on a point whose radius vector is \( 4\hat{x} + 5\hat{y} \) (in meters)?

It’s helpful to make a table of the components as shown in the figure. The results are

\[
\begin{align*}
\tau_x &= r_y F_z - F_y r_z = 15 \text{ N} \cdot \text{m} \\
\tau_y &= r_z F_x - F_z r_x = -12 \text{ N} \cdot \text{m} \\
\tau_z &= r_x F_y - F_x r_y = 3 \text{ N} \cdot \text{m}
\end{align*}
\]

Torque and angular momentum example 29

In this example, we prove explicitly the consistency of the equations involving torque and angular momentum that we proved above based purely on symmetry. Starting from the definition of torque, we have

\[
\tau = \frac{dL}{dt} = \frac{d}{dt}\sum_i r_i \times p_i
\]

The derivative of a cross product can be evaluated in the same way as the derivative of an ordinary scalar product:

\[
\tau = \sum_i \left[ \left( \frac{dr_i}{dt} \times p_i \right) + \left( r_i \times \frac{dp_i}{dt} \right) \right]
\]

The first term is zero for each particle, since the velocity vector is parallel to the momentum vector. The derivative appearing in the second term is the force acting on the particle, so

\[
\tau = \sum_i r_i \times F_i,
\]

which is the relationship we set out to prove.
Rigid-body dynamics in three dimensions

The student who is not madly in love with mathematics may wish to skip the rest of this section after absorbing the statement that, for a typical, asymmetric object, the angular momentum vector and the angular velocity vector need not be parallel. That is, only for a body that possesses symmetry about the rotation axis is it true that $\mathbf{L} = I \mathbf{\omega}$ (the rotational equivalent of $\mathbf{p} = m \mathbf{v}$) for some scalar $I$.

Let’s evaluate the angular momentum of a rigidly rotating system of particles:

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$
$$= \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i$$
$$= \sum_i m_i \mathbf{r}_i \times (\mathbf{\omega} \times \mathbf{r}_i)$$

An important mathematical skill is to know when to give up and back off. This is a complicated expression, and there is no reason to expect it to simplify and, for example, take the form of a scalar multiplied by $\mathbf{\omega}$. Instead we examine its general characteristics. If we expanded it using the equation that gives the components of a vector cross product, every term would have one of the $\mathbf{\omega}$ components raised to the first power, multiplied by a bunch of other stuff. The most general possible form for the result is

$$L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$
$$L_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z$$
$$L_z = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z,$$

which you may recognize as a case of matrix multiplication. The moment of inertia is not a scalar, and not a three-component vector. It is a matrix specified by nine numbers, called its matrix elements.

The elements of the moment of inertia matrix will depend on our choice of a coordinate system. In general, there will be some special coordinate system, in which the matrix has a simple diagonal form:

$$L_x = I_{xx} \omega_x$$
$$L_y = I_{yy} \omega_y$$
$$L_z = I_{zz} \omega_z.$$

The three special axes that cause this simplification are called the principal axes of the object, and the corresponding coordinate system is the principal axis system. For symmetric shapes such as a rectangular box or an ellipsoid, the principal axes lie along the intersections of the three symmetry planes, but even an asymmetric body has principal axes.

Section 15.8  Angular momentum in three dimensions  475
We can also generalize the plane-rotation equation \( K = (1/2)I\omega^2 \) to three dimensions as follows:

\[
K = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i (\omega \times r_i) \cdot (\omega \times r_i)
\]

We want an equation involving the moment of inertia, and this has some evident similarities to the sum we originally wrote down for the moment of inertia. To massage it into the right shape, we need the vector identity \((A \times B) \cdot C = (B \times C) \cdot A\), which we state without proof. We then write

\[
K = \frac{1}{2} \sum_i m_i [r_i \times (\omega \times r_i)] \cdot \omega
= \frac{1}{2} \omega \cdot \sum_i m_i r_i \times (\omega \times r_i)
= \frac{1}{2} L \cdot \omega
\]

As a reward for all this hard work, let’s analyze the problem of the spinning shoe that I posed at the beginning of the chapter. The three rotation axes referred to there are approximately the principal axes of the shoe. While the shoe is in the air, no external torques are acting on it, so its angular momentum vector must be constant in magnitude and direction. Its kinetic energy is also constant. That’s in the room’s frame of reference, however. The principal axis frame is attached to the shoe, and tumbles madly along with it. In the principal axis frame, the kinetic energy and the magnitude of the angular momentum stay constant, but the actual direction of the angular momentum need not stay fixed (as you saw in the case of rotation that was initially about the intermediate-length axis).

Constant \(|L|\) gives

\[
L_x^2 + L_y^2 + L_z^2 = \text{constant}.
\]

In the principal axis frame, it’s easy to solve for the components of \( \omega \) in terms of the components of \( L \), so we eliminate \( \omega \) from the expression \( 2K = L \cdot \omega \), giving

\[
\frac{1}{I_{xx}}L_x^2 + \frac{1}{I_{yy}}L_y^2 + \frac{1}{I_{zz}}L_z^2 = \text{constant} \ #2.
\]

The first equation is the equation of a sphere in the three dimensional space occupied by the angular momentum vector, while the second one is the equation of an ellipsoid. The top figure corresponds to the case of rotation about the shortest axis, which has the greatest moment of inertia element. The intersection of the two
surfaces consists only of the two points at the front and back of the sphere. The angular momentum is confined to one of these points, and can’t change its direction, i.e., its orientation with respect to the principal axis system, which is another way of saying that the shoe can’t change its orientation with respect to the angular momentum vector. In the bottom figure, the shoe is rotating about the longest axis. Now the angular momentum vector is trapped at one of the two points on the right or left. In the case of rotation about the axis with the intermediate moment of inertia element, however, the intersection of the sphere and the ellipsoid is not just a pair of isolated points but the curve shown with the dashed line. The relative orientation of the shoe and the angular momentum vector can and will change.

One application of the moment of inertia tensor is to video games that simulate car racing or flying airplanes.

One more exotic example has to do with nuclear physics. Although you have probably visualized atomic nuclei as nothing more than featureless points, or perhaps tiny spheres, they are often ellipsoids with one long axis and two shorter, equal ones. Although a spinning nucleus normally gets rid of its angular momentum via gamma ray emission within a period of time on the order of picoseconds, it may happen that a deformed nucleus gets into a state in which has a large angular momentum is along its long axis, which is a very stable mode of rotation. Such states can live for seconds or even years! (There is more to the story — this is the topic on which I wrote my Ph.D. thesis — but the basic insight applies even though the full treatment requires fancy quantum mechanics.)

Our analysis has so far assumed that the kinetic energy of rotation energy can’t be converted into other forms of energy such as heat, sound, or vibration. When this assumption fails, then rotation about the axis of least moment of inertia becomes unstable, and will eventually convert itself into rotation about the axis whose moment of inertia is greatest. This happened to the U.S.’s first artificial satellite, Explorer I, launched in 1958. Note the long, floppy antennas, which tended to dissipate kinetic energy into vibration. It had been designed to spin about its minimum-moment-of-inertia axis, but almost immediately, as soon as it was in space, it began spinning end over end. It was nevertheless able to carry out its science mission, which didn’t depend on being able to maintain a stable orientation, and it discovered the Van Allen radiation belts.

15.9 * Proof of Kepler’s elliptical orbit law

Kepler determined purely empirically that the planets’ orbits were ellipses, without understanding the underlying reason in terms of physical law. Newton’s proof of this fact based on his laws of motion
The basic idea of the proof is that we want to describe the shape of the planet’s orbit with an equation, and then show that this equation is exactly the one that represents an ellipse. Newton’s original proof had to be very complicated because it was based directly on his laws of motion, which include time as a variable. To make any statement about the shape of the orbit, he had to eliminate time from his equations, leaving only space variables. But conservation laws tell us that certain things don’t change over time, so they have already had time eliminated from them.

There are many ways of representing a curve by an equation, of which the most familiar is $y = ax + b$ for a line in two dimensions. It would be perfectly possible to describe a planet’s orbit using an $x - y$ equation like this, but remember that we are applying conservation of angular momentum, and the space variables that occur in the equation for angular momentum are the distance from the axis, $r$, and the angle between the velocity vector and the $r$ vector, which we will call $\varphi$. The planet will have $\varphi=90^\circ$ when it is moving perpendicular to the $r$ vector, i.e., at the moments when it is at its smallest or greatest distances from the sun. When $\varphi$ is less than $90^\circ$ the planet is approaching the sun, and when it is greater than $90^\circ$ it is receding from it. Describing a curve with an $r - \varphi$ equation is like telling a driver in a parking lot a certain rule for what direction to steer based on the distance from a certain streetlight in the middle of the lot.

The proof is broken into the three parts for easier digestion. The first part is a simple and intuitively reasonable geometrical fact about ellipses, whose proof we relegate to the caption of figure ba; you will not be missing much if you merely absorb the result without reading the proof.

(1) If we use one of the two foci of an ellipse as an axis for defining the variables $r$ and $\varphi$, then the angle between the tangent line and the line drawn to the other focus is the same as $\varphi$, i.e., the two angles labeled $\varphi$ in figure ba are in fact equal.

The other two parts form the meat of our proof. We state the results first and then prove them.

(2) A planet, moving under the influence of the sun’s gravity with less than the energy required to escape, obeys an equation of
the form
\[ \sin \varphi = \frac{1}{\sqrt{-pr^2 + qr}}, \]
where \( p \) and \( q \) are positive constants that depend on the planet’s energy and angular momentum.

(3) A curve is an ellipse if and only if its \( r - \varphi \) equation is of the form
\[ \sin \varphi = \frac{1}{\sqrt{-pr^2 + qr}}, \]
where \( p \) and \( q \) are positive constants that depend on the size and shape of the ellipse.

**Proof of part (2)**

The component of the planet’s velocity vector that is perpendicular to the \( r \) vector is \( v_\perp = v \sin \varphi \), so conservation of angular momentum tells us that \( L = mrv \sin \varphi \) is a constant. Since the planet’s mass is a constant, this is the same as the condition
\[ rv \sin \varphi = \text{constant}. \]

Conservation of energy gives
\[ \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}. \]

We solve the first equation for \( v \) and plug into the second equation to eliminate \( v \). Straightforward algebra then leads to the equation claimed above, with the constant \( p \) being positive because of our assumption that the planet’s energy is insufficient to escape from the sun, i.e., its total energy is negative.

**Proof of part (3)**

We define the quantities \( \alpha, d, \) and \( s \) as shown in the figure. The law of cosines gives
\[ d^2 = r^2 + s^2 - 2rs \cos \alpha. \]

Using \( \alpha = 180^\circ - 2\varphi \) and the trigonometric identities \( \cos(180^\circ - x) = -\cos x \) and \( \cos 2x = 1 - 2 \sin^2 x \), we can rewrite this as
\[ d^2 = r^2 + s^2 - 2rs (2 \sin^2 \varphi - 1). \]

Straightforward algebra transforms this into
\[ \sin \varphi = \sqrt{\frac{(r + s)^2 - d^2}{4rs}}. \]

Since \( r + s \) is constant, the top of the fraction is constant, and the denominator can be rewritten as \( 4rs = 4r(\text{constant} - r) \), which is equivalent to the desired form.
15.10 Some theorems and proofs

In this section I prove three theorems stated earlier, and state a fourth theorem whose proof is left as an exercise.

Uniqueness of the cross product

The vector cross product as we have defined it has the following properties:
(1) It does not violate rotational invariance.
(2) It has the property \( A \times (B + C) = A \times B + A \times C \).
(3) It has the property \( A \times (kB) = k(A \times B) \), where \( k \) is a scalar.

**Theorem:** The definition we have given is the only possible method of multiplying two vectors to make a third vector which has these properties, with the exception of trivial redefinitions which just involve multiplying all the results by the same constant or swapping the names of the axes. (Specifically, using a left-hand rule rather than a right-hand rule corresponds to multiplying all the results by \(-1\).)

**Proof:** We prove only the uniqueness of the definition, without explicitly proving that it has properties (1) through (3).

Using properties (2) and (3), we can break down any vector multiplication \((A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})\) into terms involving cross products of unit vectors.

A “self-term” like \(\hat{x} \times \hat{x}\) must either be zero or lie along the \(x\) axis, since any other direction would violate property (1). If it was not zero, then \((-\hat{x}) \times (-\hat{x})\) would have to lie in the opposite direction to avoid breaking rotational invariance, but property (3) says that \((-\hat{x}) \times (-\hat{x})\) is the same as \(\hat{x} \times \hat{x}\), which is a contradiction. Therefore the self-terms must be zero.

An “other-term” like \(\hat{x} \times \hat{y}\) could conceivably have components in the \(x-y\) plane and along the \(z\) axis. If there was a nonzero component in the \(x-y\) plane, symmetry would require that it lie along the diagonal between the \(x\) and \(y\) axes, and similarly the in-the-plane component of \((-\hat{x}) \times \hat{y}\) would have to be along the other diagonal in the \(x-y\) plane. Property (3), however, requires that \((-\hat{x}) \times \hat{y}\) equal \(-\hat{x} \times \hat{y}\), which would be along the original diagonal. The only way it can lie along both diagonals is if it is zero.

We now know that \(\hat{x} \times \hat{y}\) must lie along the \(z\) axis. Since we are not interested in trivial differences in definitions, we can fix \(\hat{x} \times \hat{y} = \hat{z}\), ignoring perilous possibilities such as \(\hat{x} \times \hat{y} = 7\hat{z}\) or the left-handed definition \(\hat{x} \times \hat{y} = -\hat{z}\). Given \(\hat{x} \times \hat{y} = \hat{z}\), the symmetry of space requires that similar relations hold for \(\hat{y} \times \hat{z}\) and \(\hat{z} \times \hat{x}\), with at most a difference in sign. A difference in sign could always be eliminated by swapping the names of some of the axes, so ignoring possible trivial differences in definitions we can assume that the cyclically related set of relations \(\hat{x} \times \hat{y} = \hat{z}, \hat{y} \times \hat{z} = \hat{x}\), and \(\hat{z} \times \hat{x} = \hat{y}\).
Choice of axis theorem

**Theorem:** Suppose a closed system of material particles conserves angular momentum in one frame of reference, with the axis taken to be at the origin. Then conservation of angular momentum is unaffected if the origin is relocated or if we change to a frame of reference that is in constant-velocity motion with respect to the first one. The theorem also holds in the case where the system is not closed, but the total external force is zero.

**Proof:** In the original frame of reference, angular momentum is conserved, so we have $\frac{dL}{dt}=0$. From example 29 on page 474, this derivative can be rewritten as

$$\frac{dL}{dt} = \sum_i r_i \times F_i,$$

where $F_i$ is the total force acting on particle $i$. In other words, we're adding up all the torques on all the particles.

By changing to the new frame of reference, we have changed the position vector of each particle according to $r_i \rightarrow r_i + k - ut$, where $k$ is a constant vector that indicates the relative position of the new origin at $t = 0$, and $u$ is the velocity of the new frame with respect to the old one. The forces are all the same in the new frame of reference, however. In the new frame, the rate of change of the angular momentum is

$$\frac{dL}{dt} = \sum_i (r_i + k - ut) \times F_i = \sum_i r_i \times F_i + (k - ut) \times \sum_i F_i.$$

The first term is the expression for the rate of change of the angular momentum in the original frame of reference, which is zero by assumption. The second term vanishes by Newton’s third law; since the system is closed, every force $F_i$ cancels with some force $F_j$. (If external forces act, but they add up to zero, then the sum can be broken up into a sum of internal forces and a sum of external forces, each of which is zero.) The rate of change of the angular momentum is therefore zero in the new frame of reference.

Spin theorem

**Theorem:** An object’s angular momentum with respect to some outside axis A can be found by adding up two parts:

1. The first part is the object’s angular momentum found by using its own center of mass as the axis, i.e., the angular momentum the
object has because it is spinning.

(2) The other part equals the angular momentum that the object would have with respect to the axis \( A \) if it had all its mass concentrated at and moving with its center of mass.

Proof: Let the system’s center of mass be at \( \mathbf{r}_{cm} \), and let particle \( i \) lie at position \( \mathbf{r}_{cm} + \mathbf{d}_i \). Then the total angular momentum is

\[
\mathbf{L} = \sum_i \left( \mathbf{r}_{cm} + \mathbf{d}_i \right) \times \mathbf{p}_i
\]

\[
= \mathbf{r}_{cm} \times \sum_i \mathbf{p}_i + \sum_i \mathbf{d}_i \times \mathbf{p}_i,
\]

which establishes the result claimed, since we can identify the first term with (2) and the second with (1).

**Parallel axis theorem**

Suppose an object has mass \( m \), and moment of inertia \( I_o \) for rotation about some axis \( A \) passing through its center of mass. Given a new axis \( B \), parallel to \( A \) and lying at a distance \( h \) from it, the object’s moment of inertia is given by \( I_o + mh^2 \).

The proof of this theorem is left as an exercise (problem 27, p. 490).
Summary

Selected vocabulary

angular momentum . . . a measure of rotational motion; a conserved quantity for a closed system
axis . . . . . . . . An arbitrarily chosen point used in the definition of angular momentum. Any object whose direction changes relative to the axis is considered to have angular momentum. No matter what axis is chosen, the angular momentum of a closed system is conserved.
torque . . . . . . . the rate of change of angular momentum; a numerical measure of a force’s ability to twist on an object
equilibrium . . . . . . . . a state in which an object’s momentum and angular momentum are constant
stable equilibrium one in which a force always acts to bring the object back to a certain point
unstable equilibrium one in which any deviation of the object from its equilibrium position results in a force pushing it even farther away

Notation

$L$ . . . . . . . . angular momentum
$t$ . . . . . . . . torque
$T$ . . . . . . . . the period the time required for a rigidly rotating body to complete one rotation
$\omega$ . . . . . . . . the angular velocity, $d\theta/dt$
$I$ . . . . . . . . moment of inertia, $L = I\omega$

Summary

Angular momentum is a measure of rotational motion which is conserved for a closed system. This book only discusses angular momentum for rotation of material objects in two dimensions. Not all rotation is rigid like that of a wheel or a spinning top. An example of nonrigid rotation is a cyclone, in which the inner parts take less time to complete a revolution than the outer parts. In order to define a measure of rotational motion general enough to include nonrigid rotation, we define the angular momentum of a system by dividing it up into small parts, and adding up all the angular momenta of the small parts, which we think of as tiny particles. We arbitrarily choose some point in space, the axis, and we say that anything that changes its direction relative to that point possesses angular momentum. The angular momentum of a single particle is

$$L = mv_\perp r,$$

where $v_\perp$ is the component of its velocity perpendicular to the line joining it to the axis, and $r$ is its distance from the axis. Positive and
negative signs of angular momentum are used to indicate clockwise and counterclockwise rotation.

The choice of axis theorem states that any axis may be used for defining angular momentum. If a system’s angular momentum is constant for one choice of axis, then it is also constant for any other choice of axis.

The spin theorem states that an object’s angular momentum with respect to some outside axis A can be found by adding up two parts:

1. The first part is the object’s angular momentum found by using its own center of mass as the axis, i.e., the angular momentum the object has because it is spinning.
2. The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

Torque is the rate of change of angular momentum. The torque a force can produce is a measure of its ability to twist on an object. The relationship between force and torque is

\[ |\tau| = r|F_\perp|, \]

where \( r \) is the distance from the axis to the point where the force is applied, and \( F_\perp \) is the component of the force perpendicular to the line connecting the axis to the point of application. Statics problems can be solved by setting the total force and total torque on an object equal to zero and solving for the unknowns.

In the special case of a rigid body rotating in a single plane, we define

\[ \omega = \frac{d\theta}{dt} \]  
[angular velocity]

and

\[ \alpha = \frac{d\omega}{dt}, \]  
[angular acceleration]

in terms of which we have

\[ L = I\omega \]

and

\[ \tau = I\alpha, \]

where the moment of inertia, \( I \), is defined as

\[ I = \sum m_i r_i^2, \]
summing over all the atoms in the object (or using calculus to perform a continuous sum, i.e. an integral). The relationship between the angular quantities and the linear ones is

\[ \begin{align*}
v_t & = \omega r \quad \text{[tangential velocity of a point]} \\
v_r & = 0 \quad \text{[radial velocity of a point]} \\
a_t & = \alpha r \quad \text{[radial acceleration of a point]} \\
a_r & = \omega^2 r \quad \text{[radial acceleration of a point]} \\
\end{align*} \]

at a distance \( r \) from the axis.

In three dimensions, torque and angular momentum are vectors, and are expressed in terms of the vector cross product, which is the only rotationally invariant way of defining a multiplication of two vectors that produces a third vector:

\[ \begin{align*}
\mathbf{L} & = \mathbf{r} \times \mathbf{p} \\
\tau & = \mathbf{r} \times \mathbf{F}
\end{align*} \]

In general, the cross product of vectors \( \mathbf{b} \) and \( \mathbf{c} \) has magnitude

\[ |\mathbf{b} \times \mathbf{c}| = |\mathbf{b}| |\mathbf{c}| \sin \theta_{bc}, \]

which can be interpreted geometrically as the area of the parallelogram formed by the two vectors when they are placed tail-to-tail. The direction of the cross product lies along the line which is perpendicular to both vectors; of the two such directions, we choose the one that is right-handed, in the sense that if we point the fingers of the flattened right hand along \( \mathbf{b} \), then bend the knuckles to point the fingers along \( \mathbf{c} \), the thumb gives the direction of \( \mathbf{b} \times \mathbf{c} \). In terms of components, the cross product is

\[ \begin{align*}
(\mathbf{b} \times \mathbf{c})_x & = b_y c_z - c_y b_z \\
(\mathbf{b} \times \mathbf{c})_y & = b_z c_x - c_z b_x \\
(\mathbf{b} \times \mathbf{c})_z & = b_x c_y - c_x b_y
\end{align*} \]

The cross product has the disconcerting properties

\[ \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \quad \text{[noncommutative]} \]

and

\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad \text{[nonassociative]}, \]

and there is no “cross-division.”

For rigid-body rotation in three dimensions, we define an angular velocity vector \( \mathbf{\omega} \), which lies along the axis of rotation and bears a right-hand relationship to it. Except in special cases, there is no scalar moment of inertia for which \( \mathbf{L} = I \mathbf{\omega} \); the moment of inertia must be expressed as a matrix.
Problems

Key
✓ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 A skilled motorcyclist can ride up a ramp, fly through the air, and land on another ramp. Why would it be useful for the rider to speed up or slow down the back wheel while in the air?

2 An object thrown straight up in the air is momentarily at rest when it reaches the top of its motion. Does that mean that it is in equilibrium at that point? Explain.

3 An object is observed to have constant angular momentum. Can you conclude that no torques are acting on it? Explain. [Based on a problem by Serway and Faughn.]

4 The sun turns on its axis once every 26.0 days. Its mass is \(2.0 \times 10^{30}\) kg and its radius is \(7.0 \times 10^8\) m. Assume it is a rigid sphere of uniform density.
   (a) What is the sun’s angular momentum? ✓
   In a few billion years, astrophysicists predict that the sun will use up all its sources of nuclear energy, and will collapse into a ball of exotic, dense matter known as a white dwarf. Assume that its radius becomes \(5.8 \times 10^6\) m (similar to the size of the Earth.) Assume it does not lose any mass between now and then. (Don’t be fooled by the photo, which makes it look like nearly all of the star was thrown off by the explosion. The visually prominent gas cloud is actually thinner than the best laboratory vacuum ever produced on earth. Certainly a little bit of mass is actually lost, but it is not at all unreasonable to make an approximation of zero loss of mass as we are doing.)
   (b) What will its angular momentum be?
   (c) How long will it take to turn once on its axis? ✓

5 (a) Alice says Cathy’s body has zero momentum, but Bob says Cathy’s momentum is nonzero. Nobody is lying or making a mistake. How is this possible? Give a concrete example.
   (b) Alice and Bob agree that Dong’s body has nonzero momentum, but disagree about Dong’s angular momentum, which Alice says is zero, and Bob says is nonzero. Explain.

6 Two objects have the same momentum vector. Assume that they are not spinning; they only have angular momentum due to their motion through space. Can you conclude that their angular momenta are the same? Explain. [Based on a problem by Serway and Faughn.]
7. You are trying to loosen a stuck bolt on your RV using a big wrench that is 50 cm long. If you hang from the wrench, and your mass is 55 kg, what is the maximum torque you can exert on the bolt?

8. The figure shows scale drawing of a pair of pliers being used to crack a nut, with an appropriately reduced centimeter grid. Warning: do not attempt this at home; it is bad manners. If the force required to crack the nut is 300 N, estimate the force required of the person’s hand. Solution, p. 565

9. Make a rough estimate of the mechanical advantage of the lever shown in the figure. In other words, for a given amount of force applied on the handle, how many times greater is the resulting force on the cork?

10. A physical therapist wants her patient to rehabilitate his injured elbow by laying his arm flat on a table, and then lifting a 2.1 kg mass by bending his elbow. In this situation, the weight is 33 cm from his elbow. He calls her back, complaining that it hurts him to grasp the weight. He asks if he can strap a bigger weight onto his arm, only 17 cm from his elbow. How much mass should she tell him to use so that he will be exerting the same torque? (He is raising his forearm itself, as well as the weight.)

11. Two horizontal tree branches on the same tree have equal diameters, but one branch is twice as long as the other. Give a quantitative comparison of the torques where the branches join the trunk. [Thanks to Bong Kang.]

12. A ball is connected by a string to a vertical post. The ball is set in horizontal motion so that it starts winding the string around the post. Assume that the motion is confined to a horizontal plane, i.e., ignore gravity. Michelle and Astrid are trying to predict the final velocity of the ball when it reaches the post. Michelle says that according to conservation of angular momentum, the ball has to speed up as it approaches the post. Astrid says that according to conservation of energy, the ball has to keep a constant speed. Who is right? [Hint: How is this different from the case where you whirl a rock in a circle on a string and gradually reel in the string?]

13. A person of weight $W$ stands on the ball of one foot. Find the tension in the calf muscle and the force exerted by the shinbones on the bones of the foot, in terms of $W, a,$ and $b$. For simplicity, assume that all the forces are at 90-degree angles to the foot, i.e., neglect the angle between the foot and the floor.
14. The rod in the figure is supported by the finger and the string.
(a) Find the tension, $T$, in the string, and the force, $F$, from the finger, in terms of $m$, $b$, $L$, and $g$.
(b) Comment on the cases $b = L$ and $b = L/2$.
(c) Are any values of $b$ unphysical?

15. A uniform ladder of mass $m$ and length $L$ leans against a smooth wall, making an angle $\theta$ with respect to the ground. The dirt exerts a normal force and a frictional force on the ladder, producing a force vector with magnitude $F_1$ at an angle $\phi$ with respect to the ground. Since the wall is smooth, it exerts only a normal force on the ladder; let its magnitude be $F_2$.
(a) Explain why $\phi$ must be greater than $\theta$. No math is needed.
(b) Choose any numerical values you like for $m$ and $L$, and show that the ladder can be in equilibrium (zero torque and zero total force vector) for $\theta = 45.00^\circ$ and $\phi = 63.43^\circ$.

16. Continuing problem 15, find an equation for $\phi$ in terms of $\theta$, and show that $m$ and $L$ do not enter into the equation. Do not assume any numerical values for any of the variables. You will need the trig identity $\sin(a - b) = \sin a \cos b - \sin b \cos a$. (As a numerical check on your result, you may wish to check that the angles given in part b of the previous problem satisfy your equation.)

17. (a) Find the minimum horizontal force which, applied at the axle, will pull a wheel over a step. Invent algebra symbols for whatever quantities you find to be relevant, and give your answer in symbolic form. [Hints: There are four forces on the wheel at first, but only three when it lifts off. Normal forces are always perpendicular to the surface of contact. Note that the corner of the step cannot be perfectly sharp, so the surface of contact for this force really coincides with the surface of the wheel.]
(b) Under what circumstances does your result become infinite? Give a physical interpretation.

18. In the 1950’s, serious articles began appearing in magazines like *Life* predicting that world domination would be achieved by the nation that could put nuclear bombs in orbiting space stations, from which they could be dropped at will. In fact it can be quite difficult to get an orbiting object to come down. Let the object have energy $E = KE + PE$ and angular momentum $L$. Assume that the energy is negative, i.e., the object is moving at less than escape velocity. Show that it can never reach a radius less than

$$r_{\text{min}} = \frac{GMm}{2E} \left( -1 + \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}} \right).$$

[Note that both factors are negative, giving a positive result.]
19 You wish to determine the mass of a ship in a bottle without taking it out. Show that this can be done with the setup shown in the figure, with a scale supporting the bottle at one end, provided that it is possible to take readings with the ship slid to several different locations. Note that you can’t determine the position of the ship’s center of mass just by looking at it, and likewise for the bottle. In particular, you can’t just say, “position the ship right on top of the fulcrum” or “position it right on top of the balance.”

20 Two atoms will interact via electrical forces between their protons and electrons. One fairly good approximation to the potential energy is the Lennard-Jones potential,

\[ PE(r) = k \left[ \left( \frac{a}{r} \right)^{12} - 2 \left( \frac{a}{r} \right)^{6} \right] , \]

where \( r \) is the center-to-center distance between the atoms.

Show that (a) there is an equilibrium point at \( r = a \), (b) the equilibrium is stable, and (c) the energy required to bring the atoms from their equilibrium separation to infinity is \( k \). [Hints: The first two parts can be done more easily by setting \( a = 1 \), since the value of \( a \) only changes the distance scale. One way to do part b is by graphing.]

21 Suppose that we lived in a universe in which Newton’s law of gravity gave forces proportional to \( r^{-7} \) rather than \( r^{-2} \). Which, if any, of Kepler’s laws would still be true? Which would be completely false? Which would be different, but in a way that could be calculated with straightforward algebra?

22 Show that a sphere of radius \( R \) that is rolling without slipping has angular momentum and momentum in the ratio \( L/p = (2/5)R \).

23 Suppose a bowling ball is initially thrown so that it has no angular momentum at all, i.e., it is initially just sliding down the lane. Eventually kinetic friction will get it spinning fast enough so that it is rolling without slipping. Show that the final velocity of the ball equals \( 5/7 \) of its initial velocity. [Hint: You’ll need the result of problem 22.]

24 Penguins are playful animals. Tux the Penguin invents a new game using a natural circular depression in the ice. He waddles at top speed toward the crater, aiming off to the side, and then hops into the air and lands on his belly just inside its lip. He then belly-surfs, moving in a circle around the rim. The ice is frictionless, so his speed is constant. Is Tux’s angular momentum zero, or nonzero? What about the total torque acting on him? Take the center of the crater to be the axis. Explain your answers.
25 A massless rod of length $\ell$ has weights, each of mass $m$, attached to its ends. The rod is initially put in a horizontal position, and laid on an off-center fulcrum located at a distance $b$ from the rod’s center. The rod will topple. (a) Calculate the total gravitational torque on the rod directly, by adding the two torques. (b) Verify that this gives the same result as would have been obtained by taking the entire gravitational force as acting at the center of mass.

26 Use analogies to find the equivalents of the following equations for rotation in a plane:

\[ KE = \frac{p^2}{2m} \]
\[ \Delta x = v_0 \Delta t + \left(\frac{1}{2}\right)a\Delta t^2 \]
\[ W = F \Delta x \]

Example: $v = \Delta x/\Delta t \rightarrow \omega = \Delta \theta/\Delta t$

27 Prove the parallel axis theorem stated on page 482.

28 The box shown in the figure is being accelerated by pulling on it with the rope. (a) Assume the floor is frictionless. What is the maximum force that can be applied without causing the box to tip over? [Hint, p. 550] (b) Repeat part a, but now let the coefficient of friction be $\mu$. (c) What happens to your answer to part b when the box is sufficiently tall? How do you interpret this?

29 (a) The bar of mass $m$ is attached at the wall with a hinge, and is supported on the right by a massless cable. Find the tension, $T$, in the cable in terms of the angle $\theta$. (b) Interpreting your answer to part a, what would be the best angle to use if we wanted to minimize the strain on the cable? (c) Again interpreting your answer to part a, for what angles does the result misbehave mathematically? Interpret this physically.

30 (a) The two identical rods are attached to one another with a hinge, and are supported by the two massless cables. Find the angle $\alpha$ in terms of the angle $\beta$, and show that the result is a purely geometric one, independent of the other variables involved. (b) Using your answer to part a, sketch the configurations for $\beta \to 0$, $\beta = 45^\circ$, and $\beta = 90^\circ$. Do your results make sense intuitively?

31 (a) Find the angular velocities of the earth’s rotation and of the earth’s motion around the sun. (b) Which motion involves the greater acceleration?

32 Give a numerical comparison of the two molecules’ moments of inertia for rotation in the plane of the page about their centers of mass.
33. Find the angular momentum of a particle whose position is \( \mathbf{r} = 3\hat{x} - \hat{y} + \hat{z} \) (in meters) and whose momentum is \( \mathbf{p} = -2\hat{x} + \hat{y} + \hat{z} \) (in kg m/s). √

34. Find a vector that is perpendicular to both of the following two vectors:

\[ \hat{x} + 2\hat{y} + 3\hat{z} \]
\[ 4\hat{x} + 5\hat{y} + 6\hat{z} \]

√

35. Prove property (3) of the vector cross product from the theorem on page 480.

36. Prove the anticommutative property of the vector cross product, \( \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \), using the expressions for the components of the cross product. Note that giving an example does not constitute a proof of a general rule.

37. Find three vectors with which you can demonstrate that the vector cross product need not be associative, i.e., that \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \) need not be the same as \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \).

38. Which of the following expressions make sense, and which are nonsense? For those that make sense, indicate whether the result is a vector or a scalar.
   (a) \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \)
   (b) \( (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \)
   (c) \( (\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C} \)

39. (a) As suggested in the figure, find the area of the infinitesimal region expressed in polar coordinates as lying between \( r \) and \( r + dr \) and between \( \theta \) and \( \theta + d\theta \). √
   (b) Generalize this to find the infinitesimal element of volume in cylindrical coordinates \( (r, \theta, z) \), where the Cartesian \( z \) axis is perpendicular to the directions measured by \( r \) and \( \theta \). √
   (c) Find the moment of inertia for rotation about its axis of a cone whose mass is \( M \), whose height is \( h \), and whose base has a radius \( b \). √

40. Find the moment of inertia of a solid rectangular box of mass \( M \) and uniform density, whose sides are of length \( a \), \( b \), and \( c \), for rotation about an axis through its center parallel to the edges of length \( a \). √
The nucleus $^{168}$Er (erbium-168) contains 68 protons (which is what makes it a nucleus of the element erbium) and 100 neutrons. It has an ellipsoidal shape like an American football, with one long axis and two short axes that are of equal diameter. Because this is a subatomic system, consisting of only 168 particles, its behavior shows some clear quantum-mechanical properties. It can only have certain energy levels, and it makes quantum leaps between these levels. Also, its angular momentum can only have certain values, which are all multiples of $2.109 \times 10^{-34}$ kg $\cdot$ m$^2$/s. The table shows some of the observed angular momenta and energies of $^{168}$Er, in SI units (kg $\cdot$ m$^2$/s and joules).

<table>
<thead>
<tr>
<th>$L \times 10^{34}$</th>
<th>$E \times 10^{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2.109</td>
<td>1.2786</td>
</tr>
<tr>
<td>4.218</td>
<td>4.2311</td>
</tr>
<tr>
<td>6.327</td>
<td>8.7919</td>
</tr>
<tr>
<td>8.437</td>
<td>14.8731</td>
</tr>
<tr>
<td>10.546</td>
<td>22.3798</td>
</tr>
<tr>
<td>12.655</td>
<td>31.135</td>
</tr>
<tr>
<td>14.764</td>
<td>41.206</td>
</tr>
<tr>
<td>16.873</td>
<td>52.223</td>
</tr>
</tbody>
</table>

(a) These data can be described to a good approximation as a rigid end-over-end rotation. Estimate a single best-fit value for the moment of inertia from the data, and check how well the data agree with the assumption of rigid-body rotation.  
(b) Check whether this moment of inertia is on the right order of magnitude. The moment of inertia depends on both the size and the shape of the nucleus. For the sake of this rough check, ignore the fact that the nucleus is not quite spherical. To estimate its size, use the fact that a neutron or proton has a volume of about 1 fm$^3$ (one cubic femtometer, where 1 fm = $10^{-15}$ m), and assume they are closely packed in the nucleus.

(a) Prove the identity $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$ by expanding the product in terms of its components. Note that because the $x$, $y$, and $z$ components are treated symmetrically in the definitions of the vector cross product, it is only necessary to carry out the proof for the $x$ component of the result.

(b) Applying this to the angular momentum of a rigidly rotating body, $L = \int r \times (\omega \times r) \, dm$, show that the diagonal elements of the moment of inertia tensor can be expressed as, e.g., $I_{xx} = \int (y^2 + z^2) \, dm$.

(c) Find the diagonal elements of the moment of inertia matrix of an ellipsoid with axes of lengths $a$, $b$, and $c$, in the principal-axis frame, and with the axis at the center.
When we talk about rigid-body rotation, the concept of a perfectly rigid body can only be an idealization. In reality, any object will compress, expand, or deform to some extent when subjected to the strain of rotation. However, if we let it settle down for a while, perhaps it will reach a new equilibrium. As an example, suppose we fill a centrifuge tube with some compressible substance like shaving cream or Wonder Bread. We can model the contents of the tube as a one-dimensional line of mass, extending from \( r = 0 \) to \( r = \ell \). Once the rotation starts, we expect that the contents will be most compressed near the “floor” of the tube at \( r = \ell \); this is both because the inward force required for circular motion increases with \( r \) for a fixed \( \omega \), and because the part at the floor has the greatest amount of material pressing “down” (actually outward) on it. The linear density \( \frac{dm}{dr} \), in units of kg/m, should therefore increase as a function of \( r \). Suppose that we have \( \frac{dm}{dr} = \mu e^{r/\ell} \), where \( \mu \) is a constant. Find the moment of inertia.

Two bars of length \( L \) are connected with a hinge and placed on a frictionless cylinder of radius \( r \). (a) Show that the angle \( \theta \) shown in the figure is related to the unitless ratio \( r/L \) by the equation

\[
\frac{r}{L} = \frac{\cos^2 \theta}{2 \tan \theta}.
\]

(b) Discuss the physical behavior of this equation for very large and very small values of \( r/L \).

Let two sides of a triangle be given by the vectors \( \mathbf{A} \) and \( \mathbf{B} \), with their tails at the origin, and let mass \( m \) be uniformly distributed on the interior of the triangle. (a) Show that the distance of the triangle’s center of mass from the intersection of sides \( \mathbf{A} \) and \( \mathbf{B} \) is given by \( \frac{1}{3} |\mathbf{A} + \mathbf{B}| \).

(b) Consider the quadrilateral with mass \( 2m \), and vertices at the origin, \( \mathbf{A} \), \( \mathbf{B} \), and \( \mathbf{A} + \mathbf{B} \). Show that its moment of inertia, for rotation about an axis perpendicular to it and passing through its center of mass, is \( m \left( A^2 + B^2 \right) \). (c) Show that the moment of inertia for rotation about an axis perpendicular to the plane of the original triangle, and passing through its center of mass, is \( \frac{m}{12} (A^2 + B^2 - \mathbf{A} \cdot \mathbf{B}) \). Hint: Combine the results of parts a and b with the result of problem 27.

In example 23 on page 466, prove that if the rod is sufficiently thin, it can be toppled without scraping on the floor.

> Solution, p. 565

Problems
47. A yo-yo of total mass $m$ consists of two solid cylinders of radius $R$, connected by a small spindle of negligible mass and radius $r$. The top of the string is held motionless while the string unrolls from the spindle. Show that the acceleration of the yo-yo is $g/(1 + R^2/2r^2)$. [Hint: The acceleration and the tension in the string are unknown. Use $\tau = \Delta L/\Delta t$ and $F = ma$ to determine these two unknowns.]

48. We have $n$ identical books of width $w$, and we wish to stack them at the edge of a table so that they extend the maximum possible distance $L_n$ beyond the edge. Surprisingly, it is possible to have values of $L_n$ that are greater than $w$, even with fairly small $n$. For large $n$, however, $L_n$ begins to grow very slowly. Our goal is to find $L_n$ for a given $n$. We adopt the restriction that only one book is ever used at a given height.² (a) Use proof by induction to find $L_n$, expressing your result as a sum. (b) Find a sufficiently tight lower bound on this sum, as a closed-form expression, to prove that 1,202,604 books suffice for $L > 7w$.

49. A certain function $f$ takes two vectors as inputs and gives an output that is also a vector. The function can be defined in such a way that it is rotationally invariant, and it is also well defined regardless of the units of the vectors. It takes on the following values for the following inputs:

$$f(\hat{x}, \hat{y}) = -\hat{z}$$
$$f(2\hat{x}, \hat{y}) = -8\hat{z}$$
$$f(\hat{x}, 2\hat{y}) = -2\hat{z}$$

Prove that the given information uniquely determines $f$, and give an explicit expression for it.

50. (a) Find the moment of inertia of a uniform square of mass $m$ and with sides of length $b$, for rotation in its own plane, about one of its corners. (b) The square is balanced on one corner on a frictionless surface. An infinitesimal perturbation causes it to topple. Find its angular velocity at the moment when its side slaps the surface.

²When this restriction is lifted, the calculation of $L_n$ becomes a much more difficult problem, which was partially solved in 2009 by Paterson, Peres, Thorup, Winkler, and Zwick.
51 The figure shows a microscopic view of the innermost tracks of a music CD. The pits represent the pattern of ones and zeroes that encode the musical waveform. Because the laser that reads the data has to sweep over a fixed amount of data per unit time, the disc spins at a decreasing angular velocity as the music is played from the inside out. The linear velocity \( v \), not the angular velocity, is constant. Each track is separated from its neighbors on either side by a fixed distance \( p \), called the pitch. Although the tracks are actually concentric circles, we will idealize them in this problem as a type of spiral, called an Archimedean spiral, whose turns have constant spacing, \( p \), along any radial line. Our goal is to find the angular acceleration of this idealized CD, in terms of the constants \( v \) and \( p \), and the radius \( r \) at which the laser is positioned.

(a) Use geometrical reasoning to constrain the dependence of the result on \( p \).

(b) Use units to further constrain the result up to a unitless multiplicative constant.

(c) Find the full result. [Hint: Find a differential equation involving \( r \) and its time derivative, and then solve this equation by separating variables.]

(d) Consider the signs of the variables in your answer to part c, and show that your equation still makes sense when the direction of rotation is reversed.

(e) Similarly, check that your result makes sense regardless of whether we view the CD player from the front or the back. (Clockwise seen from one side is counterclockwise from the other.)

52 Neutron stars are the collapsed remnants of dead stars. They rotate quickly, and their rotation can be measured extremely accurately by radio astronomers. Some of them rotate at such a predictable rate that they can be used to count time about as accurately as the best atomic clocks. They do decelerate slowly, but this deceleration can be taken into account. One of the best-studied stars of this type was observed continuously over a 10-year period. As of the benchmark date April 5, 2001, it was found to have

\[
\omega = 1.091313551502333 \times 10^3 \text{ s}^{-1}
\]

and

\[
\alpha = -1.085991 \times 10^{-14} \text{ s}^{-2},
\]

where the error bars in the final digit of each number are about \( \pm 1 \). Astronomers often use the Julian year as their unit of time, where one Julian year is defined to be exactly 3.15576 \( \times 10^7 \) s. Find the number of revolutions that this pulsar made over a period of 10 Julian years, starting from the benchmark date.

\footnote{Verbiest et al., Astrophysical Journal 679 (675) 2008}
53 A disk, initially rotating at 120 radians per second, is slowed down with a constant angular acceleration of magnitude $4.0 \text{ s}^{-2}$. How many revolutions does the disk make before it comes to rest? [Problem by B. Shotwell.]

54 A bell rings at the Tilden Park merry go round in Berkeley, California, and the carousel begins to move with an angular acceleration of $1.0 \times 10^{-2} \text{ s}^{-2}$. How much time does it take to perform its first revolution?

55 A gasoline-powered car has a heavy wheel called a flywheel, whose main function is to add inertia to the motion of the engine so that it keeps spinning smoothly between power strokes of the cylinders. Suppose that a certain car’s flywheel is spinning with angular velocity $\omega_0$, but the car is then turned off, so that the engine and flywheel start to slow down as a result of friction. Assume that the angular acceleration is constant. After the flywheel has made $N$ revolutions, it comes to rest. What is the magnitude of the angular acceleration? [Problem by B. Shotwell.]

56 A rigid body rotates about a line according to $\theta = At^3 - Bt$ (valid for both negative and positive $t$).

(a) What is the angular velocity as a function of time? [Problem by B. Shotwell.]

(b) What is the angular acceleration as a function of time?

(c) There are two times when the angular velocity is zero. What is the positive time for which this is true? Call this $t_+$. [Problem by B. Shotwell.]

(d) What is the average angular velocity over the time interval from 0 to $t_+$? [Problem by B. Shotwell.]

57 A bug stands on a horizontal turntable at distance $r$ from the center. The coefficient of static friction between the bug and the turntable is $\mu_s$. The turntable spins at constant angular frequency $\omega$.

(a) Is the bug more likely to slip at small values of $r$, or large values? [Problem by B. Shotwell.]

(b) If the bug walks along a radius, what is the value of $r$ at which it looses its footing? [Problem by B. Shotwell.]

58 A bug stands on a horizontal turntable at distance $r$ from the center. The coefficient of static friction between the bug and the turntable is $\mu_s$. Starting from rest, the turntable begins rotating with angular acceleration $\alpha$. What is the magnitude of the angular frequency at which the bug starts to slide? [Problem by B. Shotwell.]

Problems 57 and 58.
Problem 59. The figure shows a tabletop experiment that can be used to determine an unknown moment of inertia. A rotating platform of radius $R$ has a string wrapped around it. The string is threaded over a pulley and down to a hanging weight of mass $m$. The mass is released from rest, and its downward acceleration $a$ ($a > 0$) is measured. Find the total moment of inertia $I$ of the platform plus the object sitting on top of it. (The moment of inertia of the object itself can then be found by subtracting the value for the empty platform.)

Problem 60. The uniform cube has unit weight and sides of unit length. One corner is attached to a universal joint, i.e., a frictionless bearing that allows any type of rotation. If the cube is in equilibrium, find the magnitudes of the forces $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Problem 61. In this problem we investigate the notion of division by a vector.

(a) Given a nonzero vector $\mathbf{a}$ and a scalar $b$, suppose we wish to find a vector $\mathbf{u}$ that is the solution of $\mathbf{a} \cdot \mathbf{u} = b$. Show that the solution is not unique, and give a geometrical description of the solution set.
(b) Do the same thing for the equation $\mathbf{a} \times \mathbf{u} = \mathbf{c}$.
(c) Show that the simultaneous solution of these two equations exists and is unique.

Remark: This is one motivation for constructing the number system called the quaternions. For a certain period around 1900, quaternions were more popular than the system of vectors and scalars more commonly used today. They still have some important advantages over the scalar-vector system for certain applications, such as avoiding a phenomenon known as gimbal lock in controlling the orientation of bodies such as spacecraft.

Problem 62. The figure shows a slab of mass $M$ rolling freely down an inclined plane inclined at an angle $\theta$ to the horizontal. The slab is on top of a set of rollers, each of radius $r$, that roll without slipping at their top and bottom surfaces. The rollers may for example be cylinders, or spheres such as ball bearings. Each roller’s center of mass coincides with its geometrical center. The sum of the masses of the rollers is $m$, and the sum of their moments of inertia (each about its own center) is $I$. Find the acceleration of the slab, and verify that your expression has the correct behavior in interesting limiting cases.

Problem 63. Vector $\mathbf{A} = (3.0\hat{x} - 4.0\hat{y})$ meters, and vector $\mathbf{B} = (5.0\hat{x} + 12.0\hat{y})$ meters. Find the following:
(a) The magnitude of vector $\mathbf{A} - 2\mathbf{B}$.
(b) The dot product $\mathbf{A} \cdot \mathbf{B}$.
(c) The cross product $\mathbf{A} \times \mathbf{B}$ (expressing the result in terms of its components).
(d) The value of $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})$.
(e) The angle between the two vectors.

$[\text{problem by B. Shotwell}]$
Problem 64. A disk starts from rest and rotates about a fixed axis, subject to a constant torque. The work done by the torque during the first revolution is $W$. What is the work done by the torque during the second revolution? $\sqrt{[\text{problem by B. Shotwell}]}$

Problem 65. Show that when a thin, uniform ring rotates about a diameter, the moment of inertia is half as big as for rotation about the axis of symmetry. $\Rightarrow$ Solution, p. 566

Problem 66. A bug stands at the right end of a rod of length $\ell$, which is initially at rest in a horizontal position. The rod rests on a fulcrum which is at a distance $b$ to the left of the rod’s center, so that when the rod is released from rest, the bug’s end will drop. For what value of $b$ will the bug experience apparent weightlessness at the moment when the rod is released? $\sqrt{\text{}}$

Problem 67. The figure shows a trap door of length $\ell$, which is released at rest from a horizontal position and swings downward under its own weight. The bug stands at a distance $b$ from the hinge. Because the bug feels the floor dropping out from under it with some acceleration, it feels a change in the apparent acceleration of gravity from $g$ to some value $g_a$, at the moment when the door is released. Find $g_a$. $\sqrt{\text{}}$
Exercise 15: Torque

Equipment:

- rulers with holes in them
- spring scales (two per group)

While one person holds the pencil which forms the axle for the ruler, the other members of the group pull on the scale and take readings. In each case, calculate the total torque on the ruler, and find out whether it equals zero to roughly within the accuracy of the experiment. Finish the calculations for each part before moving on to the next one.
Vibrations and Resonance