was the “true” measure of motion. The modern student can certainly be excused for wondering why we need both quantities, when their complementary nature was not evident to the greatest minds of the 1700’s. The following table highlights their differences.

<table>
<thead>
<tr>
<th>kinetic energy . . .</th>
<th>momentum . . .</th>
</tr>
</thead>
<tbody>
<tr>
<td>is a scalar.</td>
<td>is a vector.</td>
</tr>
<tr>
<td>is not changed by a force perpendicular to the motion, which changes only the direction of the velocity vector.</td>
<td>is changed by any force, since a change in either the magnitude or the direction of the velocity vector will result in a change in the momentum vector.</td>
</tr>
<tr>
<td>is always positive, and cannot cancel out.</td>
<td>cancels with momentum in the opposite direction.</td>
</tr>
<tr>
<td>can be traded for other forms of energy that do not involve motion. KE is not a conserved quantity by itself.</td>
<td>is always conserved in a closed system.</td>
</tr>
<tr>
<td>is quadrupled if the velocity is doubled.</td>
<td>is doubled if the velocity is doubled.</td>
</tr>
</tbody>
</table>

_A spinning top_  
A spinning top has zero total momentum, because for every moving point, there is another point on the opposite side that cancels its momentum. It does, however, have kinetic energy.

_Why a tuning fork has two prongs_  
A tuning fork is made with two prongs so that they can vibrate in opposite directions, canceling their momenta. In a hypothetical version with only one prong, the momentum would have to oscillate, and this momentum would have to come from somewhere, such as the hand holding the fork. The result would be that vibrations would be transmitted to the hand and rapidly die out. In a two-prong fork, the two momenta cancel, but the energies don’t.

_Momentum and kinetic energy in firing a rifle_  
The rifle and bullet have zero momentum and zero kinetic energy to start with. When the trigger is pulled, the bullet gains some momentum in the forward direction, but this is canceled by the rifle’s backward momentum, so the total momentum is still zero. The kinetic energies of the gun and bullet are both positive scalars, however, and do not cancel. The total kinetic energy is allowed to increase, because kinetic energy is being traded for other forms of energy. Initially there is chemical energy in the gunpowder. This chemical energy is converted into heat, sound, and kinetic energy. The gun’s “backward” kinetic energy does not refrigerate the shooter’s shoulder!
As the moon completes half a circle around the earth, its motion reverses direction. This does not involve any change in kinetic energy, and the earth's gravitational force does not do any work on the moon. The reversed velocity vector does, however, imply a reversed momentum vector, so conservation of momentum in the closed earth-moon system tells us that the earth must also change its momentum. In fact, the earth wobbles in a little "orbit" about a point below its surface on the line connecting it and the moon. The two bodies' momentum vectors always point in opposite directions and cancel each other out.

Why can't the moon suddenly decide to fly off one way and the earth the other way? It is not forbidden by conservation of momentum, because the moon's newly acquired momentum in one direction could be canceled out by the change in the momentum of the earth, supposing the earth headed the opposite direction at the appropriate, slower speed. The catastrophe is forbidden by conservation of energy, because both their energies would have to increase greatly.

A cubic-kilometer glacier would have a mass of about $10^{12}$ kg. If it moves at a speed of $10^{-5}$ m/s, then its momentum is $10^7$ kg·m/s. This is the kind of heroic-scale result we expect, perhaps the equivalent of the space shuttle taking off, or all the cars in LA driving in the same direction at freeway speed. Its kinetic energy, however, is only 50 J, the equivalent of the calories contained in a poppy seed or the energy in a drop of gasoline too small to be seen without a microscope. The surprisingly small kinetic energy is because kinetic energy is proportional to the square of the velocity, and the square of a small number is an even smaller number.

Discussion questions

A If all the air molecules in the room settled down in a thin film on the floor, would that violate conservation of momentum? Conservation of energy?

B A refrigerator has coils in the back that get hot, and heat is molecular motion. These moving molecules have both energy and momentum. Why doesn’t the refrigerator need to be tied to the wall to keep it from recoiling from the momentum it loses out the back?
14.2 Collisions in one dimension

Physicists employ the term “collision” in a broader sense than ordinary usage, applying it to any situation where objects interact for a certain period of time. A bat hitting a baseball, a radioactively emitted particle damaging DNA, and a gun and a bullet going their separate ways are all examples of collisions in this sense. Physical contact is not even required. A comet swinging past the sun on a hyperbolic orbit is considered to undergo a collision, even though it never touches the sun. All that matters is that the comet and the sun exerted gravitational forces on each other.

The reason for broadening the term “collision” in this way is that all of these situations can be attacked mathematically using the same conservation laws in similar ways. In the first example, conservation of momentum is all that is required.

Getting rear-ended example 10

▷ Ms. Chang is rear-ended at a stop light by Mr. Nelson, and sues to make him pay her medical bills. He testifies that he was only going 35 miles per hour when he hit Ms. Chang. She thinks he was going much faster than that. The cars skidded together after the impact, and measurements of the length of the skid marks and the coefficient of friction show that their joint velocity immediately after the impact was 19 miles per hour. Mr. Nelson’s Nissan weighs 3100 pounds, and Ms. Chang’s Cadillac weighs 5200 pounds. Is Mr. Nelson telling the truth?

▷ Since the cars skidded together, we can write down the equation for conservation of momentum using only two velocities, \( v \) for Mr. Nelson’s velocity before the crash, and \( v' \) for their joint velocity afterward:

\[
m_N v = m_N v' + m_C v'.
\]

Solving for the unknown, \( v \), we find

\[
v = \left(1 + \frac{m_C}{m_N}\right) v'.
\]

Although we are given the weights in pounds, a unit of force, the ratio of the masses is the same as the ratio of the weights, and we find \( v = 51 \) miles per hour. He is lying.

The above example was simple because both cars had the same velocity afterward. In many one-dimensional collisions, however, the two objects do not stick. If we wish to predict the result of such a collision, conservation of momentum does not suffice, because both velocities after the collision are unknown, so we have one equation in two unknowns.

Conservation of energy can provide a second equation, but its application is not as straightforward, because kinetic energy is only the particular form of energy that has to do with motion. In many
collisions, part of the kinetic energy that was present before the collision is used to create heat or sound, or to break the objects or permanently bend them. Cars, in fact, are carefully designed to crumple in a collision. Crumpling the car uses up energy, and that’s good because the goal is to get rid of all that kinetic energy in a relatively safe and controlled way. At the opposite extreme, a superball is “super” because it emerges from a collision with almost all its original kinetic energy, having only stored it briefly as potential energy while it was being squashed by the impact.

Collisions of the superball type, in which almost no kinetic energy is converted to other forms of energy, can thus be analyzed more thoroughly, because they have $KE_f = KE_i$, as opposed to the less useful inequality $KE_f < KE_i$ for a case like a tennis ball bouncing on grass. These two types of collisions are referred to, respectively, as elastic and inelastic. The extreme inelastic case is discussed further on p. ??.

### Pool balls colliding head-on example 11

Two pool balls collide head-on, so that the collision is restricted to one dimension. Pool balls are constructed so as to lose as little kinetic energy as possible in a collision, so under the assumption that no kinetic energy is converted to any other form of energy, what can we predict about the results of such a collision?

Pool balls have identical masses, so we use the same symbol $m$ for both. Conservation of momentum and no loss of kinetic energy give us the two equations

$$m v_{1i} + m v_{2i} = m v_{1f} + m v_{2f}$$

$$\frac{1}{2} m v_{1i}^2 + \frac{1}{2} m v_{2i}^2 = \frac{1}{2} m v_{1f}^2 + \frac{1}{2} m v_{2f}^2$$

The masses and the factors of $1/2$ can be divided out, and we eliminate the cumbersome subscripts by replacing the symbols $v_{1i}$, $v_{2i}$, $v_{1f}$, $v_{2f}$ with the symbols $A$, $B$, $C$, and $D$:

$$A + B = C + D$$


A little experimentation with numbers shows that given values of $A$ and $B$, it is impossible to find $C$ and $D$ that satisfy these equations unless $C$ and $D$ equal $A$ and $B$, or $C$ and $D$ are the same as $A$ and $B$ but swapped around. A formal proof of this fact is given in the sidebar. In the special case where ball 2 is initially at rest, this tells us that ball 1 is stopped dead by the collision, and ball 2 heads off at the velocity originally possessed by ball 1. This behavior will be familiar to players of pool.

Often, as in the example above, the details of the algebra are the least interesting part of the problem, and considerable physical
insight can be gained simply by counting the number of unknowns and comparing to the number of equations. Suppose a beginner at
pool notices a case where her cue ball hits an initially stationary
bet I could never do that again in a million years.” But she tries
again, and finds that she can’t help doing it even if she doesn’t
want to. Luckily she has just learned about collisions in her physics
course. Once she has written down the equations for conservation
of energy and no loss of kinetic energy, she really doesn’t have to
complete the algebra. She knows that she has two equations in
two unknowns, so there must be a well-defined solution. Once she
has seen the result of one such collision, she knows that the same
thing must happen every time. The same thing would happen with
colliding marbles or croquet balls. It doesn’t matter if the masses or
velocities are different, because that just multiplies both equations
by some constant factor.

The discovery of the neutron

This was the type of reasoning employed by James Chadwick in
his 1932 discovery of the neutron. At the time, the atom was imag-
ined to be made out of two types of fundamental particles, protons
and electrons. The protons were far more massive, and clustered
together in the atom’s core, or nucleus. Attractive electrical forces
causd the electrons to orbit the nucleus in circles, in much the
same way that gravitational forces kept the planets from cruising
out of the solar system. Experiments showed that the helium nu-
ucleus, for instance, exerted exactly twice as much electrical force on
an electron as a nucleus of hydrogen, the smallest atom, and this was
explained by saying that helium had two protons to hydrogen’s one.
The trouble was that according to this model, helium would have
two electrons and two protons, giving it precisely twice the mass of
a hydrogen atom with one of each. In fact, helium has about four
times the mass of hydrogen.

Chadwick suspected that the helium nucleus possessed two addi-
tional particles of a new type, which did not participate in electrical
forces at all, i.e., were electrically neutral. If these particles had very
nearly the same mass as protons, then the four-to-one mass ratio of
helium and hydrogen could be explained. In 1930, a new type of
radiation was discovered that seemed to fit this description. It was
electrically neutral, and seemed to be coming from the nuclei of light
elements that had been exposed to other types of radiation. At this
time, however, reports of new types of particles were a dime a dozen,
and most of them turned out to be either clusters made of previ-
ously known particles or else previously known particles with higher
energies. Many physicists believed that the “new” particle that had
attracted Chadwick’s interest was really a previously known particle
called a gamma ray, which was electrically neutral. Since gamma
rays have no mass, Chadwick decided to try to determine the new particle’s mass and see if it was nonzero and approximately equal to the mass of a proton.

Unfortunately a subatomic particle is not something you can just put on a scale and weigh. Chadwick came up with an ingenious solution. The masses of the nuclei of the various chemical elements were already known, and techniques had already been developed for measuring the speed of a rapidly moving nucleus. He therefore set out to bombard samples of selected elements with the mysterious new particles. When a direct, head-on collision occurred between a mystery particle and the nucleus of one of the target atoms, the nucleus would be knocked out of the atom, and he would measure its velocity.

Suppose, for instance, that we bombard a sample of hydrogen atoms with the mystery particles. Since the participants in the collision are fundamental particles, there is no way for kinetic energy to be converted into heat or any other form of energy, and Chadwick thus had two equations in three unknowns:
equation #1: conservation of momentum
equation #2: no loss of kinetic energy
unknown #1: mass of the mystery particle
unknown #2: initial velocity of the mystery particle
unknown #3: final velocity of the mystery particle

The number of unknowns is greater than the number of equations, so there is no unique solution. But by creating collisions with nuclei of another element, nitrogen, he gained two more equations at the expense of only one more unknown:

equation #3: conservation of momentum in the new collision
equation #4: no loss of kinetic energy in the new collision
unknown #4: final velocity of the mystery particle in the new collision

He was thus able to solve for all the unknowns, including the mass of the mystery particle, which was indeed within 1% of the mass of a proton. He named the new particle the neutron, since it is electrically neutral.

Discussion question
A  Good pool players learn to make the cue ball spin, which can cause it not to stop dead in a head-on collision with a stationary ball. If this does not violate the laws of physics, what hidden assumption was there in the example above?

14.3  Relationship of momentum to the center of mass

In this multiple-flash photograph, we see the wrench from above as it flies through the air, rotating as it goes. Its center of mass, marked with the black cross, travels along a straight line, unlike the other points on the wrench, which execute loops.
thing like a wrench, which we think of as one object, is really made of many atoms. The center of mass is particularly easy to visualize in the case shown on the left, where two identical hockey pucks collide. It is clear on grounds of symmetry that their center of mass must be at the midpoint between them. After all, we previously defined the center of mass as the balance point, and if the two hockey pucks were joined with a very lightweight rod whose own mass was negligible, they would obviously balance at the midpoint. It doesn’t matter that the hockey pucks are two separate objects. It is still true that the motion of their center of mass is exceptionally simple, just like that of the wrench’s center of mass.

The $x$ coordinate of the hockey pucks’ center of mass is thus given by $x_{\text{cm}} = (x_1 + x_2)/2$, i.e., the arithmetic average of their $x$ coordinates. Why is its motion so simple? It has to do with conservation of momentum. Since the hockey pucks are not being acted on by any net external force, they constitute a closed system, and their total momentum is conserved. Their total momentum is

$$mv_1 + mv_2 = m(v_1 + v_2)$$

$$= m\left(\frac{\Delta x_1}{\Delta t} + \frac{\Delta x_2}{\Delta t}\right)$$

$$= \frac{m}{\Delta t} \Delta (x_1 + x_2)$$

$$= m \frac{2\Delta x_{\text{cm}}}{\Delta t}$$

$$= m_{\text{total}} v_{\text{cm}}$$

In other words, the total momentum of the system is the same as if all its mass was concentrated at the center of mass point. Since the total momentum is conserved, the $x$ component of the center of mass’s velocity vector cannot change. The same is also true for the other components, so the center of mass must move along a straight line at constant speed.

The above relationship between the total momentum and the motion of the center of mass applies to any system, even if it is not closed.

**total momentum related to center of mass motion**

The total momentum of any system is related to its total mass and the velocity of its center of mass by the equation

$$P_{\text{total}} = m_{\text{total}} v_{\text{cm}}.$$ 

What about a system containing objects with unequal masses, or containing more than two objects? The reasoning above can be generalized to a weighted average

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots},$$

with similar equations for the $y$ and $z$ coordinates.
Momentum in different frames of reference

Absolute motion is supposed to be undetectable, i.e., the laws of physics are supposed to be equally valid in all inertial frames of reference. If we first calculate some momenta in one frame of reference and find that momentum is conserved, and then rework the whole problem in some other frame of reference that is moving with respect to the first, the numerical values of the momenta will all be different. Even so, momentum will still be conserved. All that matters is that we work a single problem in one consistent frame of reference.

One way of proving this is to apply the equation $p_{total} = m_{total}v_{cm}$. If the velocity of frame B relative to frame A is $v_{BA}$, then the only effect of changing frames of reference is to change $v_{cm}$ from its original value to $v_{cm} + v_{BA}$. This adds a constant onto the momentum vector, which has no effect on conservation of momentum.

The center of mass frame of reference

A particularly useful frame of reference in many cases is the frame that moves along with the center of mass, called the center of mass (c.m.) frame. In this frame, the total momentum is zero. The following examples show how the center of mass frame can be a powerful tool for simplifying our understanding of collisions.

A collision of pool balls viewed in the c.m. frame example 12
If you move your head so that your eye is always above the point halfway in between the two pool balls, you are viewing things in the center of mass frame. In this frame, the balls come toward the center of mass at equal speeds. By symmetry, they must therefore recoil at equal speeds along the lines on which they entered. Since the balls have essentially swapped paths in the center of mass frame, the same must also be true in any other frame. This is the same result that required laborious algebra to prove previously without the concept of the center of mass frame.

The slingshot effect example 13
It is a counterintuitive fact that a spacecraft can pick up speed by swinging around a planet, if it arrives in the opposite direction compared to the planet’s motion. Although there is no physical contact, we treat the encounter as a one-dimensional collision, and analyze it in the center of mass frame. Figure j shows such a “collision,” with a space probe whipping around Jupiter. In the sun’s frame of reference, Jupiter is moving.

What about the center of mass frame? Since Jupiter is so much more massive than the spacecraft, the center of mass is essentially fixed at Jupiter’s center, and Jupiter has zero velocity in the center of mass frame, as shown in figure k. The c.m. frame is moving to the left compared to the sun-fixed frame used in j, so
the spacecraft’s initial velocity is greater in this frame.

Things are simpler in the center of mass frame, because it is more symmetric. In the complicated sun-fixed frame, the incoming leg of the encounter is rapid, because the two bodies are rushing toward each other, while their separation on the outbound leg is more gradual, because Jupiter is trying to catch up. In the c.m. frame, Jupiter is sitting still, and there is perfect symmetry between the incoming and outgoing legs, so by symmetry we have $v_{1f} = -v_{1i}$. Going back to the sun-fixed frame, the spacecraft’s final velocity is increased by the frames’ motion relative to each other. In the sun-fixed frame, the spacecraft’s velocity has increased greatly.

The result can also be understood in terms of work and energy. In Jupiter’s frame, Jupiter is not doing any work on the spacecraft as it rounds the back of the planet, because the motion is perpendicular to the force. But in the sun’s frame, the spacecraft’s velocity vector at the same moment has a large component to the left, so Jupiter is doing work on it.

Discussion questions

A Make up a numerical example of two unequal masses moving in one dimension at constant velocity, and verify the equation $p_{\text{total}} = m_{\text{total}}v_{\text{cm}}$ over a time interval of one second.

B A more massive tennis racquet or baseball bat makes the ball fly off faster. Explain why this is true, using the center of mass frame. For simplicity, assume that the racquet or bat is simply sitting still before the collision, and that the hitter’s hands do not make any force large enough to have a significant effect over the short duration of the impact.

Totally inelastic collisions

On p. 404 we discussed collisions that were totally elastic (no conversion of KE into other types of energy). A useful application of the center of mass frame of reference is to the description of the opposite extreme, a totally inelastic collision.

A totally inelastic collision cannot just be defined as one in which all the KE is converted into other forms, both because the definition would depend on our frame of reference and because there is a constraint imposed by conservation of momentum. Let’s say that a golfer hits a ball. In the frame of reference of the grass, it would violate conservation of momentum if the ball were to stay put while the club simply stopped moving. If such a complete cessation of motion is to happen, then it must occur in the center of mass frame of reference. In the c.m. frame, there is zero total momentum both before and after the collision. Thus if we observe no motion at all after the collision, we must be in the c.m. frame.

Therefore we define a totally inelastic collision as one in which there is no motion in the c.m. frame in the final state. An observer
watching such a collision, in any frame, will see that the amount of KE transformed into other forms of energy is as great as possible subject to conservation of momentum.

When objects touch physically (possibly crumpling or changing shape during the collision) in a totally elastic collision, the final state in the c.m. frame is one in which the two objects are at rest and touching. In other frames of reference, we see the objects stick to each other and travel away together after the collision. An example of this type was example 10 on p. 403, in which one car rear-ended another, and they stuck together as a unit after the crash.

14.4 Momentum transfer

The rate of change of momentum

As with conservation of energy, we need a way to measure and calculate the transfer of momentum into or out of a system when the system is not closed. In the case of energy, the answer was rather complicated, and entirely different techniques had to be used for measuring the transfer of mechanical energy (work) and the transfer of heat by conduction. For momentum, the situation is far simpler.

In the simplest case, the system consists of a single object acted on by a constant external force. Since it is only the object’s velocity that can change, not its mass, the momentum transferred is

$$\Delta p = m \Delta v,$$

which with the help of $$a = F/m$$ and the constant-acceleration equation $$a = \Delta v/\Delta t$$ becomes

$$\Delta p = ma\Delta t = F\Delta t.$$

Thus the rate of transfer of momentum, i.e., the number of kg·m/s absorbed per second, is simply the external force,

$$F = \frac{\Delta p}{\Delta t}.$$

[relationship between the force on an object and the rate of change of its momentum; valid only if the force is constant]

This is just a restatement of Newton’s second law, and in fact Newton originally stated it this way. As shown in figure 1, the relationship between force and momentum is directly analogous to that between power and energy.

The situation is not materially altered for a system composed of many objects. There may be forces between the objects, but the internal forces cannot change the system’s momentum. (If they did,
then removing the external forces would result in a closed system that could change its own momentum, like the mythical man who could pull himself up by his own bootstraps. That would violate conservation of momentum.) The equation above becomes

\[ F_{\text{total}} = \frac{\Delta p_{\text{total}}}{\Delta t}. \]

[relationship between the total external force on a system and the rate of change of its total momentum; valid only if the force is constant]

**Walking into a lamppost**

Starting from rest, you begin walking, bringing your momentum up to 100 kg·m/s. You walk straight into a lamppost. Why is the momentum change of −100 kg·m/s caused by the lamppost so much more painful than the change of +100 kg·m/s when you started walking?

The situation is one-dimensional, so we can dispense with the vector notation. It probably takes you about 1 s to speed up initially, so the ground’s force on you is \( F = \Delta p/\Delta t \approx 100 \text{ N} \). Your impact with the lamppost, however, is over in the blink of an eye, say 1/10 s or less. Dividing by this much smaller \( \Delta t \) gives a much larger force, perhaps thousands of newtons. (The negative sign simply indicates that the force is in the opposite direction.)

This is also the principle of airbags in cars. The time required for the airbag to decelerate your head is fairly long, the time required for your face to travel 20 or 30 cm. Without an airbag, your face would hit the dashboard, and the time interval would be the much shorter time taken by your skull to move a couple of centimeters while your face compressed. Note that either way, the same amount of mechanical work has to be done on your head: enough to eliminate all its kinetic energy.

**Ion drive for spacecraft**

The ion drive of the Deep Space 1 spacecraft, pictured on page 399 and discussed in example 2, produces a thrust of 90 mN (millinewtons). It carries about 80 kg of reaction mass, which it ejects at a speed of 30,000 m/s. For how long can the engine continue supplying this amount of thrust before running out of reaction mass to shove out the back?

Solving the equation \( F = \Delta p/\Delta t \) for the unknown \( \Delta t \), and treating force and momentum as scalars since the problem is one-
Example 16. Dimensional, we find

\[ \Delta t = \frac{\Delta p}{F} = \frac{m_{\text{exhaust}} \Delta v_{\text{exhaust}}}{F} = \frac{(80 \text{ kg})(30,000 \text{ m/s})}{0.090 \text{ N}} = 2.7 \times 10^7 \text{ s} = 300 \text{ days} \]

A toppling box example 16

If you place a box on a frictionless surface, it will fall over with a very complicated motion that is hard to predict in detail. We know, however, that its center of mass moves in the same direction as its momentum vector points. There are two forces, a normal force and a gravitational force, both of which are vertical. (The gravitational force is actually many gravitational forces acting on all the atoms in the box.) The total force must be vertical, so the momentum vector must be purely vertical too, and the center of mass travels vertically. This is true even if the box bounces and tumbles. [Based on an example by Kleppner and Kolenkow.]

Discussion question

Many collisions, like the collision of a bat with a baseball, appear to be instantaneous. Most people also would not imagine the bat and ball as bending or being compressed during the collision. Consider the following possibilities:

1. The collision is instantaneous.

2. The collision takes a finite amount of time, during which the ball and bat retain their shapes and remain in contact.

3. The collision takes a finite amount of time, during which the ball and bat are bending or being compressed.

How can two of these be ruled out based on energy or momentum considerations?
14.5 Momentum in three dimensions

In this section we discuss how the concepts applied previously to one-dimensional situations can be used as well in three dimensions. Often vector addition is all that is needed to solve a problem:

An explosion

Astronomers observe the planet Mars as the Martians fight a nuclear war. The Martian bombs are so powerful that they rip the planet into three separate pieces of liquified rock, all having the same mass. If one fragment flies off with velocity components

\[ v_{1x} = 0 \]
\[ v_{1y} = 1.0 \times 10^4 \text{ km/hr}, \]

and the second with

\[ v_{2x} = 1.0 \times 10^4 \text{ km/hr} \]
\[ v_{2y} = 0, \]

(all in the center of mass frame) what is the magnitude of the third one’s velocity?

In the center of mass frame, the planet initially had zero momentum. After the explosion, the vector sum of the momenta must still be zero. Vector addition can be done by adding components, so

\[ mv_{1x} + mv_{2x} + mv_{3x} = 0, \quad \text{and} \]
\[ mv_{1y} + mv_{2y} + mv_{3y} = 0, \]

where we have used the same symbol \( m \) for all the terms, because the fragments all have the same mass. The masses can be eliminated by dividing each equation by \( m \), and we find

\[ v_{3x} = -1.0 \times 10^4 \text{ km/hr} \]
\[ v_{3y} = -1.0 \times 10^4 \text{ km/hr} \]

which gives a magnitude of

\[ |v_3| = \sqrt{v_{3x}^2 + v_{3y}^2} \]
\[ = 1.4 \times 10^4 \text{ km/hr} \]

The center of mass

In three dimensions, we have the vector equations

\[ F_{total} = \frac{\Delta p_{total}}{\Delta t} \]

and

\[ p_{total} = m_{total}v_{cm}. \]

The following is an example of their use.
The bola, similar to the North American lasso, is used by South American gauchos to catch small animals by tangling up their legs in the three leather thongs. The motion of the whirling bola through the air is extremely complicated, and would be a challenge to analyze mathematically. The motion of its center of mass, however, is much simpler. The only forces on it are gravitational, so

\[ \mathbf{F}_{\text{total}} = m_{\text{total}} \mathbf{g} \]

Using the equation \( \mathbf{F}_{\text{total}} = \Delta \mathbf{p}_{\text{total}} / \Delta t \), we find

\[ \Delta \mathbf{p}_{\text{total}} / \Delta t = m_{\text{total}} \mathbf{g} \]

and since the mass is constant, the equation \( \mathbf{p}_{\text{total}} = m_{\text{total}} \mathbf{v}_{cm} \) allows us to change this to

\[ m_{\text{total}} \Delta \mathbf{v}_{cm} / \Delta t = m_{\text{total}} \mathbf{g} \]

The mass cancels, and \( \Delta \mathbf{v}_{cm} / \Delta t \) is simply the acceleration of the center of mass, so

\[ a_{cm} = g \]

In other words, the motion of the system is the same as if all its mass was concentrated at and moving with the center of mass. The bola has a constant downward acceleration equal to \( g \), and flies along the same parabola as any other projectile thrown with the same initial center of mass velocity. Throwing a bola with the correct rotation is presumably a difficult skill, but making it hit its target is no harder than it is with a ball or a single rock.

[Based on an example by Kleppner and Kolenkow.]

Counting equations and unknowns

Counting equations and unknowns is just as useful as in one dimension, but every object’s momentum vector has three components, so an unknown momentum vector counts as three unknowns. Conservation of momentum is a single vector equation, but it says that all three components of the total momentum vector stay constant, so we count it as three equations. Of course if the motion happens to be confined to two dimensions, then we need only count vectors as having two components.

A two-car crash with sticking

Suppose two cars collide, stick together, and skid off together. If we know the cars’ initial momentum vectors, we can count equations and unknowns as follows:

unknown #1: \( x \) component of cars’ final, total momentum
unknown #2: \( y \) component of cars’ final, total momentum
equation #1: conservation of the total \( p_x \)
equation #2: conservation of the total $p_y$

Since the number of equations equals the number of unknowns, there must be one unique solution for their total momentum vector after the crash. In other words, the speed and direction at which their common center of mass moves off together is unaffected by factors such as whether the cars collide center-to-center or catch each other a little off-center.

**Shooting pool example 20**

Two pool balls collide, and as before we assume there is no decrease in the total kinetic energy, i.e., no energy converted from KE into other forms. As in the previous example, we assume we are given the initial velocities and want to find the final velocities. The equations and unknowns are:

unknown #1: $x$ component of ball #1’s final momentum
unknown #2: $y$ component of ball #1’s final momentum
unknown #3: $x$ component of ball #2’s final momentum
unknown #4: $y$ component of ball #2’s final momentum

equation #1: conservation of the total $p_x$
equation #2: conservation of the total $p_y$
equation #3: no decrease in total KE

Note that we do not count the balls’ final kinetic energies as unknowns, because knowing the momentum vector, one can always find the velocity and thus the kinetic energy. The number of equations is less than the number of unknowns, so no unique result is guaranteed. This is what makes pool an interesting game. By aiming the cue ball to one side of the target ball you can have some control over the balls’ speeds and directions of motion after the collision.

It is not possible, however, to choose any combination of final speeds and directions. For instance, a certain shot may give the correct direction of motion for the target ball, making it go into a pocket, but may also have the undesired side-effect of making the cue ball go in a pocket.

**Calculations with the momentum vector**

The following example illustrates how a force is required in order to change the direction of the momentum vector, just as one would be required to change its magnitude.
Example 21.

In a hydroelectric plant, water flowing over a dam drives a turbine, which runs a generator to make electric power. The figure shows a simplified physical model of the water hitting the turbine, in which it is assumed that the stream of water comes in at a 45° angle with respect to the turbine blade, and bounces off at a 90° angle at nearly the same speed. The water flows at a rate $R$, in units of kg/s, and the speed of the water is $v$. What are the magnitude and direction of the water’s force on the turbine?

In a time interval $\Delta t$, the mass of water that strikes the blade is $R\Delta t$, and the magnitude of its initial momentum is $mv = vR\Delta t$. The water’s final momentum vector is of the same magnitude, but in the perpendicular direction. By Newton’s third law, the water’s force on the blade is equal and opposite to the blade’s force on the water. Since the force is constant, we can use the equation

$$F_{\text{blade on water}} = \frac{\Delta p_{\text{water}}}{\Delta t}.$$  

Choosing the $x$ axis to be to the right and the $y$ axis to be up, this can be broken down into components as

$$F_{\text{blade on water},x} = \frac{\Delta p_{\text{water},x}}{\Delta t} = -vR\Delta t - 0 = -vR$$

and

$$F_{\text{blade on water},y} = \frac{\Delta p_{\text{water},y}}{\Delta t} = 0 - (-vR\Delta t) = vR.$$

The water’s force on the blade thus has components

$$F_{\text{water on blade},x} = vR$$

$$F_{\text{water on blade},y} = -vR.$$  

In situations like this, it is always a good idea to check that the result makes sense physically. The $x$ component of the water’s force on the blade is positive, which is correct since we know the blade will be pushed to the right. The $y$ component is negative, which also makes sense because the water must push the blade down. The magnitude of the water’s force on the blade is

$$|F_{\text{water on blade}}| = \sqrt{2}vR$$

and its direction is at a 45-degree angle down and to the right.
Discussion questions

A  The figures show a jet of water striking two different objects. How does the total downward force compare in the two cases? How could this fact be used to create a better waterwheel? (Such a waterwheel is known as a Pelton wheel.)

B  In problem 12, p. 335, we analyzed a multiflash photograph collision between two steel balls to check for conservation of energy. The photo is reproduced below. Check conservation of momentum as well.

14.6 Applications of calculus

Few real collisions involve a constant force. For example, when a tennis ball hits a racquet, the strings stretch and the ball flattens dramatically. They are both acting like springs that obey Hooke’s law, which says that the force is proportional to the amount of stretching or flattening. The force is therefore small at first, ramps up to a maximum when the ball is about to reverse directions, and ramps back down again as the ball is on its way back out. The equation \( F = \Delta p/\Delta t \), derived under the assumption of constant acceleration, does not apply here, and the force does not even have
a single well-defined numerical value that could be plugged in to the equation.

This is like every other situation where an equation of the form \( f \circ o = \Delta \text{bar} / \Delta \text{baz} \) has to be generalized to the case where the rate of change isn't constant. We have \( F = dp/\Delta t \) and, by the fundamental theorem of calculus, \( \Delta p = \int F \Delta t \), which can be interpreted as the area under the \( F - t \) graph, figure s.

Rain falling into a moving cart example 22

\( \triangleright \) If 1 kg/s of rain falls vertically into a 10-kg cart that is rolling without friction at an initial speed of 1.0 m/s, what is the effect on the speed of the cart when the rain first starts falling?

\( \triangleright \) The rain and the cart make horizontal forces on each other, but there is no external horizontal force on the rain-plus-cart system, so the horizontal motion obeys

\[
F = \frac{d(mv)}{dt} = 0
\]

We use the product rule to find

\[
0 = \frac{dm}{dt} v + m \frac{dv}{dt}
\]

We are trying to find how \( v \) changes, so we solve for \( dv/dt \),

\[
\frac{dv}{dt} = -\frac{v \frac{dm}{dt}}{m \frac{dt}{dt}}
\]

\[
= -\left( \frac{1 \text{ m/s}}{10 \text{ kg}} \right) (1 \text{ kg/s})
\]

\[
= -0.1 \text{ m/s}^2.
\]

(This is only at the moment when the rain starts to fall.)

Finally we note that there are cases where \( F = ma \) is not just less convenient than \( F = dp/\Delta t \) but in fact \( F = ma \) is wrong and \( F = dp/\Delta t \) is right. A good example is the formation of a comet’s tail by sunlight. We cannot use \( F = ma \) to describe this process, since we are dealing with a collision of light with matter, whereas Newton’s laws only apply to matter. The equation \( F = dp/\Delta t \), on the other hand, allows us to find the force experienced by an atom of gas in the comet’s tail if we know the rate at which the momentum vectors of light rays are being turned around by reflection from the atom.
14.7 Relativistic momentum

How does momentum behave in relativity?

Newtonian mechanics has two different measures of motion, kinetic energy and momentum, and the relationship between them is nonlinear. Doubling your car’s momentum quadruples its kinetic energy.

But nonrelativistic mechanics can’t handle massless particles, which are always ultrarelativistic. We saw in section 11.6 that ultrarelativistic particles are “generic,” in the sense that they have no individual mechanical properties other than an energy and a direction of motion. Therefore the relationship between kinetic energy and momentum must be linear for ultrarelativistic particles. Indeed, experiments verify that light has momentum, and doubling the energy of a ray of light doubles its momentum rather than quadrupling it.

How can we make sense of these energy-momentum relationships, which seem to take on two completely different forms in the limiting cases of very low and very high velocities?

The first step is realize that since mass and energy are equivalent (section 12.5), we will get more of an apples-to-apples comparison if we stop talking about a material object’s kinetic energy and consider instead its total energy $E$, which includes a contribution from its mass.

On a graph of $p$ versus $E$, massless particles, which have $E \propto |p|$, lie on two diagonal lines that connect at the origin. If we like, we can pick units such that the slopes of these lines are plus and minus one. Material particles lie to the right of these lines. For example, a car sitting in a parking lot has $p = 0$ and $E = mc^2$.

Now what happens to such a graph when we change to a different frame or reference that is in motion relative to the original frame? A massless particle still has to act like a massless particle, so the diagonals are simply stretched or contracted along their own lengths. A transformation that always takes a line to a line is a linear transformation (p. 80), and if the transformation between different frames of reference preserves the linearity of the lines $p = E$ and $p = -E$, then it’s natural to suspect that it is actually some kind of linear transformation. In fact the transformation must be linear (p. 80), because conservation of energy and momentum involve addition, and we need these laws to be valid in all frames of reference. By the same reasoning as in figure am on p. 83, the transformation must be area-preserving. We then have the same three cases to consider as in figure aj on p. 82. Case I is ruled out because it would imply that particles keep the same energy when we change frames. (This is what would happen if $c$ were infinite, so that the mass-equivalent $E/c^2$ of a given energy was zero, and there-
fore $E$ would be interpreted purely as the mass.) Case II can’t be right because it doesn’t preserve the $E = |p|$ diagonals. We are left with case III, which establishes the following aesthetically appealing fact: the $p$-$E$ plane transforms according to exactly the same kind of Lorentz transformation as the $x$-$t$ plane. That is, $(E, p_x, p_y, p_z)$ is a four-vector (p. 381) just like $(t, x, y, z)$. This is a highly desirable result. If it were not true, it would be like having to learn different mathematical rules for different kinds of three-vectors in Newtonian mechanics.

The only remaining issue to settle is whether the choice of units that gives invariant 45-degree diagonals in the $x$-$t$ plane is the same as the choice of units that gives such diagonals in the $p$-$E$ plane. That is, we need to establish that the $c$ that applies to $x$ and $t$ is equal to the $c'$ needed for $p$ and $E$, i.e., that the velocity scales of the two graphs are matched up. This is true because in the Newtonian limit, the total mass-energy $E$ is essentially just the particle’s mass, and then $p/E \approx p/m \approx v$. This establishes that the velocity scales are matched at small velocities, which implies that they coincide for all velocities, since a large velocity, even one approaching $c$, can be built up from many small increments. (This also establishes that the exponent $n$ defined on p. 329 equals 1 as claimed.)

Suppose that a particle is at rest. Then it has $p = 0$ and mass-energy $E$ equal to its mass $m$. Therefore the inner product of its $(E, p)$ four-vector with itself equals $m^2$. In other words, the “magnitude” of the energy-momentum four-vector is simply equal to the particle’s mass. If we transform into a different frame of reference, the inner product stays the same. We can therefore always interpret the magnitude of an energy-momentum four-vector as the mass. In symbols,

$$m^2 = E^2 - p^2,$$

or, in units with $c \neq 1$,

$$(mc^2)^2 = E^2 - (pc)^2.$$

**self-check A**

Interpret this relationship in the case where $m = 0$. \(\triangleright\) Answer, p. 570

Since we already have an equation $E = m\gamma$ for the energy of a material particle in terms of its velocity, we can find a similar equation for the momentum:

$$p = \sqrt{E^2 - m^2} = m\sqrt{\gamma^2 - 1} = m\sqrt{\frac{1}{1 - v^2} - 1} = m\gamma v.$$
Two early high-precision tests of the relativistic equation \( p = mv \) for momentum. Graphing \( p/m \) rather than \( p \) allows the data for electrons and protons to be placed on the same graph. Natural units are used, so that the horizontal axis is the velocity in units of \( c \), and the vertical axis is the unitless quantity \( p/mc \). The very small error bars for the data point from Zrelov are represented by the height of the black rectangle.

As a material particle gets closer and closer to \( c \), its momentum approaches infinity, so that an infinite force would be required in order to reach \( c \). Figure u shows experimental data confirming the relativistic equation for momentum.

\[ \text{Light rays don't interact} \]

We observe that when two rays of light cross paths, they continue through one another without bouncing like material objects. This behavior follows directly from conservation of energy-momentum.

Any two vectors can be contained in a single plane, so we can choose our coordinates so that both rays have vanishing \( p_z \). By choosing the state of motion of our coordinate system appropriately, we can also make \( p_y = 0 \), so that the collision takes place along a single line parallel to the \( x \) axis. Since only \( p_x \) is nonzero, we write it simply as \( p \). In the resulting \( p-E \) plane, there are two possibilities: either the rays both lie along the same diagonal, or they lie along different diagonals. If they lie along the same diagonal, then there can't be a collision, because the two rays are both moving in the same direction at the same speed \( c \), and the trailing one will never catch up with the leading one.

Now suppose they lie along different diagonals. We add their energy-momentum vectors to get their total energy-momentum, which will lie in the gray area of figure t. That is, a pair of light rays taken as a single system act sort of like a material object with a nonzero mass.\(^1\) By a Lorentz transformation, we can always find a frame in which this total energy-momentum vector lies along the \( E \) axis. This is a frame in which the momenta of the two rays cancel, and we have a symmetric head-on collision between two rays of equal energy. It is the “center-of-mass” frame, although neither object has any mass on an individual basis. For convenience, let's assume that the \( x-y-z \) coordinate system was chosen so that its origin was at rest in this frame.

Since the collision occurs along the \( x \) axis, by symmetry it is not possible for the rays after the collision to depart from the \( x \) axis; for if they did, then there would be nothing to determine the orientation of the plane in which they emerged.\(^2\) Therefore we are justified in continuing to use the same \( p_x-E \) plane to analyze the four-vectors of the rays after the collision.

Let each ray have energy \( E \) in the frame described above. Given this total energy-momentum vector, how can we cook up two

---

\(^1\)If you construct a box out of mirrors and put some light inside, it has weight, and theoretically even has a gravitational field! This is an example of the fact that mass is not additive in relativity. Two objects, each with zero mass, can have an aggregate mass that is nonzero.

\(^2\)In quantum mechanics, there is a loophole here. Quantum mechanics allows certain kinds of randomness, so that the symmetry can be broken by letting the outgoing rays be observed in a plane with some random orientation.
energy-momentum vectors for the final state such that energy and momentum will have been conserved? Since there is zero total momentum, our only choice is two light rays, one with energy-momentum vector \((E, E)\) and one with \((E, -E)\). But this is exactly the same as our initial state, except that we can arbitrarily choose the roles of the two rays to have been interchanged. Such an interchanging is only a matter of labeling, so there is no observable sense in which the rays have collided.⁴

There is a second loophole here, which is that a ray of light is actually a wave, and a wave has other properties besides energy and momentum. It has a wavelength, and some waves also have a property called polarization. As a mechanical analogy for polarization, consider a rope stretched taut. Side-to-side vibrations can propagate along the rope, and these vibrations can occur in any plane that coincides with the rope. The orientation of this plane is referred to as the polarization of the wave. Returning to the case of the colliding light rays, it is possible to have nontrivial collisions in the sense that the rays could affect one another’s wavelengths and polarizations. Although this doesn’t actually happen with non-quantum-mechanical light waves, it can happen with other types of waves; see, e.g., Hu et al., arxiv.org/abs/hep-ph/9502276, figure 2. The title of example 23 is only valid if a “ray” is taken to be something that lacks wave structure. The wave nature of light is not evident in everyday life from observations with apparatus such as flashlights, mirrors, and eyeglasses, so we expect the result to hold under those circumstances, and it does. E.g., flashlight beams do pass through one another without interacting.

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Summary

Selected vocabulary
- momentum . . . a measure of motion, equal to $mv$ for material objects
- collision . . . . an interaction between moving objects that lasts for a certain time
- center of mass . . the balance point or average position of the mass in a system

Notation
- $\mathbf{p}$ . . . . . . the momentum vector
- $x_{\text{cm}}, a_{\text{cm}}$, etc.
- $\mathbf{cm}$ . . . . . . center of mass

Other terminology and notation
- impulse, $I, J$ . . the amount of momentum transferred, $\Delta p$
- elastic collision . one in which no KE is converted into other forms of energy
- inelastic collision . one in which some KE is converted to other forms of energy

Summary

If two objects interact via a force, Newton’s third law guarantees that any change in one’s velocity vector will be accompanied by a change in the other’s which is in the opposite direction. Intuitively, this means that if the two objects are not acted on by any external force, they cannot cooperate to change their overall state of motion. This can be made quantitative by saying that the quantity $m_1v_1 + m_2v_2$ must remain constant as long as the only forces are the internal ones between the two objects. This is a conservation law, called the conservation of momentum, and like the conservation of energy, it has evolved over time to include more and more phenomena unknown at the time the concept was invented. The momentum of a material object is

$$\mathbf{p} = mv,$$

but this is more like a standard for comparison of momenta rather than a definition. For instance, light has momentum, but has no mass, and the above equation is not the right equation for light. The law of conservation of momentum says that the total momentum of any closed system, i.e., the vector sum of the momentum vectors of all the things in the system, is a constant.

An important application of the momentum concept is to collisions, i.e., interactions between moving objects that last for a certain amount of time while the objects are in contact or near each other. Conservation of momentum tells us that certain outcomes of a collision are impossible, and in some cases may even be sufficient to predict the motion after the collision. In other cases, conservation of momentum does not provide enough equations to find all the unknowns. In some collisions, such as the collision of a superball with
the floor, very little kinetic energy is converted into other forms of energy, and this provides one more equation, which may suffice to predict the outcome.

The total momentum of a system can be related to its total mass and the velocity of its center of mass by the equation

\[ \mathbf{p}_{\text{total}} = m_{\text{total}} \mathbf{v}_{\text{cm}}. \]

The center of mass, introduced on an intuitive basis in book 1 as the “balance point” of an object, can be generalized to any system containing any number of objects, and is defined mathematically as the weighted average of the positions of all the parts of all the objects,

\[ x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots}, \]

with similar equations for the \( y \) and \( z \) coordinates.

The frame of reference moving with the center of mass of a closed system is always a valid inertial frame, and many problems can be greatly simplified by working them in the inertial frame. For example, any collision between two objects appears in the c.m. frame as a head-on one-dimensional collision.

When a system is not closed, the rate at which momentum is transferred in or out is simply the total force being exerted externally on the system,

\[ \mathbf{F}_{\text{total}} = \frac{d\mathbf{p}_{\text{total}}}{dt}. \]
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1. Derive a formula expressing the kinetic energy of an object in terms of its momentum and mass.

2. Two people in a rowboat wish to move around without causing the boat to move. What should be true about their total momentum? Explain.

3. A firework shoots up into the air, and just before it explodes it has a certain momentum and kinetic energy. What can you say about the momenta and kinetic energies of the pieces immediately after the explosion? [Based on a problem from PSSC Physics.]

   △ Solution, p. 564

4. A bullet leaves the barrel of a gun with a kinetic energy of 90 J. The gun barrel is 50 cm long. The gun has a mass of 4 kg, the bullet 10 g.
(a) Find the bullet’s final velocity.
(b) Find the bullet’s final momentum.
(c) Find the momentum of the recoiling gun.
(d) Find the kinetic energy of the recoiling gun, and explain why the recoiling gun does not kill the shooter.

5. The graph shows the force, in meganewtons, exerted by a rocket engine on the rocket as a function of time. If the rocket’s mass is 4000 kg, at what speed is the rocket moving when the engine stops firing? Assume it goes straight up, and neglect the force of gravity, which is much less than a meganewton.
6 Cosmic rays are particles from outer space, mostly protons and atomic nuclei, that are continually bombarding the earth. Most of them, although they are moving extremely fast, have no discernible effect even if they hit your body, because their masses are so small. Their energies vary, however, and a very small minority of them have extremely large energies. In some cases the energy is as much as several Joules, which is comparable to the KE of a well thrown rock! If you are in a plane at a high altitude and are so incredibly unlucky as to be hit by one of these rare ultra-high-energy cosmic rays, what would you notice, the momentum imparted to your body, the energy dissipated in your body as heat, or both? Base your conclusions on numerical estimates, not just random speculation. (At these high speeds, one should really take into account the deviations from Newtonian physics described by Einstein’s special theory of relativity. Don’t worry about that, though.)

7 Show that for a body made up of many equal masses, the equation for the center of mass becomes a simple average of all the positions of the masses.

8 The figure shows a view from above of a collision about to happen between two air hockey pucks sliding without friction. They have the same speed, \( v_i \), before the collision, but the big puck is 2.3 times more massive than the small one. Their sides have sticky stuff on them, so when they collide, they will stick together. At what angle will they emerge from the collision? In addition to giving a numerical answer, please indicate by drawing on the figure how your angle is defined.  

9 A learjet traveling due east at 300 mi/hr collides with a jumbo jet which was heading southwest at 150 mi/hr. The jumbo jet’s mass is 5.0 times greater than that of the learjet. When they collide, the learjet sticks into the fuselage of the jumbo jet, and they fall to earth together. Their engines stop functioning immediately after the collision. On a map, what will be the direction from the location of the collision to the place where the wreckage hits the ground? (Give an angle.)

10 A very massive object with velocity \( v \) collides head-on with an object at rest whose mass is very small. No kinetic energy is converted into other forms. Prove that the low-mass object recoils with velocity \( 2v \). [Hint: Use the center-of-mass frame of reference.]

11 A mass \( m \) moving at velocity \( v \) collides with a stationary target having the same mass \( m \). Find the maximum amount of energy that can be released as heat and sound.
12  When the contents of a refrigerator cool down, the changed molecular speeds imply changes in both momentum and energy. Why, then, does a fridge transfer power through its radiator coils, but not force?  

13  A 10-kg bowling ball moving at 2.0 m/s hits a 1.0-kg bowling pin, which is initially at rest. The other pins are all gone already, and the collision is head-on, so that the motion is one-dimensional. Assume that negligible amounts of heat and sound are produced. Find the velocity of the pin immediately after the collision.

14  A ball of mass $3m$ collides head-on with an initially stationary ball of mass $m$. No kinetic energy is transformed into heat or sound. In what direction is the mass-$3m$ ball moving after the collision, and how fast is it going compared to its original velocity?

15  Suppose a system consisting of pointlike particles has a total kinetic energy $K_{cm}$ measured in the center-of-mass frame of reference. Since they are pointlike, they cannot have any energy due to internal motion.

(a) Prove that in a different frame of reference, moving with velocity $u$ relative to the center-of-mass frame, the total kinetic energy equals $K_{cm} + M|u|^2/2$, where $M$ is the total mass. [Hint: You can save yourself a lot of writing if you express the total kinetic energy using the dot product.]  

(b) Use this to prove that if energy is conserved in one frame of reference, then it is conserved in every frame of reference. The total energy equals the total kinetic energy plus the sum of the potential energies due to the particles’ interactions with each other, which we assume depends only on the distance between particles. [For a simpler numerical example, see problem 13 on p. 335.]

16  The big difference between the equations for momentum and kinetic energy is that one is proportional to $v$ and one to $v^2$. Both, however, are proportional to $m$. Suppose someone tells you that there’s a third quantity, funkosity, defined as $f = m^2v$, and that funkosity is conserved. How do you know your leg is being pulled?
17  A rocket ejects exhaust with an exhaust velocity $u$. The rate at which the exhaust mass is used (mass per unit time) is $b$. We assume that the rocket accelerates in a straight line starting from rest, and that no external forces act on it. Let the rocket’s initial mass (fuel plus the body and payload) be $m_i$, and $m_f$ be its final mass, after all the fuel is used up. (a) Find the rocket’s final velocity, $v$, in terms of $u$, $m_i$, and $m_f$. Neglect the effects of special relativity. (b) A typical exhaust velocity for chemical rocket engines is 4000 m/s. Estimate the initial mass of a rocket that could accelerate a one-ton payload to 10% of the speed of light, and show that this design won’t work. (For the sake of the estimate, ignore the mass of the fuel tanks. The speed is fairly small compared to $c$, so it’s not an unreasonable approximation to ignore relativity.)

18  A flexible rope of mass $m$ and length $L$ slides without friction over the edge of a table. Let $x$ be the length of the rope that is hanging over the edge at a given moment in time. (a) Show that $x$ satisfies the equation of motion $d^2 x / dt^2 = gx/L$. [Hint: Use $F = dp/dt$, which allows you to handle the two parts of the rope separately even though mass is moving out of one part and into the other.] (b) Give a physical explanation for the fact that a larger value of $x$ on the right-hand side of the equation leads to a greater value of the acceleration on the left side. (c) When we take the second derivative of the function $x(t)$ we are supposed to get essentially the same function back again, except for a constant out in front. The function $e^x$ has the property that it is unchanged by differentiation, so it is reasonable to look for solutions to this problem that are of the form $x = be^{ct}$, where $b$ and $c$ are constants. Show that this does indeed provide a solution for two specific values of $c$ (and for any value of $b$). (d) Show that the sum of any two solutions to the equation of motion is also a solution. (e) Find the solution for the case where the rope starts at rest at $t = 0$ with some nonzero value of $x$.

19 (a) Find a relativistic equation for the velocity of an object in terms of its mass and momentum (eliminating $G$). Use natural units (i.e., discard factors of $c$) throughout. (b) Show that your result is approximately the same as the nonrelativistic value, $p/m$, at low velocities. (c) Show that very large momenta result in speeds close to the speed of light. (d) Insert factors of $c$ to make your result from part a usable in SI units.
The force acting on an object is $F = At^2$. The object is at rest at time $t = 0$. What is its momentum at $t = T$? 

[problem by B. Shotwell]

A bullet of mass $m$ strikes a block of mass $M$ which is hanging by a string of length $L$ from the ceiling. It is observed that, after the sticky collision, the maximum angle that the string makes with the vertical is $\theta$. This setup is called a ballistic pendulum, and it can be used to measure the speed of the bullet.

(a) What vertical height does the block reach? 
(b) What was the speed of the block just after the collision? 
(c) What was the speed of the bullet just before it struck the block? 

[problem by B. Shotwell]

A car of mass $M$ and a truck of mass $2M$ collide head-on with equal speeds $v$, and the collision is perfectly inelastic, i.e., the maximum possible amount of kinetic energy is transformed into heat and sound, consistent with conservation of momentum.

(a) What is the magnitude of the change in momentum of the car? 
(b) What is the magnitude of the change in momentum of the truck? 
(c) What is the final speed of the two vehicles? 
(d) What fraction of the initial kinetic energy was lost as a result of the collision? 

[problem by B. Shotwell]
Chapter 15
Conservation of Angular Momentum

“Sure, and maybe the sun won’t come up tomorrow.” Of course, the sun only appears to go up and down because the earth spins, so the cliche should really refer to the unlikelihood of the earth’s stopping its rotation abruptly during the night. Why can’t it stop? It wouldn’t violate conservation of momentum, because the earth’s rotation doesn’t add anything to its momentum. While California spins in one direction, some equally massive part of India goes the opposite way, canceling its momentum. A halt to Earth’s rotation would entail a drop in kinetic energy, but that energy could simply be converted into some other form, such as heat.

Other examples along these lines are not hard to find. A hydrogen atom spins at the same rate for billions of years. A high-diver who is rotating when he comes off the board does not need to make
any physical effort to continue rotating, and indeed would be unable to stop rotating before he hit the water.

These observations have the hallmarks of a conservation law:

A closed system is involved. Nothing is making an effort to twist the earth, the hydrogen atom, or the high-diver. They are isolated from rotation-changing influences, i.e., they are closed systems.

Something remains unchanged. There appears to be a numerical quantity for measuring rotational motion such that the total amount of that quantity remains constant in a closed system.

Something can be transferred back and forth without changing the total amount. In figure a, the jumper wants to get his feet out in front of him so he can keep from doing a “face plant” when he lands. Bringing his feet forward would involve a certain quantity of counterclockwise rotation, but he didn’t start out with any rotation when he left the ground. Suppose we consider counterclockwise as positive and clockwise as negative. The only way his legs can acquire some positive rotation is if some other part of his body picks up an equal amount of negative rotation. This is why he swings his arms up behind him, clockwise.

What numerical measure of rotational motion is conserved? Car engines and old-fashioned LP records have speeds of rotation measured in rotations per minute (r.p.m.), but the number of rotations per minute (or per second) is not a conserved quantity. A twirling figure skater, for instance, can pull her arms in to increase her r.p.m.’s. The first section of this chapter deals with the numerical definition of the quantity of rotation that results in a valid conservation law.
15.1 Conservation of angular momentum

When most people think of rotation, they think of a solid object like a wheel rotating in a circle around a fixed point. Examples of this type of rotation, called rigid rotation or rigid-body rotation, include a spinning top, a seated child’s swinging leg, and a helicopter’s spinning propeller. Rotation, however, is a much more general phenomenon, and includes noncircular examples such as a comet in an elliptical orbit around the sun, or a cyclone, in which the core completes a circle more quickly than the outer parts.

If there is a numerical measure of rotational motion that is a conserved quantity, then it must include nonrigid cases like these, since nonrigid rotation can be traded back and forth with rigid rotation. For instance, there is a trick for finding out if an egg is raw or hardboiled. If you spin a hardboiled egg and then stop it briefly with your finger, it stops dead. But if you do the same with a raw egg, it springs back into rotation because the soft interior was still swirling around within the momentarily motionless shell. The pattern of flow of the liquid part is presumably very complex and nonuniform due to the asymmetric shape of the egg and the different consistencies of the yolk and the white, but there is apparently some way to describe the liquid’s total amount of rotation with a single number, of which some percentage is given back to the shell when you release it.

The best strategy is to devise a way of defining the amount of rotation of a single small part of a system. The amount of rotation of a system such as a cyclone will then be defined as the total of all the contributions from its many small parts.

The quest for a conserved quantity of rotation even requires us to broaden the rotation concept to include cases where the motion doesn’t repeat or even curve around. If you throw a piece of putty at a door, the door will recoil and start rotating. The putty was traveling straight, not in a circle, but if there is to be a general conservation law that can cover this situation, it appears that we must describe the putty as having had some “rotation,” which it then gave up to the door. The best way of thinking about it is to attribute rotation to any moving object or part of an object that changes its angle in relation to the axis of rotation. In the putty-and-door example, the hinge of the door is the natural point to think of as an axis, and the putty changes its angle as seen by someone standing at the hinge. For this reason, the conserved quantity we are investigating is called angular momentum. The symbol for angular momentum can’t be $a$ or $m$, since those are used for acceleration and mass, so the symbol $L$ is arbitrarily chosen instead.

Imagine a 1-kg blob of putty, thrown at the door at a speed of 1 m/s, which hits the door at a distance of 1 m from the hinge. We define this blob to have 1 unit of angular momentum. When
it hits the door, the door will recoil and start rotating. We can use the speed at which the door recoils as a measure of the angular momentum the blob brought in.\(^1\)

Experiments show, not surprisingly, that a 2-kg blob thrown in the same way makes the door rotate twice as fast, so the angular momentum of the putty blob must be proportional to mass,

\[ L \propto m. \]

Similarly, experiments show that doubling the velocity of the blob will have a doubling effect on the result, so its angular momentum must be proportional to its velocity as well,

\[ L \propto mv. \]

You have undoubtedly had the experience of approaching a closed door with one of those bar-shaped handles on it and pushing on the wrong side, the side close to the hinges. You feel like an idiot, because you have so little leverage that you can hardly budge the door. The same would be true with the putty blob. Experiments would show that the amount of rotation the blob can give to the door is proportional to the distance, \( r \), from the axis of rotation, so angular momentum must also be proportional to \( r \),

\[ L \propto mvr. \]

We are almost done, but there is one missing ingredient. We know on grounds of symmetry that a putty ball thrown directly inward toward the hinge will have no angular momentum to give to the door. After all, there would not even be any way to decide whether the ball’s rotation was clockwise or counterclockwise in this situation. It is therefore only the component of the blob’s velocity vector perpendicular to the door that should be counted in its angular momentum,

\[ L = m v_\perp r. \]

More generally, \( v_\perp \) should be thought of as the component of the object’s velocity vector that is perpendicular to the line joining the object to the axis of rotation.

We find that this equation agrees with the definition of the original putty blob as having one unit of angular momentum, and we can now see that the units of angular momentum are \((\text{kg} \cdot \text{m/s}) \cdot \text{m}\), i.e., \(\text{kg} \cdot \text{m}^2/\text{s}\). This gives us a way of calculating the angular momentum of any material object or any system consisting of material objects:

---

\(^1\)We assume that the door is much more massive than the blob. Under this assumption, the speed at which the door recoils is much less than the original speed of the blob, so the blob has lost essentially all its angular momentum, and given it to the door.
The angular momentum of a moving particle is

\[ L = mv_\perp r, \]

where \( m \) is its mass, \( v_\perp \) is the component of its velocity vector perpendicular to the line joining it to the axis of rotation, and \( r \) is its distance from the axis. Positive and negative signs are used to describe opposite directions of rotation.

The angular momentum of a finite-sized object or a system of many objects is found by dividing it up into many small parts, applying the equation to each part, and adding to find the total amount of angular momentum.

Note that \( r \) is not necessarily the radius of a circle. (As implied by the qualifiers, matter isn’t the only thing that can have angular momentum. Light can also have angular momentum, and the above equation would not apply to light.)

Conservation of angular momentum has been verified over and over again by experiment, and is now believed to be one of the three most fundamental principles of physics, along with conservation of energy and momentum.

\[ ^{1}\text{A figure skater pulls her arms in} \]

When a figure skater is twirling, there is very little friction between her and the ice, so she is essentially a closed system, and her angular momentum is conserved. If she pulls her arms in, she is decreasing \( r \) for all the atoms in her arms. It would violate conservation of angular momentum if she then continued rotating at the same speed, i.e., taking the same amount of time for each revolution, because her arms’ contributions to her angular momentum would have decreased, and no other part of her would have increased its angular momentum. This is impossible because it would violate conservation of angular momentum. If her total angular momentum is to remain constant, the decrease in \( r \) for her arms must be compensated for by an overall increase in her rate of rotation. That is, by pulling her arms in, she substantially reduces the time for each rotation.

\[ ^{1}\text{A figure skater pulls in her arms so that she can execute a spin more rapidly.} \]
Example 3. A view of the earth-moon system from above the north pole. All distances have been highly distorted for legibility. The earth's rotation is counterclockwise from this point of view (arrow). The moon's gravity creates a bulge on the side near it, because its gravitational pull is stronger there, and an "anti-bulge" on the far side, since its gravity there is weaker. For simplicity, let's focus on the tidal bulge closer to the moon. Its frictional force is trying to slow down the earth's rotation, so its force on the earth's solid crust is toward the bottom of the figure. By Newton's third law, the crust must thus make a force on the bulge which is toward the top of the figure. This causes the bulge to be pulled forward at a slight angle, and the bulge's gravity therefore pulls the moon forward, accelerating its orbital motion about the earth and flinging it outward.

Example 2. Changing the axis

An object's angular momentum can be different depending on the axis about which it rotates. Figure g shows two double-exposure photographs a viola player tipping the bow in order to cross from one string to another. Much more angular momentum is required when playing near the bow's handle, called the frog, as in the panel on the right; not only are most of the atoms in the bow at greater distances, \( r \), from the axis of rotation, but the ones in the tip also have more momentum, \( p \). It is difficult for the player to quickly transfer a large angular momentum into the bow, and then transfer it back out just as quickly. (In the language of section 15.4, large torques are required.) This is one of the reasons that string players tend to stay near the middle of the bow as much as possible.

Earth's slowing rotation and the receding moon

As noted in chapter 1, the earth's rotation is actually slowing down very gradually, with the kinetic energy being dissipated as heat by friction between the land and the tidal bulges raised in the seas by the earth's gravity. Does this mean that angular momentum is not really perfectly conserved? No, it just means that the earth is not quite a closed system by itself. If we consider the earth and moon as a system, then the angular momentum lost by the earth must be gained by the moon somehow. In fact very precise measurements of the distance between the earth and the moon have been carried out by bouncing laser beams off of a mirror left there by astronauts, and these measurements show that the moon is receding from the earth at a rate of 4 centimeters per year! The moon's greater value of \( r \) means that it has a greater
angular momentum, and the increase turns out to be exactly the amount lost by the earth. In the days of the dinosaurs, the days were significantly shorter, and the moon was closer and appeared bigger in the sky.

But what force is causing the moon to speed up, drawing it out into a larger orbit? It is the gravitational forces of the earth’s tidal bulges. The effect is described qualitatively in the caption of the figure. The result would obviously be extremely difficult to calculate directly, and this is one of those situations where a conservation law allows us to make precise quantitative statements about the outcome of a process when the calculation of the process itself would be prohibitively complex.

**Restriction to rotation in a plane**

Is angular momentum a vector, or is it a scalar? On p. 218, we defined the distinction between a vector and a scalar in terms of the quantity’s behavior when rotated. If rotation doesn’t change it, it’s a scalar. If rotation affects it in the same way that it would affect an arrow, then it’s a vector. Using these definitions, figure i shows that angular momentum cannot be a scalar.

It turns out that there is a way of defining angular momentum as a vector, but until section 15.8 the examples will be confined to a single plane of rotation, i.e., effectively two-dimensional situations. In this special case, we can choose to visualize the plane of rotation from one side or the other, and to define clockwise and counterclockwise rotation as having opposite signs of angular momentum.

Figure j shows a can rolling down a board. Although the can is three-dimensional, we can view it from the side and project out the third dimension, reducing the motion to rotation in a plane. This means that the axis is a point, even though the word “axis” often connotes a line in students’ minds, as in an \( x \) or \( y \) axis.
The planet’s angular momentum is related to the rate at which it sweeps out area.

We reduce the motion to rotation in a plane, and the axis is then a point.

Discussion question

A Conservation of plain old momentum, \( p \), can be thought of as the greatly expanded and modified descendant of Galileo’s original principle of inertia, that no force is required to keep an object in motion. The principle of inertia is counterintuitive, and there are many situations in which it appears superficially that a force is needed to maintain motion, as maintained by Aristotle. Think of a situation in which conservation of angular momentum, \( L \), also seems to be violated, making it seem incorrectly that something external must act on a closed system to keep its angular momentum from “running down.”

15.2 Angular momentum in planetary motion

We now discuss the application of conservation of angular momentum to planetary motion, both because of its intrinsic importance and because it is a good way to develop a visual intuition for angular momentum.

Kepler’s law of equal areas states that the area swept out by a planet in a certain length of time is always the same. Angular momentum had not been invented in Kepler’s time, and he did not even know the most basic physical facts about the forces at work. He thought of this law as an entirely empirical and unexpectedly simple way of summarizing his data, a rule that succeeded in describing and predicting how the planets sped up and slowed down in their elliptical paths. It is now fairly simple, however, to show that the equal area law amounts to a statement that the planet’s angular momentum stays constant.

There is no simple geometrical rule for the area of a pie wedge cut out of an ellipse, but if we consider a very short time interval, as shown in figure k, the shaded shape swept out by the planet is very nearly a triangle. We do know how to compute the area of a triangle. It is one half the product of the base and the height:

\[
\text{area} = \frac{1}{2}bh.
\]

We wish to relate this to angular momentum, which contains the variables \( r \) and \( v_\perp \). If we consider the sun to be the axis of rotation, then the variable \( r \) is identical to the base of the triangle,
Discussion question A. Suppose an object is simply traveling in a straight line at constant speed. If we pick some point not on the line and call it the axis of rotation, is area swept out by the object at a constant rate? Would it matter if we chose a different axis?

B The figure is a strobe photo of a pendulum bob, taken from underneath the pendulum looking straight up. The black string can’t be seen
in the photograph. The bob was given a slight sideways push when it was released, so it did not swing in a plane. The bright spot marks the center, i.e., the position the bob would have if it hung straight down at us. Does the bob’s angular momentum appear to remain constant if we consider the center to be the axis of rotation? What if we choose a different axis?

Discussion question B.

15.3 Two theorems about angular momentum

With plain old momentum, \( p \), we had the freedom to work in any inertial frame of reference we liked. The same object could have different values of momentum in two different frames, if the frames were not at rest with respect to each other. Conservation of momentum, however, would be true in either frame. As long as we employed a single frame consistently throughout a calculation, everything would work.

The same is true for angular momentum, and in addition there is an ambiguity that arises from the definition of an axis of rotation. For a wheel, the natural choice of an axis of rotation is obviously the axle, but what about an egg rotating on its side? The egg has an asymmetric shape, and thus no clearly defined geometric center. A similar issue arises for a cyclone, which does not even have a sharply defined shape, or for a complicated machine with many gears. The following theorem, the first of two presented in this section without proof, explains how to deal with this issue. Although I have put descriptive titles above both theorems, they have no generally accepted names.

the choice of axis theorem

It is entirely arbitrary what point one defines as the axis for purposes of calculating angular momentum. If a closed system’s angular momentum is conserved when calculated with one choice of axis, then it will also be conserved for any other choice. Likewise, any inertial frame of reference may be used.
Colliding asteroids described with different axes  example 4

Observers on planets A and B both see the two asteroids colliding. The asteroids are of equal mass and their impact speeds are the same. Astronomers on each planet decide to define their own planet as the axis of rotation. Planet A is twice as far from the collision as planet B. The asteroids collide and stick. For simplicity, assume planets A and B are both at rest.

With planet A as the axis, the two asteroids have the same amount of angular momentum, but one has positive angular momentum and the other has negative. Before the collision, the total angular momentum is therefore zero. After the collision, the two asteroids will have stopped moving, and again the total angular momentum is zero. The total angular momentum both before and after the collision is zero, so angular momentum is conserved if you choose planet A as the axis.

The only difference with planet B as axis is that \( r \) is smaller by a factor of two, so all the angular momenta are halved. Even though the angular momenta are different than the ones calculated by planet A, angular momentum is still conserved.

The earth spins on its own axis once a day, but simultaneously travels in its circular one-year orbit around the sun, so any given part of it traces out a complicated loopy path. It would seem difficult to calculate the earth’s angular momentum, but it turns out that there is an intuitively appealing shortcut: we can simply add up the angular momentum due to its spin plus that arising from its center of mass’s circular motion around the sun. This is a special case of the following general theorem:

the spin theorem

An object’s angular momentum with respect to some outside axis A can be found by adding up two parts:
(1) The first part is the object’s angular momentum found by using its own center of mass as the axis, i.e., the angular momentum the object has because it is spinning.
(2) The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

A system with its center of mass at rest  example 5

In the special case of an object whose center of mass is at rest, the spin theorem implies that the object’s angular momentum is the same regardless of what axis we choose. (This is an even stronger statement than the choice of axis theorem, which only guarantees that angular momentum is conserved for any given choice of axis, without specifying that it is the same for all such choices.)
Discussion question

A In the example of the colliding asteroids, suppose planet A was moving toward the top of the page, at the same speed as the bottom asteroid. How would planet A’s astronomers describe the angular momenta of the asteroids? Would angular momentum still be conserved?

15.4 Torque: the rate of transfer of angular momentum

Force can be interpreted as the rate of transfer of momentum. The equivalent in the case of angular momentum is called torque (rhymes with “fork”). Where force tells us how hard we are pushing or pulling on something, torque indicates how hard we are twisting on it. Torque is represented by the Greek letter tau, \( \tau \), and the rate of change of an object’s angular momentum equals the total torque acting on it,

\[
\tau_{\text{total}} = \frac{dL}{dt}.
\]

As with force and momentum, it often happens that angular momentum recedes into the background and we focus our interest on the torques. The torque-focused point of view is exemplified by the fact that many scientifically untrained but mechanically apt people know all about torque, but none of them have heard of angular momentum. Car enthusiasts eagerly compare engines’ torques, and there is a tool called a torque wrench which allows one to apply a desired amount of torque to a screw and avoid overtightening it.

Torque distinguished from force

Of course a force is necessary in order to create a torque — you can’t twist a screw without pushing on the wrench — but force and torque are two different things. One distinction between them is direction. We use positive and negative signs to represent forces in the two possible directions along a line. The direction of a torque, however, is clockwise or counterclockwise, not a linear direction.

The other difference between torque and force is a matter of leverage. A given force applied at a door’s knob will change the door’s angular momentum twice as rapidly as the same force applied halfway between the knob and the hinge. The same amount of force produces different amounts of torque in these two cases.

It is possible to have a zero total torque with a nonzero total force. An airplane with four jet engines, p, would be designed so that their forces are balanced on the left and right. Their forces are all in the same direction, but the clockwise torques of two of the engines are canceled by the counterclockwise torques of the other two, giving zero total torque.

Conversely we can have zero total force and nonzero total torque.
A merry-go-round’s engine needs to supply a nonzero torque on it to bring it up to speed, but there is zero total force on it. If there was not zero total force on it, its center of mass would accelerate!

**Relationship between force and torque**

How do we calculate the amount of torque produced by a given force? Since it depends on leverage, we should expect it to depend on the distance between the axis and the point of application of the force. We’ll derive an equation relating torque to force for a particular very simple situation, and state without proof that the equation actually applies to all situations.

To try to pin down this relationship more precisely, let’s imagine hitting a tetherball, figure q. The boy applies a force $F$ to the ball for a short time $\Delta t$, accelerating the ball from rest to a velocity $v$. Since force is the rate of transfer of momentum, we have

$$F = \frac{m\Delta v}{\Delta t}.$$ 

Since the initial velocity is zero, $\Delta v$ is the same as the final velocity $v$. Multiplying both sides by $r$ gives

$$Fr = \frac{mvr}{\Delta t}.$$ 

But $mvr$ is simply the amount of angular momentum he’s given the ball, so $mvr/\Delta t$ also equals the amount of torque he applied. The result of this example is

$$\tau = Fr.$$ 

Figure q was drawn so that the force $F$ was in the direction tangent to the circle, i.e., perpendicular to the radius $r$. If the boy had applied a force parallel to the radius line, either directly inward or outward, then the ball would not have picked up any clockwise or counterclockwise angular momentum.

If a force acts at an angle other than 0 or 90° with respect to the line joining the object and the axis, it would be only the component of the force perpendicular to the line that would produce a torque,

$$\tau = F_{\perp}r.$$ 

Although this result was proved under a simplified set of circumstances, it is more generally valid:

**relationship between force and torque**

The rate at which a force transfers angular momentum to an object, i.e., the torque produced by the force, is given by

$$|\tau| = r|F_{\perp}|,$$ 

Section 15.4  Torque: the rate of transfer of angular momentum
where \( r \) is the distance from the axis to the point of application of the force, and \( F_\perp \) is the component of the force that is perpendicular to the line joining the axis to the point of application.

The equation is stated with absolute value signs because the positive and negative signs of force and torque indicate different things, so there is no useful relationship between them. The sign of the torque must be found by physical inspection of the case at hand.

From the equation, we see that the units of torque can be written as newtons multiplied by meters. Metric torque wrenches are calibrated in N·m, but American ones use foot-pounds, which is also a unit of distance multiplied by a unit of force. We know from our study of mechanical work that newtons multiplied by meters equal joules, but torque is a completely different quantity from work, and nobody writes torques with units of joules, even though it would be technically correct.

**self-check A**

Compare the magnitudes and signs of the four torques shown in the figure.

\[ \text{Answer, p. 570} \]

\[ (1) \quad (2) \quad (3) \quad (4) \]

**How torque depends on the direction of the force example 6**

How can the torque applied to the wrench in the figure be expressed in terms of \( r \), \( |F| \), and the angle \( \theta \) between these two vectors?

The force vector and its \( F_\perp \) component form the hypotenuse and one leg of a right triangle,

\[ F_\perp = |F| \sin \theta, \]

and the interior angle opposite to \( F_\perp \) equals \( \theta \). The absolute value of \( F_\perp \) can thus be expressed as
leading to

\[ |\tau| = r|\mathbf{F}| \sin \theta. \]

Sometimes torque can be more neatly visualized in terms of the quantity \( r_\perp \) shown in figure s, which gives us a third way of expressing the relationship between torque and force:

\[ |\tau| = r_\perp |\mathbf{F}|. \]

Of course you would not want to go and memorize all three equations for torque. Starting from any one of them you could easily derive the other two using trigonometry. Familiarizing yourself with them can however clue you in to easier avenues of attack on certain problems.

**The torque due to gravity**

Up until now we’ve been thinking in terms of a force that acts at a single point on an object, such as the force of your hand on the wrench. This is of course an approximation, and for an extremely realistic calculation of your hand’s torque on the wrench you might need to add up the torques exerted by each square millimeter where your skin touches the wrench. This is seldom necessary. But in the case of a gravitational force, there is never any single point at which the force is applied. Our planet is exerting a separate tug on every brick in the Leaning Tower of Pisa, and the total gravitational torque on the tower is the sum of the torques contributed by all the little forces. Luckily there is a trick that allows us to avoid such a massive calculation. It turns out that for purposes of computing the total gravitational torque on an object, you can get the right answer by just pretending that the whole gravitational force acts at the object’s center of mass.

**Gravitational torque on an outstretched arm**  

Example 7

> Your arm has a mass of 3.0 kg, and its center of mass is 30 cm from your shoulder. What is the gravitational torque on your arm when it is stretched out horizontally to one side, taking the shoulder to be the axis?

> The total gravitational force acting on your arm is

\[ |\mathbf{F}| = (3.0 \text{ kg})(9.8 \text{ m/s}^2) = 29 \text{ N}. \]

For the purpose of calculating the gravitational torque, we can treat the force as if it acted at the arm’s center of mass. The force is straight down, which is perpendicular to the line connecting the shoulder to the center of mass, so

\[ F_\perp = |\mathbf{F}| = 29 \text{ N}. \]

Continuing to pretend that the force acts at the center of the arm, \( r \) equals 30 cm = 0.30 m, so the torque is

\[ \tau = rF_\perp = 9 \text{ N} \cdot \text{m}. \]
In 2005, Dr. Margo Lillie and her graduate student Tracy Boechler published a study claiming to debunk cow tipping. Their claim was based on an analysis of the torques that would be required to tip a cow, which showed that one person wouldn’t be able to make enough torque to do it. A lively discussion ensued on the popular web site slashdot.org (“news for nerds, stuff that matters”) concerning the validity of the study. Personally, I had always assumed that cow-tipping was a group sport anyway, but as a physicist, I also had some quibbles with their calculation. Here’s my own analysis.

There are three forces on the cow: the force of gravity \( \mathbf{F}_W \), the ground’s normal force \( \mathbf{F}_N \), and the tippers’ force \( \mathbf{F}_A \).

As soon as the cow’s left hooves (on the right from our point of view) break contact with the ground, the ground’s force is being applied only to hooves on the other side. We don’t know the ground’s force, and we don’t want to find it. Therefore we take the axis to be at its point of application, so that its torque is zero.

For the purpose of computing torques, we can pretend that gravity acts at the cow’s center of mass, which I’ve placed a little lower than the center of its torso, since its legs and head also have some mass, and the legs are more massive than the head and stick out farther, so they lower the c.m. more than the head raises it. The angle \( \theta_W \) between the vertical gravitational force and the line \( r_W \) is about 14°. (An estimate by Matt Semke at the University of Nebraska-Lincoln gives 20°, which is in the same ballpark.)

To generate the maximum possible torque with the least possible force, the tippers want to push at a point as far as possible from the axis, which will be the shoulder on the other side, and they want to push at a 90 degree angle with respect to the radius line \( r_A \).

When the tippers are just barely applying enough force to raise the cow’s hooves on one side, the total torque has to be just slightly more than zero. (In reality, they want to push a lot harder than this — hard enough to impart a lot of angular momentum to the cow fair in a short time, before it gets mad and hurts them. We’re just trying to calculate the bare minimum force they can possibly use, which is the question that science can answer.) Setting the total torque equal to zero,

\[
\tau_N + \tau_W + \tau_A = 0,
\]

and letting counterclockwise torques be positive, we have

\[
0 - mg r_W \sin \theta_W + F_A r_A \sin 90^\circ = 0
\]
\[ F_A = \frac{r_W}{r_A} mg \sin \theta_W \]
\[ \approx \frac{1}{1.5} (680 \text{ kg})(9.8 \text{ m/s}^2) \sin 14^\circ \]
\[ = 1100 \text{ N}. \]

The 680 kg figure for the typical mass of a cow is due to Lillie and Boechler, who are veterinarians, so I assume it’s fairly accurate. My estimate of 1100 N comes out significantly lower than their 1400 N figure, mainly because their incorrect placement of the center of mass gives \( \theta_W = 24^\circ \). I don’t think 1100 N is an impossible amount of force to require of one big, strong person (it’s equivalent to lifting about 110 kg, or 240 pounds), but given that the tippers need to impart a large angular momentum fairly quickly, it’s probably true that several people would be required.

The main practical issue with cow tipping is that cows generally sleep lying down. Falling on its side can also seriously injure a cow.

**Discussion questions**

A This series of discussion questions deals with past students’ incorrect reasoning about the following problem.

Suppose a comet is at the point in its orbit shown in the figure. The only force on the comet is the sun’s gravitational force.

![Diagram of comet and sun](image)

Throughout the question, define all torques and angular momenta using the sun as the axis.

(1) Is the sun producing a nonzero torque on the comet? Explain.
(2) Is the comet’s angular momentum increasing, decreasing, or staying the same? Explain.

Explain what is wrong with the following answers. In some cases, the answer is correct, but the reasoning leading up to it is wrong. (a) Incorrect answer to part (1): “Yes, because the sun is exerting a force on the comet, and the comet is a certain distance from the sun.”
(b) Incorrect answer to part (1): “No, because the torques cancel out.”
(c) Incorrect answer to part (2): “Increasing, because the comet is speeding up.”

B Which claw hammer would make it easier to get the nail out of the wood if the same force was applied in the same direction?

C You whirl a rock over your head on the end of a string, and gradually pull in the string, eventually cutting the radius in half. What happens to the rock’s angular momentum? What changes occur in its speed, the time required for one revolution, and its acceleration? Why might the string break?

D A helicopter has, in addition to the huge fan blades on top, a smaller propeller mounted on the tail that rotates in a vertical plane. Why?
The photo shows an amusement park ride whose two cars rotate in opposite directions. Why is this a good design?
15.5 Statics

Equilibrium

There are many cases where a system is not closed but maintains constant angular momentum. When a merry-go-round is running at constant angular momentum, the engine’s torque is being canceled by the torque due to friction.

When an object has constant momentum and constant angular momentum, we say that it is in equilibrium. In symbols,

\[
\frac{dp}{dt} = 0 \quad \text{and} \quad \frac{dL}{dt} = 0 \quad \text{[conditions for equilibrium]}
\]

(or equivalently, zero total force and zero total torque). This is a scientific redefinition of the common English word, since in ordinary speech nobody would describe a car spinning out on an icy road as being in equilibrium.

Very commonly, however, we are interested in cases where an object is not only in equilibrium but also at rest, and this corresponds more closely to the usual meaning of the word. Trees and bridges have been designed by evolution and engineers to stay at rest, and to do so they must have not just zero total force acting on them but zero total torque. It is not enough that they don’t fall down, they also must not tip over. Statics is the branch of physics concerned with problems such as these.

Solving statics problems is now simply a matter of applying and combining some things you already know:

- You know the behaviors of the various types of forces, for example that a frictional force is always parallel to the surface of contact.
- You know about vector addition of forces. It is the vector sum of the forces that must equal zero to produce equilibrium.
- You know about torque. The total torque acting on an object must be zero if it is to be in equilibrium.
- You know that the choice of axis is arbitrary, so you can make a choice of axis that makes the problem easy to solve.

In general, this type of problem could involve four equations in four unknowns: three equations that say the force components add up to zero, and one equation that says the total torque is zero. Most cases you’ll encounter will not be this complicated. In the following example, only the equation for zero total torque is required in order to get an answer.
Example 9.

The abstract sculpture shown in figure w contains a cube of mass \( m \) and sides of length \( b \). The cube rests on top of a cylinder, which is off-center by a distance \( a \). Find the tension in the cable.

There are four forces on the cube: a gravitational force \( mg \), the force \( F_T \) from the cable, the upward normal force from the cylinder, \( F_N \), and the horizontal static frictional force from the cylinder, \( F_s \).

The total force on the cube in the vertical direction is zero:

\[ F_N - mg = 0. \]

As our axis for defining torques, let's choose the center of the cube. The cable's torque is counterclockwise, the torque due to \( F_N \) clockwise. Letting counterclockwise torques be positive, and using the convenient equation \( \tau = r \times F \), we find the equation for the total torque:

\[ b F_T - a F_N = 0. \]

We could also write down the equation saying that the total horizontal force is zero, but that would bring in the cylinder's frictional force on the cube, which we don't know and don't need to find. We already have two equations in the two unknowns \( F_T \) and \( F_N \), so there's no need to make it into three equations in three unknowns. Solving the first equation for \( F_N = mg \), we then substitute into the second equation to eliminate \( F_N \), and solve for \( F_T = \left(\frac{a}{b}\right) mg \).

As a check, our result makes sense when \( a = 0 \); the cube is balanced on the cylinder, so the cable goes slack.

Example 10.

A flagpole

A 10-kg flagpole is being held up by a lightweight horizontal cable, and is propped against the foot of a wall as shown in the figure. If the cable is only capable of supporting a tension of 70 N, how great can the angle \( \alpha \) be without breaking the cable?

All three objects in the figure are supposed to be in equilibrium: the pole, the cable, and the wall. Whichever of the three objects we pick to investigate, all the forces and torques on it have to cancel out. It is not particularly helpful to analyze the forces and torques on the wall, since it has forces on it from the ground that are not given and that we don't want to find. We could study the forces and torques on the cable, but that doesn't let us use the given information about the pole. The object we need to analyze is the pole.

The pole has three forces on it, each of which may also result in a torque: (1) the gravitational force, (2) the cable's force, and (3) the wall's force.

We are free to define an axis of rotation at any point we wish, and it is helpful to define it to lie at the bottom end of the pole, since...