26. (a) Let \( R \) be the radius of the Earth and \( T \) the time (one day) that it takes for one rotation. Find the speed at which a point on the equator moves due to the rotation of the earth. (b) Check the units of your equation using the method shown in example 1 on p. 22. (c) Check that your answer to part a makes sense in the case where the Earth stops rotating completely, so that \( T \) is infinitely long. (d) Nairobi, Kenya, is very close to the equator. Plugging in numbers to your answer from part a, find Nairobi’s speed in meters per second. See the table in the back of the book for the relevant data. For comparison, the speed of sound is about 340 m/s.

27. (a) Let \( \theta \) be the latitude of a point on the Earth’s surface. Derive an algebra equation for the distance, \( L \), traveled by that point during one rotation of the Earth about its axis, i.e., over one day, expressed in terms of \( \theta \) and \( R \), the radius of the earth. You may find it helpful to draw one or more diagrams in the style of figure h on p. 29. (b) Generalize the result of problem 26a to points not necessarily on the equator. (c) Check the units of your equation using the method shown in example 1 on p. 22. (d) Check that your equation in part b gives zero for the North Pole, and also that it agrees with problem 26a in the special case of a point on the equator. (e) At what speed is Fullerton, California, at latitude \( \theta = 34^\circ \), moving with the rotation of the Earth about its axis?

28. In running races at distances of 800 meters and longer, runners do not have their own lanes, so in order to pass, they have to go around their opponents. Suppose we adopt the simplified geometrical model suggested by the figure, in which the two runners take equal times to trace out the sides of an isosceles triangle, deviating from parallelism by the angle \( \theta \). The runner going straight runs at speed \( v \), while the one who is passing must run at a greater speed. Let the difference in speeds be \( \Delta v \). (a) Find \( \Delta v \) in terms of \( v \) and \( \theta \). (b) Check the units of your equation using the method shown in example 1 on p. 22. (c) Check that your answer makes sense in the special case where \( \theta = 0 \), i.e., in the case where the runners are on an extremely long straightaway. (d) Suppose that \( \theta = 1.0 \) degrees, which is about the smallest value that will allow a runner to pass in the distance available on the straightaway of a track, and let \( v = 7.06 \) m/s, which is the women’s world record pace at 800 meters. Plug numbers into your equation from part a to determine \( \Delta v \), and comment on the result.
29 In 1849, Fizeau carried out the first terrestrial measurement of the speed of light; previous measurements by Roemer and Bradley had involved astronomical observation. The figure shows a simplified conceptual representation of Fizeau’s experiment. A ray of light from a bright source was directed through the teeth at the edge of a spinning cogwheel. After traveling a distance $L$, it was reflected from a mirror and returned along the same path. The figure shows the case in which the ray passes between two teeth, but when it returns, the wheel has rotated by half the spacing of the teeth, so that the ray is blocked. When this condition is achieved, the observer looking through the teeth toward the far-off mirror sees it go completely dark. Fizeau adjusted the speed of the wheel to achieve this condition and recorded the rate of rotation to be $f$ rotations per second. Let the number of teeth on the wheel be $n$.

(a) Find the speed of light $c$ in terms of $L$, $n$, and $f$. $\sqrt{\text{Check the units of your equation using the method shown in example 1 on p. 22. (Here $f$’s units of rotations per second should be taken as inverse seconds, } s^{-1}, \text{ since the number of rotations in a second is a unitless count.)}}$

(b) Imagine that you are Fizeau trying to design this experiment. The speed of light is a huge number in ordinary units. Use your equation from part a to determine whether increasing $c$ requires an increase in $L$, or a decrease. Do the same for $n$ and $f$. Based on this, decide for each of these variables whether you want a value that is as big as possible, or as small as possible.

(c) Fizeau used $L = 8633 \text{ m}$, $f = 12.6 \text{ s}^{-1}$, and $n = 720$. Plug in to your equation from part a and extract the speed of light from his data.
Galileo's contradiction of Aristotle had serious consequences. He was interrogated by the Church authorities and convicted of teaching that the earth went around the sun as a matter of fact and not, as he had promised previously, as a mere mathematical hypothesis. He was placed under permanent house arrest, and forbidden to write about or teach his theories. Immediately after being forced to recant his claim that the earth revolved around the sun, the old man is said to have muttered defiantly “and yet it does move.” The story is dramatic, but there are some omissions in the commonly taught heroic version. There was a rumor that the Simplicio character represented the Pope. Also, some of the ideas Galileo advocated had controversial religious overtones. He believed in the existence of atoms, and atomism was thought by some people to contradict the Church’s doctrine of transubstantiation, which said that in the Catholic mass, the blessing of the bread and wine literally transformed them into the flesh and blood of Christ. His support for a cosmology in which the earth circled the sun was also disreputable because one of its supporters, Giordano Bruno, had also proposed a bizarre synthesis of Christianity with the ancient Egyptian religion.

Chapter 3

**Acceleration and Free Fall**

### 3.1 The motion of falling objects

The motion of falling objects is the simplest and most common example of motion with changing velocity. The early pioneers of
According to Galileo’s student Viviani, Galileo dropped a cannonball and a musketball simultaneously from the leaning tower of Pisa, and observed that they hit the ground at nearly the same time. This contradicted Aristotle’s long-accepted idea that heavier objects fell faster.

Physics had a correct intuition that the way things drop was a message directly from Nature herself about how the universe worked. Other examples seem less likely to have deep significance. A walking person who speeds up is making a conscious choice. If one stretch of a river flows more rapidly than another, it may be only because the channel is narrower there, which is just an accident of the local geography. But there is something impressively consistent, universal, and inexorable about the way things fall.

Stand up now and simultaneously drop a coin and a bit of paper side by side. The paper takes much longer to hit the ground. That’s why Aristotle wrote that heavy objects fell more rapidly. Europeans believed him for two thousand years.

Now repeat the experiment, but make it into a race between the coin and your shoe. My own shoe is about 50 times heavier than the nickel I had handy, but it looks to me like they hit the ground at exactly the same moment. So much for Aristotle! Galileo, who had a flair for the theatrical, did the experiment by dropping a bullet and a heavy cannonball from a tall tower. Aristotle’s observations had been incomplete, his interpretation a vast oversimplification.

It is inconceivable that Galileo was the first person to observe a discrepancy with Aristotle’s predictions. Galileo was the one who changed the course of history because he was able to assemble the observations into a coherent pattern, and also because he carried out systematic quantitative (numerical) measurements rather than just describing things qualitatively.

Why is it that some objects, like the coin and the shoe, have similar motion, but others, like a feather or a bit of paper, are different? Galileo speculated that in addition to the force that always pulls objects down, there was an upward force exerted by the air. Anyone can speculate, but Galileo went beyond speculation and came up with two clever experiments to probe the issue. First, he experimented with objects falling in water, which probed the same issues but made the motion slow enough that he could take time measurements with a primitive pendulum clock. With this technique, he established the following facts:

- All heavy, streamlined objects (for example a steel rod dropped point-down) reach the bottom of the tank in about the same amount of time, only slightly longer than the time they would take to fall the same distance in air.

- Objects that are lighter or less streamlined take a longer time to reach the bottom.

This supported his hypothesis about two contrary forces. He imagined an idealized situation in which the falling object did not
have to push its way through any substance at all. Falling in air would be more like this ideal case than falling in water, but even a thin, sparse medium like air would be sufficient to cause obvious effects on feathers and other light objects that were not streamlined. Today, we have vacuum pumps that allow us to suck nearly all the air out of a chamber, and if we drop a feather and a rock side by side in a vacuum, the feather does not lag behind the rock at all.

**How the speed of a falling object increases with time**

Galileo’s second stroke of genius was to find a way to make quantitative measurements of how the speed of a falling object increased as it went along. Again it was problematic to make sufficiently accurate time measurements with primitive clocks, and again he found a tricky way to slow things down while preserving the essential physical phenomena: he let a ball roll down a slope instead of dropping it vertically. The steeper the incline, the more rapidly the ball would gain speed. Without a modern video camera, Galileo had invented a way to make a slow-motion version of falling.

Although Galileo’s clocks were only good enough to do accurate experiments at the smaller angles, he was confident after making a systematic study at a variety of small angles that his basic conclusions were generally valid. Stated in modern language, what he found was that the velocity-versus-time graph was a line. In the language of algebra, we know that a line has an equation of the form \( y = ax + b \), but our variables are \( v \) and \( t \), so it would be \( v = at + b \).

(The constant \( b \) can be interpreted simply as the initial velocity of the object, i.e., its velocity at the time when we started our clock, which we conventionally write as \( v_o \).

### Self-check A

An object is rolling down an incline. After it has been rolling for a short time, it is found to travel 13 cm during a certain one-second interval. During the second after that, it goes 16 cm. How many cm will it travel in the second after that?

*Answer, p. 567*
A contradiction in Aristotle’s reasoning

Galileo’s inclined-plane experiment disproved the long-accepted claim by Aristotle that a falling object had a definite “natural falling speed” proportional to its weight. Galileo had found that the speed just kept on increasing, and weight was irrelevant as long as air friction was negligible. Not only did Galileo prove experimentally that Aristotle had been wrong, but he also pointed out a logical contradiction in Aristotle’s own reasoning. Simplicio, the stupid character, mouths the accepted Aristotelian wisdom:

SIMPlicio: There can be no doubt but that a particular body . . . has a fixed velocity which is determined by nature . . .

SALVIATI: If then we take two bodies whose natural speeds are different, it is clear that, [according to Aristotle], on uniting the two, the more rapid one will be partly held back by the slower, and the slower will be somewhat hastened by the swifter. Do you not agree with me in this opinion?

SIMPlicio: You are unquestionably right.

SALVIATI: But if this is true, and if a large stone moves with a speed of, say, eight [unspecified units] while a smaller moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. Thus you see how, from your assumption that the heavier body moves more rapidly than the lighter one, I infer that the heavier body moves more slowly.

What is gravity?

The physicist Richard Feynman liked to tell a story about how when he was a little kid, he asked his father, “Why do things fall?” As an adult, he praised his father for answering, “Nobody knows why things fall. It’s a deep mystery, and the smartest people in the world don’t know the basic reason for it.” Contrast that with the average person’s off-the-cuff answer, “Oh, it’s because of gravity.” Feynman liked his father’s answer, because his father realized that simply giving a name to something didn’t mean that you understood it. The radical thing about Galileo’s and Newton’s approach to science was that they concentrated first on describing mathematically what really did happen, rather than spending a lot of time on untestable speculation such as Aristotle’s statement that “Things fall because they are trying to reach their natural place in contact with the earth.” That doesn’t mean that science can never answer the “why” questions. Over the next month or two as you delve deeper into physics, you will learn that there are more fundamental reasons why all falling objects have \( v - t \) graphs with the same slope, regardless
of their mass. Nevertheless, the methods of science always impose limits on how deep our explanation can go.

### 3.2 Acceleration

**Definition of acceleration for linear \( v - t \) graphs**

Galileo’s experiment with dropping heavy and light objects from a tower showed that all falling objects have the same motion, and his inclined-plane experiments showed that the motion was described by \( v = at + v_0 \). The initial velocity \( v_0 \) depends on whether you drop the object from rest or throw it down, but even if you throw it down, you cannot change the slope, \( a \), of the \( v - t \) graph.

Since these experiments show that all falling objects have linear \( v - t \) graphs with the same slope, the slope of such a graph is apparently an important and useful quantity. We use the word acceleration, and the symbol \( a \), for the slope of such a graph. In symbols, \( a = \Delta v/\Delta t \). The acceleration can be interpreted as the amount of speed gained in every second, and it has units of velocity divided by time, i.e., “meters per second per second,” or m/s/s. Continuing to treat units as if they were algebra symbols, we simplify “m/s/s” to read “m/s².” Acceleration can be a useful quantity for describing other types of motion besides falling, and the word and the symbol “\( a \)” can be used in a more general context. We reserve the more specialized symbol “\( g \)” for the acceleration of falling objects, which on the surface of our planet equals 9.8 m/s². Often when doing approximate calculations or merely illustrative numerical examples it is good enough to use \( g = 10 \) m/s², which is off by only 2%.

---

**Example 1.** Finding final speed, given time

- A despondent physics student jumps off a bridge, and falls for three seconds before hitting the water. How fast is he going when he hits the water?

- Approximating \( g \) as 10 m/s², he will gain 10 m/s of speed each second. After one second, his velocity is 10 m/s, after two seconds it is 20 m/s, and on impact, after falling for three seconds, he is moving at 30 m/s.

**Example 2.** Extracting acceleration from a graph

- The \( x - t \) and \( v - t \) graphs show the motion of a car starting from a stop sign. What is the car’s acceleration?

- Acceleration is defined as the slope of the \( v\)-t graph. The graph rises by 3 m/s during a time interval of 3 s, so the acceleration is \( (3 \text{ m/s})/(3 \text{ s}) = 1 \text{ m/s}^2 \).

Incorrect solution #1: The final velocity is 3 m/s, and acceleration is velocity divided by time, so the acceleration is \( (3 \text{ m/s})/(10 \text{ s}) = 0.3 \text{ m/s}^2 \).
x The solution is incorrect because you can’t find the slope of a graph from one point. This person was just using the point at the right end of the v-t graph to try to find the slope of the curve.

Incorrect solution #2: Velocity is distance divided by time so \( v = \frac{(4.5 \text{ m})}{(3 \text{ s})} = 1.5 \text{ m/s} \). Acceleration is velocity divided by time, so \( a = \frac{(1.5 \text{ m/s})}{(3 \text{ s})} = 0.5 \text{ m/s}^2 \).

x The solution is incorrect because velocity is the slope of the tangent line. In a case like this where the velocity is changing, you can’t just pick two points on the x-t graph and use them to find the velocity.

\( \text{Converting } g \text{ to different units} \) example 3

▷ What is \( g \) in units of cm/s\(^2\)?

▷ The answer is going to be how many cm/s of speed a falling object gains in one second. If it gains 9.8 m/s in one second, then it gains 980 cm/s in one second, so \( g = 980 \text{ cm/s}^2 \). Alternatively, we can use the method of fractions that equal one:

\[
\frac{9.8 \text{ m/s}^2}{\text{s}^2} \times \frac{100 \text{ cm}}{1 \text{ m}} = 980 \text{ cm/s}^2
\]

▷ What is \( g \) in units of miles/hour\(^2\)?

▷

\[
\frac{9.8 \text{ m/s}^2}{\text{s}^2} \times \frac{1 \text{ mile}}{1600 \text{ m}} \times \left(\frac{3600 \text{ s}}{1 \text{ hour}}\right)^2 = 7.9 \times 10^4 \text{ mile/hour}^2
\]

This large number can be interpreted as the speed, in miles per hour, that you would gain by falling for one hour. Note that we had to square the conversion factor of 3600 s/hour in order to cancel out the units of seconds squared in the denominator.

▷ What is \( g \) in units of miles/hour/s?

▷

\[
\frac{9.8 \text{ m/s}^2}{\text{s}^2} \times \frac{1 \text{ mile}}{1600 \text{ m}} \times \frac{3600 \text{ s}}{1 \text{ hour}} = 22 \text{ mile/hour/s}
\]

This is a figure that Americans will have an intuitive feel for. If your car has a forward acceleration equal to the acceleration of a falling object, then you will gain 22 miles per hour of speed every second. However, using mixed time units of hours and seconds like this is usually inconvenient for problem-solving. It would be like using units of foot-inches for area instead of ft\(^2\) or in\(^2\).

The acceleration of gravity is different in different locations.

Everyone knows that gravity is weaker on the moon, but actually it is not even the same everywhere on Earth, as shown by the sampling of numerical data in the following table.
The main variables that relate to the value of $g$ on Earth are latitude and elevation. Although you have not yet learned how $g$ would be calculated based on any deeper theory of gravity, it is not too hard to guess why $g$ depends on elevation. Gravity is an attraction between things that have mass, and the attraction gets weaker with increasing distance. As you ascend from the seaport of Guayaquil to the nearby top of Mt. Cotopaxi, you are distancing yourself from the mass of the planet. The dependence on latitude occurs because we are measuring the acceleration of gravity relative to the earth’s surface, but the earth’s rotation causes the earth’s surface to fall out from under you. (We will discuss both gravity and rotation in more detail later in the course.)

<table>
<thead>
<tr>
<th>Location</th>
<th>Latitude</th>
<th>Elevation (m)</th>
<th>$g$ (m/s$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>North Pole</td>
<td>90°N</td>
<td>0</td>
<td>9.8322</td>
</tr>
<tr>
<td>Reykjavik, Iceland</td>
<td>64°N</td>
<td>0</td>
<td>9.8225</td>
</tr>
<tr>
<td>Guayaquil, Ecuador</td>
<td>2°S</td>
<td>0</td>
<td>9.7806</td>
</tr>
<tr>
<td>Mt. Cotopaxi, Ecuador</td>
<td>1°S</td>
<td>5896</td>
<td>9.7624</td>
</tr>
<tr>
<td>Mt. Everest</td>
<td>28°N</td>
<td>8848</td>
<td>9.7643</td>
</tr>
</tbody>
</table>

This false-color map shows variations in the strength of the earth’s gravity. Purple areas have the strongest gravity, yellow the weakest. The overall trend toward weaker gravity at the equator and stronger gravity at the poles has been artificially removed to allow the weaker local variations to show up. The map covers only the oceans because of the technique used to make it: satellites look for bulges and depressions in the surface of the ocean. A very slight bulge will occur over an undersea mountain, for instance, because the mountain’s gravitational attraction pulls water toward it. The US government originally began collecting data like these for military use, to correct for the deviations in the paths of missiles. The data have recently been released for scientific and commercial use (e.g., searching for sites for off-shore oil wells).
<table>
<thead>
<tr>
<th>Location</th>
<th>( g ) (( m/s^2 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asteroid Vesta (surface)</td>
<td>0.3</td>
</tr>
<tr>
<td>Earth’s moon (surface)</td>
<td>1.6</td>
</tr>
<tr>
<td>Mars (surface)</td>
<td>3.7</td>
</tr>
<tr>
<td>Earth (surface)</td>
<td>9.8</td>
</tr>
<tr>
<td>Jupiter (cloud-tops)</td>
<td>26</td>
</tr>
<tr>
<td>Sun (visible surface)</td>
<td>270</td>
</tr>
<tr>
<td>Typical neutron star (surface)</td>
<td>( 10^{12} )</td>
</tr>
<tr>
<td>Black hole (center)</td>
<td>Infinite according to some theories, on the order of ( 10^{52} ) according to others</td>
</tr>
</tbody>
</table>

A typical neutron star is not so different in size from a large asteroid, but is orders of magnitude more massive, so the mass of a body definitely correlates with the \( g \) it creates. On the other hand, a neutron star has about the same mass as our Sun, so why is its \( g \) billions of times greater? If you had the misfortune of being on the surface of a neutron star, you’d be within a few thousand miles of all its mass, whereas on the surface of the Sun, you’d still be millions of miles from most of its mass.

**Discussion questions**

**A** What is wrong with the following definitions of \( g \)?

1. “\( g \) is gravity.”
2. “\( g \) is the speed of a falling object.”
3. “\( g \) is how hard gravity pulls on things.”

**B** When advertisers specify how much acceleration a car is capable of, they do not give an acceleration as defined in physics. Instead, they usually specify how many seconds are required for the car to go from rest to 60 miles/hour. Suppose we use the notation “\( a \)” for the acceleration as defined in physics, and “\( a_{\text{car ad}} \)” for the quantity used in advertisements for cars. In the US’s non-metric system of units, what would be the units of \( a \) and \( a_{\text{car ad}} \)? How would the use and interpretation of large and small, positive and negative values be different for \( a \) as opposed to \( a_{\text{car ad}} \)?

**C** Two people stand on the edge of a cliff. As they lean over the edge, one person throws a rock down, while the other throws one straight up with an exactly opposite initial velocity. Compare the speeds of the rocks on impact at the bottom of the cliff.

### 3.3 Positive and negative acceleration

Gravity always pulls down, but that does not mean it always speeds things up. If you throw a ball straight up, gravity will first slow it down to \( v = 0 \) and then begin increasing its speed. When I took physics in high school, I got the impression that positive signs of acceleration indicated speeding up, while negative accelerations represented slowing down, i.e., deceleration. Such a definition would be inconvenient, however, because we would then have to say that the same downward tug of gravity could produce either a positive...
or a negative acceleration. As we will see in the following example, such a definition also would not be the same as the slope of the $v-t$ graph.

Let’s study the example of the rising and falling ball. In the example of the person falling from a bridge, I assumed positive velocity values without calling attention to it, which meant I was assuming a coordinate system whose $x$ axis pointed down. In this example, where the ball is reversing direction, it is not possible to avoid negative velocities by a tricky choice of axis, so let’s make the more natural choice of an axis pointing up. The ball’s velocity will initially be a positive number, because it is heading up, in the same direction as the $x$ axis, but on the way back down, it will be a negative number. As shown in the figure, the $v-t$ graph does not do anything special at the top of the ball’s flight, where $v$ equals 0. Its slope is always negative. In the left half of the graph, there is a negative slope because the positive velocity is getting closer to zero. On the right side, the negative slope is due to a negative velocity that is getting farther from zero, so we say that the ball is speeding up, but its velocity is decreasing!

To summarize, what makes the most sense is to stick with the original definition of acceleration as the slope of the $v-t$ graph, $\Delta v/\Delta t$. By this definition, it just isn’t necessarily true that things speeding up have positive acceleration while things slowing down have negative acceleration. The word “deceleration” is not used much by physicists, and the word “acceleration” is used unblushingly to refer to slowing down as well as speeding up: “There was a red light, and we accelerated to a stop.”

1Numerical calculation of a negative acceleration example 4

In figure i, what happens if you calculate the acceleration between $t = 1.0$ and 1.5 s?

Reading from the graph, it looks like the velocity is about $-1$ m/s at $t = 1.0$ s, and around $-6$ m/s at $t = 1.5$ s. The acceleration, figured between these two points, is

$$a = \frac{\Delta v}{\Delta t} = \frac{(-6 \text{ m/s}) - (-1 \text{ m/s})}{(1.5 \text{ s}) - (1.0 \text{ s})} = -10 \text{ m/s}^2.$$ 

Even though the ball is speeding up, it has a negative acceleration.

Another way of convincing you that this way of handling the plus and minus signs makes sense is to think of a device that measures acceleration. After all, physics is supposed to use operational definitions, ones that relate to the results you get with actual measuring devices. Consider an air freshener hanging from the rear-view mirror of your car. When you speed up, the air freshener swings backward. Suppose we define this as a positive reading. When you slow down, the air freshener swings forward, so we’ll call this a negative reading.
on our accelerometer. But what if you put the car in reverse and start speeding up backwards? Even though you’re speeding up, the accelerometer responds in the same way as it did when you were going forward and slowing down. There are four possible cases:

<table>
<thead>
<tr>
<th>motion of car</th>
<th>accelerometer slope of direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>forward, speeding up</td>
<td>backward +</td>
</tr>
<tr>
<td>forward, slowing down</td>
<td>forward −</td>
</tr>
<tr>
<td>backward, speeding up</td>
<td>forward −</td>
</tr>
<tr>
<td>backward, slowing down</td>
<td>backward +</td>
</tr>
</tbody>
</table>

Note the consistency of the three right-hand columns — nature is trying to tell us that this is the right system of classification, not the left-hand column.

Because the positive and negative signs of acceleration depend on the choice of a coordinate system, the acceleration of an object under the influence of gravity can be either positive or negative. Rather than having to write things like “$g = 9.8 \text{ m/s}^2$” or “$-9.8 \text{ m/s}^2$” every time we want to discuss $g$’s numerical value, we simply define $g$ as the absolute value of the acceleration of objects moving under the influence of gravity. We consistently let $g = 9.8 \text{ m/s}^2$, but we may have either $a = g$ or $a = -g$, depending on our choice of a coordinate system.

\[ \text{Acceleration with a change in direction of motion example 5} \]

\[ \text{A person kicks a ball, which rolls up a sloping street, comes to a halt, and rolls back down again. The ball has constant acceleration. The ball is initially moving at a velocity of 4.0 m/s, and after 10.0 s it has returned to where it started. At the end, it has sped back up to the same speed it had initially, but in the opposite direction. What was its acceleration?} \]

\[ \text{By giving a positive number for the initial velocity, the statement of the question implies a coordinate axis that points up the slope of the hill. The “same” speed in the opposite direction should therefore be represented by a negative number, -4.0 m/s. The acceleration is} \]

\[ a = \frac{\Delta v}{\Delta t} \]
\[ = \frac{(v_f - v_o)}{10.0 \text{ s}} \]
\[ = \frac{[(-4.0 \text{ m/s}) - (4.0 \text{ m/s})]}{10.0 \text{ s}} \]
\[ = -0.80 \text{ m/s}^2. \]

The acceleration was no different during the upward part of the roll than on the downward part of the roll.

Incorrect solution: Acceleration is $\Delta v/\Delta t$, and at the end it’s not moving any faster or slower than when it started, so $\Delta v=0$ and
Discussion question C.

The velocity does change, from a positive number to a negative number.

Discussion question B.

Discussion questions

A A child repeatedly jumps up and down on a trampoline. Discuss the sign and magnitude of his acceleration, including both the time when he is in the air and the time when his feet are in contact with the trampoline.

B The figure shows a refugee from a Picasso painting blowing on a rolling water bottle. In some cases the person’s blowing is speeding the bottle up, but in others it is slowing it down. The arrow inside the bottle shows which direction it is going, and a coordinate system is shown at the bottom of each figure. In each case, figure out the plus or minus signs of the velocity and acceleration. It may be helpful to draw a $v - t$ graph in each case.

C Sally is on an amusement park ride which begins with her chair being hoisted straight up a tower at a constant speed of 60 miles/hour. Despite stern warnings from her father that he’ll take her home the next time she misbehaves, she decides that as a scientific experiment she really needs to release her corndog over the side as she’s on the way up. She does not throw it. She simply sticks it out of the car, lets it go, and watches it against the background of the sky, with no trees or buildings as reference points. What does the corndog’s motion look like as observed by Sally? Does its speed ever appear to her to be zero? What acceleration does she observe it to have: is it ever positive? negative? zero? What would her enraged father answer if asked for a similar description of its motion as it appears to him, standing on the ground?

D Can an object maintain a constant acceleration, but meanwhile reverse the direction of its velocity?

E Can an object have a velocity that is positive and increasing at the same time that its acceleration is decreasing?
3.4 Varying acceleration

So far we have only been discussing examples of motion for which the acceleration is constant. As always, an expression of the form $\Delta \ldots / \Delta \ldots$ for a rate of change must be generalized to a derivative when the rate of change isn’t constant. We therefore define the acceleration as $a = \frac{dv}{dt}$, which is the same as the second derivative, which Leibniz notated as

$$a = \frac{d^2 x}{dt^2}.$$  

The seemingly inconsistent placement of the twos on the top and bottom confuses all beginning calculus students. The motivation for this funny notation is that acceleration has units of $\text{m/s}^2$, and the notation correctly suggests that: the top looks like it has units of meters, the bottom seconds$^2$. The notation is not meant, however, to suggest that $t$ is really squared.

3.5 Algebraic results for constant acceleration

When an object is accelerating, the variables $x$, $v$, and $t$ are all changing continuously. It is often of interest to eliminate one of these and relate the other two to each other.

<table>
<thead>
<tr>
<th>Constant acceleration</th>
<th>example 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ How high does a diving board have to be above the water if the diver is to have as much as 1.0 s in the air?</td>
<td></td>
</tr>
<tr>
<td>▶ The diver starts at rest, and has an acceleration of 9.8 m/s$^2$. We need to find a connection between the distance she travels and time it takes. In other words, we’re looking for information about the function $x(t)$, given information about the acceleration. To go from acceleration to position, we need to integrate twice:</td>
<td></td>
</tr>
</tbody>
</table>

$$\begin{align*}
x &= \int \int a \, dt \, dt \\
&= \int (at + v_0) \, dt \quad [v_0 \text{ is a constant of integration.}] \\
&= \int at \, dt \quad [v_0 \text{ is zero because she’s dropping from rest.}] \\
&= \frac{1}{2}at^2 + x_0 \quad [x_0 \text{ is a constant of integration.}] \\
&= \frac{1}{2}at^2 \quad [x_0 \text{ can be zero if we define it that way.}]
\end{align*}$$

Note some of the good problem-solving habits demonstrated here. We solve the problem symbolically, and only plug in numbers at the very end, once all the algebra and calculus are done. One should also make a habit, after finding a symbolic result, of checking whether the dependence on the variables make sense. A greater value of $t$ in this expression would lead to a greater value
for $x$; that makes sense, because if you want more time in the air, you’re going to have to jump from higher up. A greater acceleration also leads to a greater height; this also makes sense, because the stronger gravity is, the more height you’ll need in order to stay in the air for a given amount of time. Now we plug in numbers.

$$x = \frac{1}{2} \left(9.8 \text{ m/s}^2\right) (1.0 \text{ s})^2$$

$$= 4.9 \text{ m}$$

Note that when we put in the numbers, we check that the units work out correctly, $(\text{m/s}^2)(\text{s})^2 = \text{m}$. We should also check that the result makes sense: 4.9 meters is pretty high, but not unreasonable.

Under conditions of constant acceleration, we can relate velocity and time,

$$a = \frac{\Delta v}{\Delta t},$$

or, as in the example 6, position and time,

$$x = \frac{1}{2}at^2 + v_0 t + x_0.$$

It can also be handy to have a relation involving velocity and position, eliminating time. Straightforward algebra gives

$$v_f^2 = v_o^2 + 2a\Delta x,$$

where $v_f$ is the final velocity, $v_0$ the initial velocity, and $\Delta x$ the distance traveled.

> **Solved problem: Dropping a rock on Mars** page 120, problem 13
> **Solved problem: The Dodge Viper** page 120, problem 11

### 3.6 ⋆ A test of the principle of inertia

Historically, the first quantitative and well documented experimental test of the principle of inertia (p. 71) was performed by Galileo around 1590 and published decades later when he managed to find a publisher in the Netherlands that was beyond the reach of the Roman Inquisition. It was ingenious but somewhat indirect, and required a layer of interpretation and extrapolation on top of the actual observations. As described on p. 105, he established that

---

Galileo, *Discourses and Mathematical Demonstrations Relating to Two New Sciences*, 1638. The experiments are described in the Third Day, and their support for the principle of inertia is discussed in the Scholium following Theorems I-XIV. Another experiment involving a ship is described in Galileo’s 1624 reply to a letter from Fr. Ingoli, but although Galileo vigorously asserts that he really did carry it out, no detailed description or quantitative results are given.
objects rolling on inclined planes moved according to mathematical laws that we would today describe as in section 3.5. He knew that his rolling balls were subject to friction, as well as random errors due to the limited precision of the water clock that he used, but he took the approximate agreement of his equations with experiment to indicate that they gave the results that would be exact in the absence of friction. He also showed, purely empirically, that when a ball went up or down a ramp inclined at an angle \( \theta \), its acceleration was proportional to \( \sin \theta \). Again, this required extrapolation to idealized conditions of zero friction. He then reasoned that if a ball was rolled on a horizontal ramp, with \( \theta = 0 \), its acceleration would be zero. This is exactly what is required by the principle of inertia: in the absence of friction, motion continues indefinitely.

Reversing the logic

If we assume that the principle of inertia holds, then we can reverse the direction of Galileo’s reasoning and show that the acceleration of an object moving frictionlessly on an inclined plane is proportional to \( \sin \theta \), which is a useful practical result for skateboarders or for mountaineers worried about sliding down an icy slope.

Figure k/1 shows beads sliding frictionlessly down three wires inclined at different angles, each losing the same amount of height as it travels from one end of its wire to the other. How do their speeds compare when they reach the bottom? It seems likely that the bead on the vertical wire will have the greatest acceleration, but that doesn’t necessarily mean that its final speed will be greater, because it will also have less time available to accelerate. In fact, we will show that the final speed is the same regardless of the angle.

To demonstrate this, we assume that it is not true and show that it leads to an impossible result. Suppose that the final speed is greater, for the same vertical drop, when the slope is steeper (which is probably what most people intuitively expect, and is true when there is friction). In figure k/2, we join together two pieces of wire with different slopes, and let a bead ride down the steep side and then coast back up the less steep one. Under our assumption, the loss of speed on the way back up must be smaller than the gain on the way down, so that there is a net gain in speed. If we released the bead on the right and let it travel to the left, there would be a net loss.

Now suppose that we join together a series of these shapes, making a kind of asymmetric sawtooth pattern, k/3. A bead released on the left near the top of one of the steep sections will speed up indefinitely as it travels to the right. There is no limit to how much speed can be gained if we extend the wire far enough. We could use this arrangement to launch satellites into orbit, without
the need for fuel or rocket engines. If we instead start the bead heading to the left with an initial push, it will gradually slow down, turn around, and then speed up indefinitely to the right as in the case considered earlier.

This is not just an absurd conclusion but one that in some sense violates the principle of inertia, if we state the principle loosely as a rule that motion doesn’t naturally speed up or slow down in the absence of friction. We therefore conclude that the assumption was false: an object moving frictionlessly down an inclined plane does not gain a different amount of speed, for a fixed loss of height, depending on the slope.

It is now straightforward to show that the acceleration equals \(g \sin \theta\). Applying the appropriate constant-acceleration equation, we have \(v_f^2 = 2a\ell\), where \(\ell\) is the distance traveled along the slope. Since \(v_f\) is the same regardless of angle, we have \(a \propto \ell^{-1}\), but for a change in height \(h\), we have \(h/\ell = \sin \theta\), so \(a \propto \sin \theta\), which is what we wanted to prove.

For the bead on the wire, or for an object sliding down a ramp, we can fix the constant of proportionality because \(\theta = 90^\circ\) should give \(a = g\), so we have \(a = g \sin \theta\).

For an object such as a sphere or a cylinder that rolls down a ramp without slipping, a similar proportionality holds between \(a\) and \(\sin \theta\), but the constant of proportionality is smaller by a unit-less factor that depends on the object’s shape and how its mass is distributed. The difference comes about because of the frictional force (the “traction”) between the object and the ramp.
Summary

Selected vocabulary

gravity . . . . . A general term for the phenomenon of attraction between things having mass. The attraction between our planet and a human-sized object causes the object to fall.

acceleration . . . The rate of change of velocity; the slope of the tangent line on a $v - t$ graph.

Notation

$v_o$ . . . . . . . . . initial velocity
$v_f$ . . . . . . . . . final velocity
$a$ . . . . . . . . . acceleration
$g$ . . . . . . . . . the acceleration of objects in free fall; the strength of the local gravitational field

Summary

Galileo showed that when air resistance is negligible all falling bodies have the same motion regardless of mass. Moreover, their $v - t$ graphs are straight lines. We therefore define a quantity called acceleration as the derivative $dv/dt$. This definition has the advantage that a force with a given sign, representing its direction, always produces an acceleration with the same sign. The acceleration of objects in free fall varies slightly across the surface of the earth, and greatly on other planets.

For motion with constant acceleration, the following three equations hold:

$$\Delta x = v_o \Delta t + \frac{1}{2} a \Delta t^2$$
$$v_f^2 = v_o^2 + 2a\Delta x$$
$$a = \frac{\Delta v}{\Delta t}$$

They are not valid if the acceleration is changing.
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
★ A difficult problem.

1 On New Year’s Eve, a stupid person fires a pistol straight up. The bullet leaves the gun at a speed of 100 m/s. How long does it take before the bullet hits the ground? [Solution, p. 553]

2 What is the acceleration of a car that moves at a steady velocity of 100 km/h for 100 seconds? Explain your answer. [Based on a problem by Hewitt.]

3 You are looking into a deep well. It is dark, and you cannot see the bottom. You want to find out how deep it is, so you drop a rock in, and you hear a splash 3.0 seconds later. How deep is the well?

4 A honeybee’s position as a function of time is given by \( x = 10t - t^3 \), where \( t \) is in seconds and \( x \) in meters. What is its acceleration at \( t = 3.0 \) s? [Solution, p. 553]

5 Alice drops a rock off a cliff. Bubba shoots a gun straight down from the edge of the same cliff. Compare the accelerations of the rock and the bullet while they are in the air on the way down. [Based on a problem by Serway and Faughn.]

6 The top part of the figure shows the position-versus-time graph for an object moving in one dimension. On the bottom part of the figure, sketch the corresponding v-versus-t graph.

7 (a) The ball is released at the top of the ramp shown in the figure. Friction is negligible. Use physical reasoning to draw \( v - t \) and \( a - t \) graphs. Assume that the ball doesn’t bounce at the point where the ramp changes slope. (b) Do the same for the case where the ball is rolled up the slope from the right side, but doesn’t quite have enough speed to make it over the top. [Solution, p. 554]

8 You throw a rubber ball up, and it falls and bounces several times. Draw graphs of position, velocity, and acceleration as functions of time. [Solution, p. 554]

9 A ball rolls down the ramp shown in the figure, consisting of a curved knee, a straight slope, and a curved bottom. For each part of the ramp, tell whether the ball’s velocity is increasing, decreasing, or constant, and also whether the ball’s acceleration is increasing, decreasing, or constant. Explain your answers. Assume there is no air friction or rolling resistance.
Consider the following passage from Alice in Wonderland, in which Alice has been falling for a long time down a rabbit hole:

Down, down, down. Would the fall never come to an end? “I wonder how many miles I’ve fallen by this time?” she said aloud. “I must be getting somewhere near the center of the earth. Let me see: that would be four thousand miles down, I think” (for, you see, Alice had learned several things of this sort in her lessons in the schoolroom, and though this was not a very good opportunity for showing off her knowledge, as there was no one to listen to her, still it was good practice to say it over)... 

Alice doesn’t know much physics, but let’s try to calculate the amount of time it would take to fall four thousand miles, starting from rest with an acceleration of $10 \text{ m/s}^2$. This is really only a lower limit; if there really was a hole that deep, the fall would actually take a longer time than the one you calculate, both because there is air friction and because gravity gets weaker as you get deeper (at the center of the earth, $g$ is zero, because the earth is pulling you equally in every direction at once).

In July 1999, Popular Mechanics carried out tests to find which car sold by a major auto maker could cover a quarter mile (402 meters) in the shortest time, starting from rest. Because the distance is so short, this type of test is designed mainly to favor the car with the greatest acceleration, not the greatest maximum speed (which is irrelevant to the average person). The winner was the Dodge Viper, with a time of 12.08 s. The car’s top (and presumably final) speed was 118.51 miles per hour (52.98 m/s). (a) If a car, starting from rest and moving with constant acceleration, covers a quarter mile in this time interval, what is its acceleration? (b) What would be the final speed of a car that covered a quarter mile with the constant acceleration you found in part a? (c) Based on the discrepancy between your answer in part b and the actual final speed of the Viper, what do you conclude about how its acceleration changed over time?

The photo shows Apollo 16 astronaut John Young jumping on the moon and saluting at the top of his jump. The video footage of the jump shows him staying aloft for 1.45 seconds. Gravity on the moon is 1/6 as strong as on the earth. Compute the height of the jump.

If the acceleration of gravity on Mars is 1/3 that on Earth, how many times longer does it take for a rock to drop the same distance on Mars? Ignore air resistance.

You climb half-way up a tree, and drop a rock. Then you climb to the top, and drop another rock. How many times greater is the velocity of the second rock on impact? Explain. (The answer is not two times greater.)
Starting from rest, a ball rolls down a ramp, traveling a distance \(L\) and picking up a final speed \(v\). How much of the distance did the ball have to cover before achieving a speed of \(v/2\)? [Based on a problem by Arnold Arons.

A toy car is released on one side of a piece of track that is bent into an upright \(U\) shape. The car goes back and forth. When the car reaches the limit of its motion on one side, its velocity is zero. Is its acceleration also zero? Explain using a \(v - t\) graph. [Based on a problem by Serway and Faughn.]

A physics homework question asks, “If you start from rest and accelerate at 1.54 m/s\(^2\) for 3.29 s, how far do you travel by the end of that time?” A student answers as follows:

\[
1.54 \times 3.29 = 5.07 \text{ m}
\]

His Aunt Wanda is good with numbers, but has never taken physics. She doesn’t know the formula for the distance traveled under constant acceleration over a given amount of time, but she tells her nephew his answer cannot be right. How does she know?

Find the error in the following calculation. A student wants to find the distance traveled by a car that accelerates from rest for 5.0 s with an acceleration of 2.0 m/s\(^2\). First he solves \(a = \Delta v/\Delta t\) for \(\Delta v = 10 \text{ m/s}\). Then he multiplies to find \((10 \text{ m/s})(5.0 \text{ s}) = 50 \text{ m}\). Do not just recalculate the result by a different method; if that was all you did, you’d have no way of knowing which calculation was correct, yours or his.

Acceleration could be defined either as \(\Delta v/\Delta t\) or as the slope of the tangent line on the \(v - t\) graph. Is either one superior as a definition, or are they equivalent? If you say one is better, give an example of a situation where it makes a difference which one you use.

If an object starts accelerating from rest, we have \(v^2 = 2a\Delta x\) for its speed after it has traveled a distance \(\Delta x\). Explain in words why it makes sense that the equation has velocity squared, but distance only to the first power. Don’t recapitulate the derivation in the book, or give a justification based on units. The point is to explain what this feature of the equation tells us about the way speed increases as more distance is covered.

The graph shows the acceleration of a chipmunk in a TV cartoon. It consists of two circular arcs and two line segments. At \(t = 0.00 \text{ s}\), the chipmunk’s velocity is \(-3.10 \text{ m/s}\). What is its velocity at \(t = 10.00 \text{ s}\)?
You take a trip in your spaceship to another star. Setting off, you increase your speed at a constant acceleration. Once you get half-way there, you start decelerating, at the same rate, so that by the time you get there, you have slowed down to zero speed. You see the tourist attractions, and then head home by the same method.

(a) Find a formula for the time, $T$, required for the round trip, in terms of $d$, the distance from our sun to the star, and $a$, the magnitude of the acceleration. Note that the acceleration is not constant over the whole trip, but the trip can be broken up into constant-acceleration parts.

(b) The nearest star to the Earth (other than our own sun) is Proxima Centauri, at a distance of $d = 4 \times 10^{16}$ m. Suppose you use an acceleration of $a = 10$ m/s$^2$, just enough to compensate for the lack of true gravity and make you feel comfortable. How long does the round trip take, in years?

(c) Using the same numbers for $d$ and $a$, find your maximum speed. Compare this to the speed of light, which is $3.0 \times 10^8$ m/s. (Later in this course, you will learn that there are some new things going on in physics when one gets close to the speed of light, and that it is impossible to exceed the speed of light. For now, though, just use the simpler ideas you’ve learned so far.)

Problem 23. This spectacular series of photos from a 2011 paper by Burrows and Sutton (“Biomechanics of jumping in the flea,” J. Exp. Biology 214:836) shows the flea jumping at about a 45-degree angle, but for the sake of this estimate just consider the case of a flea jumping vertically.

Some fleas can jump as high as 30 cm. The flea only has a short time to build up speed — the time during which its center of mass is accelerating upward but its feet are still in contact with the ground. Make an order-of-magnitude estimate of the acceleration the flea needs to have while straightening its legs, and state your answer in units of $g$, i.e., how many “$g$’s it pulls.” (For comparison, fighter pilots black out or die if they exceed about 5 or 10 $g$’s.)
The speed required for a low-earth orbit is $7.9 \times 10^3 \text{ m/s}$. When a rocket is launched into orbit, it goes up a little at first to get above almost all of the atmosphere, but then tips over horizontally to build up to orbital speed. Suppose the horizontal acceleration is limited to $3g$ to keep from damaging the cargo (or hurting the crew, for a crewed flight). (a) What is the minimum distance the rocket must travel downrange before it reaches orbital speed? How much does it matter whether you take into account the initial eastward velocity due to the rotation of the earth? (b) Rather than a rocket ship, it might be advantageous to use a railgun design, in which the craft would be accelerated to orbital speeds along a railroad track. This has the advantage that it isn’t necessary to lift a large mass of fuel, since the energy source is external. Based on your answer to part a, comment on the feasibility of this design for crewed launches from the earth’s surface.

A person is parachute jumping. During the time between when she leaps out of the plane and when she opens her chute, her altitude is given by an equation of the form

$$y = b - c\left(t + ke^{-t/k}\right),$$

where $e$ is the base of natural logarithms, and $b$, $c$, and $k$ are constants. Because of air resistance, her velocity does not increase at a steady rate as it would for an object falling in vacuum.

(a) What units would $b$, $c$, and $k$ have to have for the equation to make sense?
(b) Find the person’s velocity, $v$, as a function of time. [You will need to use the chain rule, and the fact that $d(e^x)/dx = e^x$.]
(c) Use your answer from part (b) to get an interpretation of the constant $c$. [Hint: $e^{-x}$ approaches zero for large values of $x$.]
(d) Find the person’s acceleration, $a$, as a function of time.
(e) Use your answer from part (d) to show that if she waits long enough to open her chute, her acceleration will become very small.
26. The figure shows a practical, simple experiment for determining \( g \) to high precision. Two steel balls are suspended from electromagnets, and are released simultaneously when the electric current is shut off. They fall through unequal heights \( \Delta x_1 \) and \( \Delta x_2 \). A computer records the sounds through a microphone as first one ball and then the other strikes the floor. From this recording, we can accurately determine the quantity \( T \) defined as \( T = \Delta t_2 - \Delta t_1 \), i.e., the time lag between the first and second impacts. Note that since the balls do not make any sound when they are released, we have no way of measuring the individual times \( \Delta t_2 \) and \( \Delta t_1 \).

(a) Find an equation for \( g \) in terms of the measured quantities \( T, \Delta x_1 \) and \( \Delta x_2 \).

(b) Check the units of your equation.

(c) Check that your equation gives the correct result in the case where \( \Delta x_1 \) is very close to zero. However, is this case realistic?

(d) What happens when \( \Delta x_1 = \Delta x_2 \)? Discuss this both mathematically and physically.

27. Most people don’t know that *Spinosaurus aegyptiacus*, not *Tyrannosaurus rex*, was the biggest theropod dinosaur. We can’t put a dinosaur on a track and time it in the 100 meter dash, so we can only infer from physical models how fast it could have run. When an animal walks at a normal pace, typically its legs swing more or less like pendulums of the same length \( \ell \). As a further simplification of this model, let’s imagine that the leg simply moves at a fixed acceleration as it falls to the ground. That is, we model the time for a quarter of a stride cycle as being the same as the time required for free fall from a height \( \ell \). *S. aegyptiacus* had legs about four times longer than those of a human. (a) Compare the time required for a human’s stride cycle to that for *S. aegyptiacus*.

(b) Compare their running speeds.

28. Engineering professor Qingming Li used sensors and video cameras to study punches delivered in the lab by British former welterweight boxing champion Ricky “the Hitman” Hatton. For comparison, Li also let a TV sports reporter put on the gloves and throw punches. The time it took for Hatton’s best punch to arrive, i.e., the time his opponent would have had to react, was about 0.47 of that for the reporter. Let’s assume that the fist starts from rest and moves with constant acceleration all the way up until impact, at some fixed distance (arm’s length). Compare Hatton’s acceleration to the reporter’s.
29. Aircraft carriers originated in World War I, and the first landing on a carrier was performed by E.H. Dunning in a Sopwith Pup biplane, landing on HMS Furious. (Dunning was killed the second time he attempted the feat.) In such a landing, the pilot slows down to just above the plane’s stall speed, which is the minimum speed at which the plane can fly without stalling. The plane then lands and is caught by cables and decelerated as it travels the length of the flight deck. Comparing a modern US F-14 fighter jet landing on an Enterprise-class carrier to Dunning’s original exploit, the stall speed is greater by a factor of 4.8, and to accommodate this, the length of the flight deck is greater by a factor of 1.9. Which deceleration is greater, and by what factor?

30. In college-level women’s softball in the U.S., typically a pitcher is expected to be at least 1.75 m tall, but Virginia Tech pitcher Jasmin Harrell is 1.62 m. Although a pitcher actually throws by stepping forward and swinging her arm in a circle, let’s make a simplified physical model to estimate how much of a disadvantage Harrell has had to overcome due to her height. We’ll pretend that the pitcher gives the ball a constant acceleration in a straight line, and that the length of this line is proportional to the pitcher’s height. Compare the acceleration Harrell would have to supply with the acceleration that would suffice for a pitcher of the nominal minimum height, if both were to throw a pitch at the same speed.

31. When the police engage in a high-speed chase on city streets, it can be extremely dangerous both to the police and to other motorists and pedestrians. Suppose that the police car must travel at a speed that is limited by the need to be able to stop before hitting a baby carriage, and that the distance at which the driver first sees the baby carriage is fixed. Tests show that in a panic stop from high speed, a police car based on a Chevy Impala has a deceleration 9% greater than that of a Dodge Intrepid. Compare the maximum safe speeds for the two cars.

32. When an object slides frictionlessly down a plane inclined at an angle $\theta$, its acceleration equals $g \sin \theta$ (example 7, p. 116). Suppose that a person on a bike is to coast down a ramp, starting from rest, and then coast back up an identical ramp, tracing a “V.” The horizontal distance is fixed to be $2w$, and we want to set the depth of the “V” so as to achieve the minimal possible value $t^*$ for the total time.

(a) Based only on units, infer the form of the expression for $t^*$ in terms of $w$, up to a unitless multiplicative constant.
(b) Find the angle that minimizes the time.
(c) Complete the determination of $t^*$ by finding the unitless constant.

$\triangleright$ Solution, p. 555  $\star$
33. The figure shows a circle in a vertical plane, with two wires positioned along chords of the circle. The top of each wire coincides with the top of the circle. Beads slide frictionlessly on the wires. If the beads are released simultaneously at the top, which one wins the race? You will need the fact that the acceleration equals $g \sin \theta$ (example 7, p. 116).

34. You shove a box with initial velocity 2.0 m/s, and it stops after sliding 1.3 m. What is the magnitude of the deceleration, assuming it is constant? [problem by B. Shotwell]

35. You’re an astronaut, and you’ve arrived on planet X, which is airless. You drop a hammer from a height of 1.00 m and find that it takes 350 ms to fall to the ground. What is the acceleration due to gravity on planet X? [problem by B. Shotwell]

36. A naughty child drops a golf ball from the roof of your apartment building, and you see it drop past your window. It takes the ball time $T$ to traverse the window’s height $H$. Find the initial speed of the ball when it first came into view. [problem by B. Shotwell]
Chapter 4
Force and Motion

If I have seen farther than others, it is because I have stood on the shoulders of giants.

*Newton, referring to Galileo*

Even as great and skeptical a genius as Galileo was unable to make much progress on the causes of motion. It was not until a generation later that Isaac Newton (1642-1727) was able to attack the problem successfully. In many ways, Newton’s personality was the opposite of Galileo’s. Where Galileo aggressively publicized his ideas,
Newton had to be coaxed by his friends into publishing a book on his physical discoveries. Where Galileo’s writing had been popular and dramatic, Newton originated the stilted, impersonal style that most people think is standard for scientific writing. (Scientific journals today encourage a less ponderous style, and papers are often written in the first person.) Galileo’s talent for arousing animosity among the rich and powerful was matched by Newton’s skill at making himself a popular visitor at court. Galileo narrowly escaped being burned at the stake, while Newton had the good fortune of being on the winning side of the revolution that replaced King James II with William and Mary of Orange, leading to a lucrative post running the English royal mint.

Newton discovered the relationship between force and motion, and revolutionized our view of the universe by showing that the same physical laws applied to all matter, whether living or nonliving, on or off of our planet’s surface. His book on force and motion, the Mathematical Principles of Natural Philosophy, was uncontradicted by experiment for 200 years, but his other main work, Optics, was on the wrong track, asserting that light was composed of particles rather than waves. Newton was also an avid alchemist, a fact that modern scientists would like to forget.

4.1 Force

We need only explain changes in motion, not motion itself.

So far you’ve studied the measurement of motion in some detail, but not the reasons why a certain object would move in a certain way. This chapter deals with the “why” questions. Aristotle’s ideas about the causes of motion were completely wrong, just like all his other ideas about physical science, but it will be instructive to start with them, because they amount to a road map of modern students’ incorrect preconceptions.

Aristotle thought he needed to explain both why motion occurs and why motion might change. Newton inherited from Galileo the important counter-Aristotelian idea that motion needs no explanation, that it is only changes in motion that require a physical cause. Aristotle’s needlessly complex system gave three reasons for motion:

Natural motion, such as falling, came from the tendency of objects to go to their “natural” place, on the ground, and come to rest.

Voluntary motion was the type of motion exhibited by animals, which moved because they chose to.

Forced motion occurred when an object was acted on by some other object that made it move.
Motion changes due to an interaction between two objects.

In the Aristotelian theory, natural motion and voluntary motion are one-sided phenomena: the object causes its own motion. Forced motion is supposed to be a two-sided phenomenon, because one object imposes its “commands” on another. Where Aristotle conceived of some of the phenomena of motion as one-sided and others as two-sided, Newton realized that a change in motion was always a two-sided relationship of a force acting between two physical objects.

The one-sided “natural motion” description of falling makes a crucial omission. The acceleration of a falling object is not caused by its own “natural” tendencies but by an attractive force between it and the planet Earth. Moon rocks brought back to our planet do not “want” to fly back up to the moon because the moon is their “natural” place. They fall to the floor when you drop them, just like our homegrown rocks. As we’ll discuss in more detail later in this course, gravitational forces are simply an attraction that occurs between any two physical objects. Minute gravitational forces can even be measured between human-scale objects in the laboratory.

The idea of natural motion also explains incorrectly why things come to rest. A basketball rolling across a beach slows to a stop because it is interacting with the sand via a frictional force, not because of its own desire to be at rest. If it was on a frictionless surface, it would never slow down. Many of Aristotle’s mistakes stemmed from his failure to recognize friction as a force.

The concept of voluntary motion is equally flawed. You may have been a little uneasy about it from the start, because it assumes a clear distinction between living and nonliving things. Today, however, we are used to having the human body likened to a complex machine. In the modern world-view, the border between the living and the inanimate is a fuzzy no-man’s land inhabited by viruses, prions, and silicon chips. Furthermore, Aristotle’s statement that you can take a step forward “because you choose to” inappropriately mixes two levels of explanation. At the physical level of explanation, the reason your body steps forward is because of a frictional force acting between your foot and the floor. If the floor was covered with a puddle of oil, no amount of “choosing to” would enable you to take a graceful stride forward.

Forces can all be measured on the same numerical scale.

In the Aristotelian-scholastic tradition, the description of motion as natural, voluntary, or forced was only the broadest level of classification, like splitting animals into birds, reptiles, mammals, and amphibians. There might be thousands of types of motion, each of which would follow its own rules. Newton’s realization that all changes in motion were caused by two-sided interactions made
it seem that the phenomena might have more in common than had been apparent. In the Newtonian description, there is only one cause for a change in motion, which we call force. Forces may be of different types, but they all produce changes in motion according to the same rules. Any acceleration that can be produced by a magnetic force can equally well be produced by an appropriately controlled stream of water. We can speak of two forces as being equal if they produce the same change in motion when applied in the same situation, which means that they pushed or pulled equally hard in the same direction.

The idea of a numerical scale of force and the newton unit were introduced in chapter 0. To recapitulate briefly, a force is when a pair of objects push or pull on each other, and one newton is the force required to accelerate a 1-kg object from rest to a speed of 1 m/s in 1 second.

More than one force on an object

As if we hadn’t kicked poor Aristotle around sufficiently, his theory has another important flaw, which is important to discuss because it corresponds to an extremely common student misconception. Aristotle conceived of forced motion as a relationship in which one object was the boss and the other “followed orders.” It therefore would only make sense for an object to experience one force at a time, because an object couldn’t follow orders from two sources at once. In the Newtonian theory, forces are numbers, not orders, and if more than one force acts on an object at once, the result is found by adding up all the forces. It is unfortunate that the use of the English word “force” has become standard, because to many people it suggests that you are “forcing” an object to do something. The force of the earth’s gravity cannot “force” a boat to sink, because there are other forces acting on the boat. Adding them up gives a total of zero, so the boat accelerates neither up nor down.

Objects can exert forces on each other at a distance.

Aristotle declared that forces could only act between objects that were touching, probably because he wished to avoid the type of occult speculation that attributed physical phenomena to the influence of a distant and invisible pantheon of gods. He was wrong, however, as you can observe when a magnet leaps onto your refrigerator or when the planet earth exerts gravitational forces on objects that are in the air. Some types of forces, such as friction, only operate between objects in contact, and are called contact forces. Magnetism, on the other hand, is an example of a noncontact force. Although the magnetic force gets stronger when the magnet is closer to your refrigerator, touching is not required.
Weight

In physics, an object’s weight, \( F_W \), is defined as the earth’s gravitational force on it. The SI unit of weight is therefore the Newton. People commonly refer to the kilogram as a unit of weight, but the kilogram is a unit of mass, not weight. Note that an object’s weight is not a fixed property of that object. Objects weigh more in some places than in others, depending on the local strength of gravity. It is their mass that always stays the same. A baseball pitcher who can throw a 90-mile-per-hour fastball on earth would not be able to throw any faster on the moon, because the ball’s inertia would still be the same.

Positive and negative signs of force

We’ll start by considering only cases of one-dimensional center-of-mass motion in which all the forces are parallel to the direction of motion, i.e., either directly forward or backward. In one dimension, plus and minus signs can be used to indicate directions of forces, as shown in figure c. We can then refer generically to addition of forces, rather than having to speak sometimes of addition and sometimes of subtraction. We add the forces shown in the figure and get 11 N. In general, we should choose a one-dimensional coordinate system with its \( x \) axis parallel the direction of motion. Forces that point along the positive \( x \) axis are positive, and forces in the opposite direction are negative. Forces that are not directly along the \( x \) axis cannot be immediately incorporated into this scheme, but that’s OK, because we’re avoiding those cases for now.

Discussion questions

A In chapter 0, I defined 1 N as the force that would accelerate a 1-kg mass from rest to 1 m/s in 1 s. Anticipating the following section, you might guess that 2 N could be defined as the force that would accelerate the same mass to twice the speed, or twice the mass to the same speed. Is there an easier way to define 2 N based on the definition of 1 N?

4.2 Newton’s first law

We are now prepared to make a more powerful restatement of the principle of inertia.\(^1\)

\[\text{Newton’s first law}\]

If the total force acting on an object is zero, its center of mass continues in the same state of motion.

In other words, an object initially at rest is predicted to remain at rest if the total force acting on it is zero, and an object in motion

\(^1\)Page 71 lists places in this book where we describe experimental tests of the principle of inertia and Newton’s first law.
remains in motion with the same velocity in the same direction. The converse of Newton’s first law is also true: if we observe an object moving with constant velocity along a straight line, then the total force on it must be zero.

In a future physics course or in another textbook, you may encounter the term “net force,” which is simply a synonym for total force.

What happens if the total force on an object is not zero? It accelerates. Numerical prediction of the resulting acceleration is the topic of Newton’s second law, which we’ll discuss in the following section.

This is the first of Newton’s three laws of motion. It is not important to memorize which of Newton’s three laws are numbers one, two, and three. If a future physics teacher asks you something like, “Which of Newton’s laws are you thinking of?,” a perfectly acceptable answer is “The one about constant velocity when there’s zero total force.” The concepts are more important than any specific formulation of them. Newton wrote in Latin, and I am not aware of any modern textbook that uses a verbatim translation of his statement of the laws of motion. Clear writing was not in vogue in Newton’s day, and he formulated his three laws in terms of a concept now called momentum, only later relating it to the concept of force. Nearly all modern texts, including this one, start with force and do momentum later.

An elevator example 1

An elevator has a weight of 5000 N. Compare the forces that the cable must exert to raise it at constant velocity, lower it at constant velocity, and just keep it hanging.

In all three cases the cable must pull up with a force of exactly 5000 N. Most people think you’d need at least a little more than 5000 N to make it go up, and a little less than 5000 N to let it down, but that’s incorrect. Extra force from the cable is only necessary for speeding the car up when it starts going up or slowing it down when it finishes going down. Decreased force is needed to speed the car up when it gets going down and to slow it down when it finishes going up. But when the elevator is cruising at constant velocity, Newton’s first law says that you just need to cancel the force of the earth’s gravity.

To many students, the statement in the example that the cable’s upward force “cancels” the earth’s downward gravitational force implies that there has been a contest, and the cable’s force has won, vanquishing the earth’s gravitational force and making it disappear. That is incorrect. Both forces continue to exist, but because they add up numerically to zero, the elevator has no center-of-mass acceleration. We know that both forces continue to exist because they both have side-effects other than their effects on the car’s center-of-
mass motion. The force acting between the cable and the car continues to produce tension in the cable and keep the cable taut. The earth’s gravitational force continues to keep the passengers (whom we are considering as part of the elevator-object) stuck to the floor and to produce internal stresses in the walls of the car, which must hold up the floor.

**Terminal velocity for falling objects example 2**

An object like a feather that is not dense or streamlined does not fall with constant acceleration, because air resistance is nonnegligible. In fact, its acceleration tapers off to nearly zero within a fraction of a second, and the feather finishes dropping at constant speed (known as its terminal velocity). Why does this happen?

Newton’s first law tells us that the total force on the feather must have been reduced to nearly zero after a short time. There are two forces acting on the feather: a downward gravitational force from the planet earth, and an upward frictional force from the air. As the feather speeds up, the air friction becomes stronger and stronger, and eventually it cancels out the earth’s gravitational force, so the feather just continues with constant velocity without speeding up any more.

The situation for a skydiver is exactly analogous. It’s just that the skydiver experiences perhaps a million times more gravitational force than the feather, and it is not until she is falling very fast that the force of air friction becomes as strong as the gravitational force. It takes her several seconds to reach terminal velocity, which is on the order of a hundred miles per hour.

**More general combinations of forces**

It is too constraining to restrict our attention to cases where all the forces lie along the line of the center of mass’s motion. For one thing, we can’t analyze any case of horizontal motion, since any object on earth will be subject to a vertical gravitational force! For instance, when you are driving your car down a straight road, there are both horizontal forces and vertical forces. However, the vertical forces have no effect on the center of mass motion, because the road’s upward force simply counteracts the earth’s downward gravitational force and keeps the car from sinking into the ground.

Later in the book we’ll deal with the most general case of many forces acting on an object at any angles, using the mathematical technique of vector addition, but the following slight generalization of Newton’s first law allows us to analyze a great many cases of interest:

Suppose that an object has two sets of forces acting on it, one set along the line of the object’s initial motion and another set perpendicular to the first set. If both sets of forces cancel, then the object’s center of mass continues in the same state of motion.
A passenger riding the subway

Example 3

Describe the forces acting on a person standing in a subway train that is cruising at constant velocity.

No force is necessary to keep the person moving relative to the ground. He will not be swept to the back of the train if the floor is slippery. There are two vertical forces on him, the earth’s downward gravitational force and the floor’s upward force, which cancel. There are no horizontal forces on him at all, so of course the total horizontal force is zero.

Forces on a sailboat

Example 4

If a sailboat is cruising at constant velocity with the wind coming from directly behind it, what must be true about the forces acting on it?

The forces acting on the boat must be canceling each other out. The boat is not sinking or leaping into the air, so evidently the vertical forces are canceling out. The vertical forces are the downward gravitational force exerted by the planet earth and an upward force from the water.

The air is making a forward force on the sail, and if the boat is not accelerating horizontally then the water’s backward frictional force must be canceling it out.

Contrary to Aristotle, more force is not needed in order to maintain a higher speed. Zero total force is always needed to maintain constant velocity. Consider the following made-up numbers:

<table>
<thead>
<tr>
<th></th>
<th>boat moving at a low, constant velocity</th>
<th>boat moving at a high, constant velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>forward force of the wind on the sail . . .</td>
<td>10,000 N</td>
<td>20,000 N</td>
</tr>
<tr>
<td>backward force of the water on the hull . . .</td>
<td>−10,000 N</td>
<td>−20,000 N</td>
</tr>
<tr>
<td>total force on the boat . . .</td>
<td>0 N</td>
<td>0 N</td>
</tr>
</tbody>
</table>

The faster boat still has zero total force on it. The forward force on it is greater, and the backward force smaller (more negative), but that’s irrelevant because Newton’s first law has to do with the total force, not the individual forces.

This example is quite analogous to the one about terminal velocity of falling objects, since there is a frictional force that increases with speed. After casting off from the dock and raising the sail, the boat will accelerate briefly, and then reach its terminal velocity, at which the water’s frictional force has become as great as the wind’s force on the sail.
A car crash example 5

If you drive your car into a brick wall, what is the mysterious force that slams your face into the steering wheel?

Your surgeon has taken physics, so she is not going to believe your claim that a mysterious force is to blame. She knows that your face was just following Newton’s first law. Immediately after your car hit the wall, the only forces acting on your head were the same canceling-out forces that had existed previously: the earth’s downward gravitational force and the upward force from your neck. There were no forward or backward forces on your head, but the car did experience a backward force from the wall, so the car slowed down and your face caught up.

Discussion questions

A Newton said that objects continue moving if no forces are acting on them, but his predecessor Aristotle said that a force was necessary to keep an object moving. Why does Aristotle’s theory seem more plausible, even though we now believe it to be wrong? What insight was Aristotle missing about the reason why things seem to slow down naturally? Give an example.

B In the figure what would have to be true about the saxophone’s initial motion if the forces shown were to result in continued one-dimensional motion of its center of mass?

C This figure requires an ever further generalization of the preceding discussion. After studying the forces, what does your physical intuition tell you will happen? Can you state in words how to generalize the conditions for one-dimensional motion to include situations like this one?

4.3 Newton’s second law

What about cases where the total force on an object is not zero, so that Newton’s first law doesn’t apply? The object will have an acceleration. The way we’ve defined positive and negative signs of force and acceleration guarantees that positive forces produce positive accelerations, and likewise for negative values. How much acceleration will it have? It will clearly depend on both the object’s mass and on the amount of force.

Experiments with any particular object show that its acceleration is directly proportional to the total force applied to it. This may seem wrong, since we know of many cases where small amounts of force fail to move an object at all, and larger forces get it going. This apparent failure of proportionality actually results from forgetting that there is a frictional force in addition to the force we apply to move the object. The object’s acceleration is exactly proportional to the total force on it, not to any individual force on it. In the absence of friction, even a very tiny force can slowly change the velocity of a very massive object.
Experiments (e.g., the one described in example 12 on p. 140) also show that the acceleration is inversely proportional to the object’s mass, and combining these two proportionalities gives the following way of predicting the acceleration of any object:

\[ a = \frac{F_{\text{total}}}{m}, \]

where

- \( m \) is an object’s mass, a measure of its resistance to changes in its motion
- \( F_{\text{total}} \) is the sum of the forces acting on it, and
- \( a \) is the acceleration of the object’s center of mass.

We are presently restricted to the case where the forces of interest are parallel to the direction of motion.

We have already encountered the SI unit of force, which is the newton (N). It is designed so that the units in Newton’s second law all work out if we use SI units: \( m/s^2 \) for acceleration and kg (not grams!) for mass.

**Rocket science example 6**

The Falcon 9 launch vehicle, built and operated by the private company SpaceX, has mass \( m = 5.1 \times 10^5 \) kg. At launch, it has two forces acting on it: an upward thrust \( F_t = 5.9 \times 10^6 \) N and a downward gravitational force of \( F_g = 5.0 \times 10^6 \) N. Find its acceleration.

Let’s choose our coordinate system such that positive is up. Then the downward force of gravity is considered negative. Using Newton’s second law,

\[
\begin{align*}
a &= \frac{F_{\text{total}}}{m} \\
&= \frac{F_t - F_g}{m} \\
&= \frac{(5.9 \times 10^6 \text{ N}) - (5.0 \times 10^6 \text{ N})}{5.1 \times 10^5 \text{ kg}} \\
&= 1.6 \text{ m/s}^2,
\end{align*}
\]

where as noted above, units of N/kg (newtons per kilogram) are the same as \( m/s^2 \).
An accelerating bus example 7

A VW bus with a mass of 2000 kg accelerates from 0 to 25 m/s (freeway speed) in 34 s. Assuming the acceleration is constant, what is the total force on the bus?

We solve Newton’s second law for $F_{total} = ma$, and substitute $\Delta v/\Delta t$ for $a$, giving

$$F_{total} = m\Delta v/\Delta t$$
$$= (2000 \text{ kg})(25 \text{ m/s} - 0 \text{ m/s})/(34 \text{ s})$$
$$= 1.5 \text{ kN}.$$

Some applications of calculus

Newton doesn’t care what frame of reference you use his laws in, and this makes him different from Aristotle, who says there is something special about the frame of reference attached firmly to the dirt underfoot. Suppose that an object obeys Newton’s second law in the dirt’s frame. It has some velocity that is a function of time, and differentiating this function gives $dv/dt = F/m$. Suppose we change to the frame of reference of a train that is in motion relative to the dirt at constant velocity $c$. Looking out the window of the train, we see the object moving with velocity $v - c$. But the derivative of a constant is zero, so when we differentiate $v - c$, the constant goes away, and we get exactly the same result. Newton is still happy, although Aristotle feels a great disturbance in the force.

Often we know the forces acting on an object, and we want to find its motion, i.e., its position as a function of time, $x(t)$. Since Newton’s second law predicts the acceleration $d^2 x/dt^2$, we need to integrate twice to find $x$. The first integration gives the velocity, and the constant of integration is also a velocity, which can be fixed by giving the object’s velocity at some initial time. In the second integration we pick up a second constant of integration, this one related to an initial position.

A force that tapers off to zero example 8

An object of mass $m$ starts at rest at $t = t_0$. A force varying as $F = bt^{-2}$, where $b$ is a constant, begins acting on it. Find the greatest speed it will ever have.

$$F = m\frac{dv}{dt}$$
$$dv = \frac{F}{m} dt$$
$$\int dv = \int \frac{F}{m} dt$$
$$v = -\frac{b}{m} t^{-1} + v^*,$$
where $v^*$ is a constant of integration with units of velocity. The given initial condition is that $v = 0$ at $t = t_0$, so we find that $v^* = b/mt_o$. The negative term gets closer to zero with increasing time, so the maximum velocity is achieved by letting $t$ approach infinity. That is, the object will never stop speeding up, but it will also never surpass a certain speed. In the limit $t \to \infty$, we identify $v^*$ as the velocity that the object will approach asymptotically.

A generalization

As with the first law, the second law can be easily generalized to include a much larger class of interesting situations:

Suppose an object is being acted on by two sets of forces, one set lying parallel to the object’s initial direction of motion and another set acting along a perpendicular line. If the forces perpendicular to the initial direction of motion cancel out, then the object accelerates along its original line of motion according to $a = F\parallel/m$, where $F\parallel$ is the sum of the forces parallel to the line.

A coin sliding across a table

Suppose a coin is sliding to the right across a table, f, and let’s choose a positive $x$ axis that points to the right. The coin’s velocity is positive, and we expect based on experience that it will slow down, i.e., its acceleration should be negative.

Although the coin’s motion is purely horizontal, it feels both vertical and horizontal forces. The Earth exerts a downward gravitational force $F_2$ on it, and the table makes an upward force $F_3$ that prevents the coin from sinking into the wood. In fact, without these vertical forces the horizontal frictional force wouldn’t exist: surfaces don’t exert friction against one another unless they are being pressed together.

Although $F_2$ and $F_3$ contribute to the physics, they do so only indirectly. The only thing that directly relates to the acceleration along the horizontal direction is the horizontal force: $a = F_1/m$.

The relationship between mass and weight

Mass is different from weight, but they’re related. An apple’s mass tells us how hard it is to change its motion. Its weight measures the strength of the gravitational attraction between the apple and the planet earth. The apple’s weight is less on the moon, but its mass is the same. Astronauts assembling the International Space Station in zero gravity couldn’t just pitch massive modules back and forth with their bare hands; the modules were weightless, but not massless.

We have already seen the experimental evidence that when weight (the force of the earth’s gravity) is the only force acting on an object, its acceleration equals the constant $g$, and $g$ depends on where
A simple double-pan balance works by comparing the weight forces exerted by the earth on the contents of the two pans. Since the two pans are at almost the same location on the earth's surface, the value of \( g \) is essentially the same for each one, and equality of weight therefore also implies equality of mass.

**Example 10.** You are on the surface of the earth, but not on the mass of the object. Applying Newton’s second law then allows us to calculate the magnitude of the gravitational force on any object in terms of its mass:

\[
|F_W| = mg.
\]

(The equation only gives the magnitude, i.e. the absolute value, of \( F_W \), because we're defining \( g \) as a positive number, so it equals the absolute value of a falling object’s acceleration.)

The readings.

Let’s start with the single kilogram. It’s not accelerating, so evidently the total force on it is zero: the spring scale’s upward force on it is canceling out the earth’s downward gravitational force. The spring scale tells us how much force it is being obliged to supply, but since the two forces are equal in strength, the spring scale’s reading can also be interpreted as measuring the strength of the gravitational force, i.e., the weight of the one-kilogram mass. The weight of a one-kilogram mass should be

\[
F_W = mg
\]

\[
= (1.0 \text{ kg})(9.8 \text{ m/s}^2) = 9.8 \text{ N},
\]

and that’s indeed the reading on the spring scale.

Similarly for the two-kilogram mass, we have

\[
F_W = mg
\]

\[
= (2.0 \text{ kg})(9.8 \text{ m/s}^2) = 19.6 \text{ N}.
\]

Calculating terminal velocity

Experiments show that the force of air friction on a falling object such as a skydiver or a feather can be approximated fairly well with the equation \( |F_{air}| = c \rho A v^2 \), where \( c \) is a constant, \( \rho \) is the density of the air, \( A \) is the cross-sectional area of the object as seen from below, and \( v \) is the object’s velocity. Predict the object’s terminal velocity, i.e., the final velocity it reaches after a long time.

As the object accelerates, its greater \( v \) causes the upward force of the air to increase until finally the gravitational force and the force of air friction cancel out, after which the object continues at constant velocity. We choose a coordinate system in which positive is up, so that the gravitational force is negative and the
force of air friction is positive. We want to find the velocity at which

\[ F_{\text{air}} + F_W = 0, \quad \text{i.e.,} \]
\[ c_p Av^2 - mg = 0. \]

Solving for \( v \) gives

\[ v_{\text{terminal}} = \sqrt{\frac{mg}{c_pA}} \]

**self-check A**

It is important to get into the habit of interpreting equations. This may be difficult at first, but eventually you will get used to this kind of reasoning.

1. Interpret the equation \( v_{\text{terminal}} = \sqrt{\frac{mg}{c_pA}} \) in the case of \( \rho = 0 \).
2. How would the terminal velocity of a 4-cm steel ball compare to that of a 1-cm ball?
3. In addition to teasing out the *mathematical* meaning of an equation, we also have to be able to place it in its *physical* context. How generally important is this equation?  

[A test of the second law](#)

Because the force \( mg \) of gravity on an object of mass \( m \) is proportional to \( m \), the acceleration predicted by Newton's second law is \( a = F/m = mg/m = g \), in which the mass cancels out. It is therefore an ironclad prediction of Newton's laws of motion that free fall is universal: in the absence of other forces such as air resistance, heavier objects do not fall with a greater acceleration than lighter ones. The experiment by Galileo at the Leaning Tower of Pisa (p. 104) is therefore consistent with Newton's second law. Since Galileo's time, experimental methods have had several centuries in which to improve, and the second law has been subjected to similar tests with exponentially improving precision. For such an experiment in 1993,\(^2\) physicists at the University of Pisa (!) built a metal disk out of copper and tungsten semicircles joined together at their flat edges. They evacuated the air from a vertical shaft and dropped the disk down it 142 times, using lasers to measure any tiny rotation that would result if the accelerations of the copper and tungsten were very slightly different. The results were statistically consistent with zero rotation, and put an upper limit of \( 1 \times 10^{-9} \) on the fractional difference in acceleration \( |g_{\text{copper}} - g_{\text{tungsten}}|/g \). A more recent experiment using test masses in orbit\(^3\) has refined this bound to \( 10^{-14} \).


\(^3\)Touboul *et al.*, “The MICROSCOPE mission: first results of a space test of the Equivalence Principle,” arxiv.org/abs/1712.01176
A failure of the second law  

The graph in the figure displays data from a 1964 experiment by Bortozzi that shows how Newton’s second law fails if you keep on applying a force to an object indefinitely. Electrons were accelerated by a constant electrical force through a certain distance. Applying Newton’s laws gives Newtonian predictions $a_N$ for the acceleration and $t_N$ for the time required. The electrons were then allowed to fly down a pipe for a further distance of 8.4 m without being acted on by any force. The time of flight for this second distance was used to find the final velocity $v$ to which they had actually been accelerated.

According to Newton, an acceleration $a_N$ acting for a time $t_N$ should produce a final velocity $a_N t_N$. The solid line in the graph shows the prediction of Newton’s laws, which is that a constant force exerted steadily over time will produce a velocity that rises linearly and without limit.

The experimental data, shown as black dots, clearly tell a different story. The velocity never goes above a certain maximum value, which turns out to be the speed of light. The dashed line shows the predictions of Einstein’s theory of special relativity, which are in good agreement with the experimental results. This experiment is an example of a general fact, which is that Newton’s laws are only good approximations when objects move at velocities that are small compared to the speed of light. This is discussed further on p. 154.

Discussion questions

A Show that the Newton can be reexpressed in terms of the three basic mks units as the combination $\text{kg} \cdot \text{m} / \text{s}^2$.

B What is wrong with the following statements?

(1) “g is the force of gravity.”

(2) “Mass is a measure of how much space something takes up.”

C Criticize the following incorrect statement:

“If an object is at rest and the total force on it is zero, it stays at rest. There can also be cases where an object is moving and keeps on moving without having any total force on it, but that can only happen when there’s no friction, like in outer space.”

D Table k gives laser timing data for Ben Johnson’s 100 m dash at the 1987 World Championship in Rome. (His world record was later revoked because he tested positive for steroids.) How does the total force on him change over the duration of the race?
4.4 What force is not

Violin teachers have to endure their beginning students’ screeching. A frown appears on the woodwind teacher’s face as she watches her student take a breath with an expansion of his ribcage but none in his belly. What makes physics teachers cringe is their students’ verbal statements about forces. Below I have listed six dicta about what force is not.

1. Force is not a property of one object.

A great many of students’ incorrect descriptions of forces could be cured by keeping in mind that a force is an interaction of two objects, not a property of one object.

*Incorrect statement:* “That magnet has a lot of force.”

× If the magnet is one millimeter away from a steel ball bearing, they may exert a very strong attraction on each other, but if they were a meter apart, the force would be virtually undetectable. The magnet’s strength can be rated using certain electrical units (ampere − meters$^2$), but not in units of force.

2. Force is not a measure of an object’s motion.

If force is not a property of a single object, then it cannot be used as a measure of the object’s motion.

*Incorrect statement:* “The freight train rumbled down the tracks with awesome force.”

× Force is not a measure of motion. If the freight train collides with a stalled cement truck, then some awesome forces will occur, but if it hits a fly the force will be small.

3. Force is not energy.

There are two main approaches to understanding the motion of objects, one based on force and one on a different concept, called energy. The SI unit of energy is the Joule, but you are probably more familiar with the calorie, used for measuring food’s energy, and the kilowatt-hour, the unit the electric company uses for billing you. Physics students’ previous familiarity with calories and kilowatt-hours is matched by their universal unfamiliarity with measuring forces in units of Newtons, but the precise operational definitions of the energy concepts are more complex than those of the force concepts, and textbooks, including this one, almost universally place the force description of physics before the energy description. During the long period after the introduction of force and before the careful definition of energy, students are therefore vulnerable to situations in which, without realizing it, they are imputing the properties of energy to phenomena of force.

*Incorrect statement:* “How can my chair be making an upward force on my rear end? It has no power!”

× Power is a concept related to energy, e.g., a 100-watt lightbulb uses
up 100 joules per second of energy. When you sit in a chair, no energy is used up, so forces can exist between you and the chair without any need for a source of power.

4. Force is not stored or used up.

Because energy can be stored and used up, people think force also can be stored or used up.

Incorrect statement: “If you don’t fill up your tank with gas, you’ll run out of force.”

X Energy is what you’ll run out of, not force.

5. Forces need not be exerted by living things or machines.

Transforming energy from one form into another usually requires some kind of living or mechanical mechanism. The concept is not applicable to forces, which are an interaction between objects, not a thing to be transferred or transformed.

Incorrect statement: “How can a wooden bench be making an upward force on my rear end? It doesn’t have any springs or anything inside it.”

X No springs or other internal mechanisms are required. If the bench didn’t make any force on you, you would obey Newton’s second law and fall through it. Evidently it does make a force on you!

6. A force is the direct cause of a change in motion.

I can click a remote control to make my garage door change from being at rest to being in motion. My finger’s force on the button, however, was not the force that acted on the door. When we speak of a force on an object in physics, we are talking about a force that acts directly. Similarly, when you pull a reluctant dog along by its leash, the leash and the dog are making forces on each other, not your hand and the dog. The dog is not even touching your hand.

self-check B

Which of the following things can be correctly described in terms of force?

(1) A nuclear submarine is charging ahead at full steam.

(2) A nuclear submarine’s propellers spin in the water.

(3) A nuclear submarine needs to refuel its reactor periodically.

Answer, p. 568

Discussion questions

A Criticize the following incorrect statement: “If you shove a book across a table, friction takes away more and more of its force, until finally it stops.”

B You hit a tennis ball against a wall. Explain any and all incorrect ideas in the following description of the physics involved: “The ball gets some force from you when you hit it, and when it hits the wall, it loses part of that force, so it doesn’t bounce back as fast. The muscles in your arm are the only things that a force can come from.”
4.5 Inertial and noninertial frames of reference

One day, you’re driving down the street in your pickup truck, on your way to deliver a bowling ball. The ball is in the back of the truck, enjoying its little jaunt and taking in the fresh air and sunshine. Then you have to slow down because a stop sign is coming up. As you brake, you glance in your rearview mirror, and see your trusty companion accelerating toward you. Did some mysterious force push it forward? No, it only seems that way because you and the car are slowing down. The ball is faithfully obeying Newton’s first law, and as it continues at constant velocity it gets ahead relative to the slowing truck. No forces are acting on it (other than the same canceling-out vertical forces that were always acting on it). The ball only appeared to violate Newton’s first law because there was something wrong with your frame of reference, which was based on the truck.

1. In a frame of reference that moves with the truck, the bowling ball appears to violate Newton’s first law by accelerating despite having no horizontal forces on it. 2. In an inertial frame of reference, which the surface of the earth approximately is, the bowling ball obeys Newton’s first law. It moves equal distances in equal time intervals, i.e., maintains constant velocity. In this frame of reference, it is the truck that appears to have a change in velocity, which makes sense, since the road is making a horizontal force on it.

How, then, are we to tell in which frames of reference Newton’s laws are valid? It’s no good to say that we should avoid moving frames of reference, because there is no such thing as absolute rest or absolute motion. All frames can be considered as being either at rest or in motion. According to an observer in India, the strip mall that constituted the frame of reference in panel (b) of the figure was moving along with the earth’s rotation at hundreds of miles per hour.

The reason why Newton’s laws fail in the truck’s frame of reference is that the friction is not considered.

\(^4\)Let’s assume for simplicity that there is no friction.
ence is not because the truck is *moving* but because it is *accelerating*. (Recall that physicists use the word to refer either to speeding up or slowing down.) Newton’s laws were working just fine in the moving truck’s frame of reference as long as the truck was moving at constant velocity. It was only when its speed changed that there was a problem. How, then, are we to tell which frames are accelerating and which are not? What if you claim that your truck is not accelerating, and the sidewalk, the asphalt, and the Burger King are accelerating? The way to settle such a dispute is to examine the motion of some object, such as the bowling ball, which we know has zero total force on it. Any frame of reference in which the ball appears to obey Newton’s first law is then a valid frame of reference, and to an observer in that frame, Mr. Newton assures us that all the other objects in the universe will obey his laws of motion, not just the ball.

Valid frames of reference, in which Newton’s laws are obeyed, are called inertial frames of reference. Frames of reference that are not inertial are called noninertial frames. In those frames, objects violate the principle of inertia and Newton’s first law. While the truck was moving at constant velocity, both it and the sidewalk were valid inertial frames. The truck became an invalid frame of reference when it began changing its velocity.

You usually assume the ground under your feet is a perfectly inertial frame of reference, and we made that assumption above. It isn’t perfectly inertial, however. Its motion through space is quite complicated, being composed of a part due to the earth’s daily rotation around its own axis, the monthly wobble of the planet caused by the moon’s gravity, and the rotation of the earth around the sun. Since the accelerations involved are numerically small, the earth is approximately a valid inertial frame.

Noninertial frames are avoided whenever possible, and we will seldom, if ever, have occasion to use them in this course. Sometimes, however, a noninertial frame can be convenient. Naval gunners, for instance, get all their data from radars, human eyeballs, and other detection systems that are moving along with the earth’s surface. Since their guns have ranges of many miles, the small discrepancies between their shells’ actual accelerations and the accelerations predicted by Newton’s second law can have effects that accumulate and become significant. In order to kill the people they want to kill, they have to add small corrections onto the equation \( a = \frac{F_{\text{total}}}{m} \). Doing their calculations in an inertial frame would allow them to use the usual form of Newton’s second law, but they would have to convert all their data into a different frame of reference, which would require cumbersome calculations.
Discussion question

A If an object has a linear $x - t$ graph in a certain inertial frame, what is the effect on the graph if we change to a coordinate system with a different origin? What is the effect if we keep the same origin but reverse the positive direction of the $x$ axis? How about an inertial frame moving alongside the object? What if we describe the object's motion in a noninertial frame?
4.6 * Numerical techniques

Engineering majors are a majority of the students in the kind of physics course for which this book is designed, so most likely you fall into that category. Although you surely recognize that physics is an important part of your training, if you’ve had any exposure to how engineers really work, you’re probably skeptical about the flavor of problem-solving taught in most science courses. You realize that not very many practical engineering calculations fall into the narrow range of problems for which an exact solution can be calculated with a piece of paper and a sharp pencil. Real-life problems are usually complicated, and typically they need to be solved by number-crunching on a computer, although we can often gain insight by working simple approximations that have algebraic solutions. Not only is numerical problem-solving more useful in real life, it’s also educational; as a beginning physics student, I really only felt like I understood projectile motion after I had worked it both ways, using algebra and then a computer program.

In this section, we’ll start by seeing how to apply numerical techniques to some simple problems for which we know the answer in “closed form,” i.e., a single algebraic expression without any calculus or infinite sums. After that, we’ll solve a problem that would have made you world-famous if you could have done it in the seventeenth century using paper and a quill pen! Before you continue, you should read Appendix 1 on page 548 that introduces you to the Python programming language.

First let’s solve the trivial problem of finding the distance traveled by an object moving at speed $v$ to in time $t$. This closed-form answer is, of course, $vt$, but the point is to introduce the techniques we can use to solve other problems of this type. The basic idea is to divide the time up into $n$ equal parts, and add up the distances traveled in all the parts. The following Python function does the job. Note that you shouldn’t type in the line numbers on the left, and you don’t need to type in the comments, either.

```python
1 import math
2 def dist(n):
3     t = 1.0 # seconds
4     v = 1.0 # m/s
5     x = 0 # Initialize the position.
6     dt = t/n # Divide t into n equal parts.
7     for i in range(n):
8         dx = v*dt # tiny distance traveled in dt
9         x = x+dx # Change x.
10     return x
```

Of course line 8 shows how silly this example is — if we knew $dx = vdt$, then presumably we knew $x = vt$, which was the answer to
the whole problem — but the point is to illustrate the technique with the simplest possible example. How far do we move in 1 s at a constant speed of 1 m/s? If we do this,

```python
>>> print(dist(10))
1.0
```

Python produces the expected answer by dividing the time into ten equal 0.1-second intervals, and adding up the ten 0.1-meter segments traversed in them. Since the object moves at constant speed, it doesn’t even matter whether we set \( n \) to 10, 1, or a million:

```python
>>> print(dist(1))
1.0
```

Now let’s do an example where the answer isn’t obvious to people who don’t know calculus: through what distance does an object fall in 1.0 s, starting from rest? By integrating \( a = g \) to find \( v = gt \) and the integrating again to get \( x = \frac{1}{2}gt^2 \), we know that the exact answer is 4.9 m. Let’s see if we can reproduce that answer numerically, as suggested by figure m. The main difference between this program and the previous one is that now the velocity isn’t constant, so we need to update it as we go along.

```python
1 import math
2 def dist2(n):
3     t = 1.0 # seconds
4     g=9.8 # strength of gravity, in m/s^2
5     x=0 # Initialize the distance fallen.
6     v=0 # Initialize the velocity.
7     dt = t/n # Divide t into n equal parts.
8     for i in range(n):
9         dx = v*dt # tiny distance traveled during tiny time dt
10        x = x+dx # Change x.
11        dv = g*dt # tiny change in vel. during tiny time dt
12        v = v+dv
13     return x
```

With the drop split up into only 10 equal height intervals, the numerical technique provides a decent approximation:

```python
>>> print(dist2(10))
4.41
```

By increasing \( n \) to ten thousand, we get an answer that’s as close as we need, given the limited accuracy of the raw data:

```python
>>> print(dist2(10000))
4.89951
```
Now let’s use these techniques to solve the following somewhat whimsical problem, which cannot be solved in closed form using elementary functions such as sines, exponentials, roots, etc.

Ann E. Hodges of Sylacauga, Alabama is the only person ever known to have been injured by a meteorite. In 1954, she was struck in the hip by a meteorite that crashed through the roof of her house while she was napping on the couch. Since Hodges was asleep, we do not have direct evidence on the following silly trivia question: if you’re going to be hit by a meteorite, will you hear it coming, or will it approach at more than the speed of sound? To answer this question, we start by constructing a physical model that is as simple as possible. We take the meteor as entering the earth’s atmosphere directly along the vertical. The atmosphere does not cut off suddenly at a certain height; its density can be approximated as being proportional to $e^{-x/H}$, where $x$ is the altitude and $H \approx 7.6$ km is called the scale height. The force of air friction is proportional to the density and to the square of the velocity, so

$$F = bv^2e^{-x/H}$$

where $b$ is a constant and $F$ is positive in the coordinate system we’ve chosen, where $+x$ is up. The constant $b$ depends on the size of the object, and its mass also affects the acceleration through Newton’s second law, $a = F/m$. The answer to the question therefore depends on the size of the meteorite. However, it is reasonable to take the results for the Sylacauga meteorite as constituting a general answer to our question, since larger ones are very rare, while the much more common pebble-sized ones do not make it through the atmosphere before they disintegrate. The object’s initial velocity as it entered the atmosphere is not known, so we assume a typical value of 20 km/s. The Sylacauga meteorite was seen breaking up into three pieces, only two of which were recovered. The complete object’s mass was probably about 7 kg and its radius about 9 cm. For an object with this radius, we expect $b \approx 1.5 \times 10^{-3}$ kg/m. Using Newton’s second law, we find

$$a = \frac{F_{total}}{m} = \frac{bv^2e^{-x/H} - mg}{m}.$$  

I don’t know of any way to solve this to find the function $x(t)$ closed form, so let’s solve it numerically.

This problem is of a slightly different form than the ones above, where we knew we wanted to follow the motion up until a certain time. This problem is more of an “Are we there yet?” We want to stop the calculation where the altitude reaches zero. If it started at an initial position $x$ at velocity $v$ ($v < 0$) and maintained that velocity all the way down to sea level, the time interval would be
\[ \Delta t = \Delta x/v = (0 - x)/v = -x/v. \] Since it actually slows down, \( \Delta t \) will be greater than that. We guess ten times that as a maximum, and then have the program check each time through the loop to see if we’ve hit the ground yet. When this happens, we bail out of the loop at line 15 before completing all \( n \) iterations.

```python
import math

def meteor(n):
    r = .09
    m = 7  # mass in kg
    b = 1.5e-3  # const. of prop. for friction, kg/m
    x = 200.*1000.  # start at 200 km altitude, far above air
    v = -20.*1000.  # 20 km/s
    H = 7.6*1000.  # scale height in meters
    g = 9.8  # m/s²
    t_max = -x/v*10.  # guess the longest time it could take
    dt = t_max/n  # Divide t into n equal parts.
    for i in range(n):
        dx = v*dt  # Change x.
        x = x+dx
        if x<0.:  # If we’ve hit the ground...
            return v  # ...quit.
        F = b*v**2*math.exp(-x/H)-m*g
        a = F/m
        dv = a*dt
        v = v+dv
    return -999.  # If we get here, t_max was too short.
```

The result is:

```
>>> print(meteor(100000))
-3946.95754982
```

For comparison, the speed of sound is about 340 m/s. The answer is that if you are hit by a meteorite, you will not be able to hear its sound before it hits you.