Summary

Nature behaves differently on large and small scales. Galileo showed that this results fundamentally from the way area and volume scale. Area scales as the second power of length, \( A \propto L^2 \), while volume scales as length to the third power, \( V \propto L^3 \).

An order of magnitude estimate is one in which we do not attempt or expect an exact answer. The main reason why the uninitiated have trouble with order-of-magnitude estimates is that the human brain does not intuitively make accurate estimates of area and volume. Estimates of area and volume should be approached by first estimating linear dimensions, which one’s brain has a feel for.
Problems

Key
✓ A computerized answer check is available online.
∫ A problem that requires calculus.
☆ A difficult problem.

1 The one-liter cube in the photo has been marked off into smaller cubes, with linear dimensions one tenth those of the big one. What is the volume of each of the small cubes?

2 How many cm² is 1 mm²?

3 Compare the light-gathering powers of a 3-cm-diameter telescope and a 30-cm telescope.

4 The traditional Martini glass is shaped like a cone with the point at the bottom. Suppose you make a Martini by pouring vermouth into the glass to a depth of 3 cm, and then adding gin to bring the depth to 6 cm. What are the proportions of gin and vermouth?

5 How many cubic inches are there in a cubic foot? The answer is not 12.

6 Assume a dog’s brain is twice as great in diameter as a cat’s, but each animal’s brain cells are the same size and their brains are the same shape. In addition to being a far better companion and much nicer to come home to, how many times more brain cells does a dog have than a cat? The answer is not 2.

7 The population density of Los Angeles is about 4000 people/km². That of San Francisco is about 6000 people/km². How many times farther away is the average person’s nearest neighbor in LA than in San Francisco? The answer is not 1.5.

8 One step on the Richter scale corresponds to a factor of 100 in terms of the energy absorbed by something on the surface of the Earth, e.g., a house. For instance, a 9.3-magnitude quake would release 100 times more energy than an 8.3. The energy spreads out from the epicenter as a wave, and for the sake of this problem we’ll assume we’re dealing with seismic waves that spread out in three dimensions, so that we can visualize them as hemispheres spreading out under the surface of the earth. If a certain 7.6-magnitude earthquake and a certain 5.6-magnitude earthquake produce the same amount of vibration where I live, compare the distances from my house to the two epicenters.
9 The central portion of a CD is taken up by the hole and some surrounding clear plastic, and this area is unavailable for storing data. The radius of the central circle is about 35% of the outer radius of the data-storing area. What percentage of the CD’s area is therefore lost? √

10 A taxon (plural taxa) is a group of living things. For example, *Homo sapiens* and *Homo neanderthalensis* are both taxa — specifically, they are two different species within the genus *Homo*. Surveys by botanists show that the number of plant taxa native to a given contiguous land area $A$ is usually approximately proportional to $A^{1/3}$. (a) There are 70 different species of lupine native to Southern California, which has an area of about 200,000 km$^2$. The San Gabriel Mountains cover about 1,600 km$^2$. Suppose that you wanted to learn to identify all the species of lupine in the San Gabriels. Approximately how many species would you have to familiarize yourself with? ▶ Answer, p. 571 √
(b) What is the interpretation of the fact that the exponent, $1/3$, is less than one?

11 X-ray images aren’t only used with human subjects but also, for example, on insects and flowers. In 2003, a team of researchers at Argonne National Laboratory used x-ray imagery to find for the first time that insects, although they do not have lungs, do not necessarily breathe completely passively, as had been believed previously; many insects rapidly compress and expand their trachea, head, and thorax in order to force air in and out of their bodies. One difference between x-raying a human and an insect is that if a medical x-ray machine was used on an insect, virtually 100% of the x-rays would pass through its body, and there would be no contrast in the image produced. Less penetrating x-rays of lower energies have to be used. For comparison, a typical human body mass is about 70 kg, whereas a typical ant is about 10 mg. Estimate the ratio of the thicknesses of tissue that must be penetrated by x-rays in one case compared to the other. √

12 Radio was first commercialized around 1920, and ever since then, radio signals from our planet have been spreading out across our galaxy. It is possible that alien civilizations could detect these signals and learn that there is life on earth. In the 90 years that the signals have been spreading at the speed of light, they have created a sphere with a radius of 90 light-years. To show an idea of the size of this sphere, I’ve indicated it in the figure as a tiny white circle on an image of a spiral galaxy seen edge on. (We don’t have similar photos of our own Milky Way galaxy, because we can’t see it from the outside.) So far we haven’t received answering signals from aliens within this sphere, but as time goes on, the sphere will expand as suggested by the dashed outline, reaching more and more stars that might harbor extraterrestrial life. Approximately what year will it be when the sphere has expanded to fill a volume 100 times greater than the volume it fills today in 2010? √
13 The Earth’s surface is about 70% water. Mars’s diameter is about half the Earth’s, but it has no surface water. Compare the land areas of the two planets. ✓

14 In Europe, a piece of paper of the standard size, called A4, is a little narrower and taller than its American counterpart. The ratio of the height to the width is the square root of 2, and this has some useful properties. For instance, if you cut an A4 sheet from left to right, you get two smaller sheets that have the same proportions. You can even buy sheets of this smaller size, and they’re called A5. There is a whole series of sizes related in this way, all with the same proportions. (a) Compare an A5 sheet to an A4 in terms of area and linear size. (b) The series of paper sizes starts from an A0 sheet, which has an area of one square meter. Suppose we had a series of boxes defined in a similar way: the B0 box has a volume of one cubic meter, two B1 boxes fit exactly inside an B0 box, and so on. What would be the dimensions of a B0 box? ✓

15 Estimate the volume of a human body, in cm³.

16 Estimate the number of blades of grass on a football field.

17 In a computer memory chip, each bit of information (a 0 or a 1) is stored in a single tiny circuit etched onto the surface of a silicon chip. The circuits cover the surface of the chip like lots in a housing development. A typical chip stores 64 Mb (megabytes) of data, where a byte is 8 bits. Estimate (a) the area of each circuit, and (b) its linear size.

18 Suppose someone built a gigantic apartment building, measuring 10 km × 10 km at the base. Estimate how tall the building would have to be to have space in it for the entire world’s population to live.

19 (a) Using the microscope photo in the figure, estimate the mass of a one cell of the *E. coli* bacterium, which is one of the most common ones in the human intestine. Note the scale at the lower right corner, which is 1 µm. Each of the tubular objects in the column is one cell. (b) The feces in the human intestine are mostly bacteria (some dead, some alive), of which *E. coli* is a large and typical component. Estimate the number of bacteria in your intestines, and compare with the number of human cells in your body, which is believed to be roughly on the order of $10^{13}$. (c) Interpreting your result from part b, what does this tell you about the size of a typical human cell compared to the size of a typical bacterial cell?
20 A hamburger chain advertises that it has sold 10 billion Bongo Burgers. Estimate the total mass of feed required to raise the cows used to make the burgers.

21 Estimate the mass of one of the hairs in Albert Einstein’s moustache, in units of kg.

22 Estimate the number of man-hours required for building the Great Wall of China.  

23 According to folklore, every time you take a breath, you are inhaling some of the atoms exhaled in Caesar’s last words. Is this true? If so, how many?

24 Estimate the number of jellybeans in figure o on p. 48.

25 At the grocery store you will see oranges packed neatly in stacks. Suppose we want to pack spheres as densely as possible, so that the greatest possible fraction of the space is filled by the spheres themselves, not by empty space. Let’s call this fraction \( f \). Mathematicians have proved that the best possible result is \( f \approx 0.7405 \), which requires a systematic pattern of stacking. If you buy ball bearings or golf balls, however, the seller is probably not going to go to the trouble of stacking them neatly. Instead they will probably pour the balls into a box and vibrate the box vigorously for a while to make them settle. This results in a random packing. The closest random packing has \( f \approx 0.64 \). Suppose that golf balls, with a standard diameter of 4.27 cm, are sold in bulk with the closest random packing. What is the diameter of the largest ball that could be sold in boxes of the same size, packed systematically, so that there would be the same number of balls per box?

26 Plutonium-239 is one of a small number of important long-lived forms of high-level radioactive nuclear waste. The world’s waste stockpiles have about \( 10^3 \) metric tons of plutonium. Drinking water is considered safe by U.S. government standards if it contains less than \( 2 \times 10^{-13} \text{ g/cm}^3 \) of plutonium. The amount of radioactivity to which you were exposed by drinking such water on a daily basis would be very small compared to the natural background radiation that you are exposed to every year. Suppose that the world’s inventory of plutonium-239 were ground up into an extremely fine dust and then dispersed over the world’s oceans, thereby becoming mixed uniformly into the world’s water supplies over time. Estimate the resulting concentration of plutonium, and compare with the government standard.
Exercise 1: Scaling applied to leaves

Equipment:

leaves of three sizes, having roughly similar proportions of length, width, and thickness
balance

Each group will have one leaf, and should measure its surface area and volume, and determine its surface-to-volume ratio. For consistency, every group should use units of cm\(^2\) and cm\(^3\), and should only find the area of one side of the leaf. The area can be found by tracing the area of the leaf on graph paper and counting squares. The volume can be found by weighing the leaf and assuming that its density is 1 g/cm\(^3\) (the density of water). What implications do your results have for the plants’ abilities to survive in different environments?
Motion in One Dimension
Chapter 2
Velocity and Relative Motion

2.1 Types of motion

If you had to think consciously in order to move your body, you would be severely disabled. Even walking, which we consider to be no great feat, requires an intricate series of motions that your cerebrum would be utterly incapable of coordinating. The task of putting one foot in front of the other is controlled by the more primitive parts of your brain, the ones that have not changed much since the mammals and reptiles went their separate evolutionary ways. The thinking part of your brain limits itself to general directives such as “walk faster,” or “don’t step on her toes,” rather than micromanaging every contraction and relaxation of the hundred or so muscles of your hips, legs, and feet.

Physics is all about the conscious understanding of motion, but we’re obviously not immediately prepared to understand the most complicated types of motion. Instead, we’ll use the divide-and-conquer technique. We’ll first classify the various types of motion, and then begin our campaign with an attack on the simplest cases. To make it clear what we are and are not ready to consider, we need to examine and define carefully what types of motion can exist.

Rigid-body motion distinguished from motion that changes an object’s shape

Nobody, with the possible exception of Fred Astaire, can simply glide forward without bending their joints. Walking is thus an example in which there is both a general motion of the whole object and a change in the shape of the object. Another example is the motion of a jiggling water balloon as it flies through the air. We are not presently attempting a mathematical description of the way in which the shape of an object changes. Motion without a change in shape is called rigid-body motion. (The word “body” is often used in physics as a synonym for “object.”)

Center-of-mass motion as opposed to rotation

A ballerina leaps into the air and spins around once before landing. We feel intuitively that her rigid-body motion while her feet are off the ground consists of two kinds of motion going on simul-
No matter what point you hang the pear from, the string lines up with the pear’s center of mass. The center of mass can therefore be defined as the intersection of all the lines made by hanging the pear in this way. Note that the X in the figure should not be interpreted as implying that the center of mass is on the surface — it is actually inside the pear.

The circus performers hang with the ropes passing through their centers of mass.

The leaping dancer’s motion is complicated, but the motion of her center of mass is simple.

It turns out that there is one particularly natural and useful way to make a clear definition, but it requires a brief digression. Every object has a balance point, referred to in physics as the center of mass. For a two-dimensional object such as a cardboard cutout, the center of mass is the point at which you could hang the object from a string and make it balance. In the case of the ballerina (who is likely to be three-dimensional unless her diet is particularly severe), it might be a point either inside or outside her body, depending on how she holds her arms. Even if it is not practical to attach a string to the balance point itself, the center of mass can be defined as shown in figure e.

Why is the center of mass concept relevant to the question of classifying rotational motion as opposed to motion through space? As illustrated in figures d and g, it turns out that the motion of an object’s center of mass is nearly always far simpler than the motion of any other part of the object. The ballerina’s body is a large object with a complex shape. We might expect that her motion would be

taneously: a rotation and a motion of her body as a whole through space, along an arc. It is not immediately obvious, however, what is the most useful way to define the distinction between rotation and motion through space. Imagine that you attempt to balance a chair and it falls over. One person might say that the only motion was a rotation about the chair’s point of contact with the floor, but another might say that there was both rotation and motion down and to the side.
An improperly balanced wheel has a center of mass that is not at its geometric center. When you get a new tire, the mechanic clamps little weights to the rim to balance the wheel.

This toy was intentionally designed so that the mushroom-shaped piece of metal on top would throw off the center of mass. When you wind it up, the mushroom spins, but the center of mass doesn’t want to move, so the rest of the toy tends to counter the mushroom’s motion, causing the whole thing to jump around.

Ballerinas and professional basketball players can create an illusion of flying horizontally through the air because our brains intuitively expect them to have rigid-body motion, but the body does not stay rigid while executing a grand jete or a slam dunk. The legs...
A fixed point on the dancer’s body follows a trajectory that is flatter than what we expect, creating an illusion of flight. Are low at the beginning and end of the jump, but come up higher at the middle. Regardless of what the limbs do, the center of mass will follow the same arc, but the low position of the legs at the beginning and end means that the torso is higher compared to the center of mass, while in the middle of the jump it is lower compared to the center of mass. Our eye follows the motion of the torso and tries to interpret it as the center-of-mass motion of a rigid body. But since the torso follows a path that is flatter than we expect, this attempted interpretation fails, and we experience an illusion that the person is flying horizontally.

Example 1. Explain how we know that the center of mass of each object is at the location shown in figure k.

The center of mass is a sort of average, so the height of the centers of mass in 1 and 2 has to be midway between the two squares, because that height is the average of the heights of the two squares. Example 3 is a combination of examples 1 and 2, so we can find its center of mass by averaging the horizontal positions of their centers of mass. In example 4, each square has been skewed a little, but just as much mass has been moved up as down, so the average vertical position of the mass hasn’t changed. Example 5 is clearly not all that different from example...
4, the main difference being a slight clockwise rotation, so just as in example 4, the center of mass must be hanging in empty space, where there isn’t actually any mass. Horizontally, the center of mass must be between the heels and toes, or else it wouldn’t be possible to stand without tipping over.

Another interesting example from the sports world is the high jump, in which the jumper’s curved body passes over the bar, but the center of mass passes under the bar! Here the jumper lowers his legs and upper body at the peak of the jump in order to bring his waist higher compared to the center of mass.

Later in this course, we’ll find that there are more fundamental reasons (based on Newton’s laws of motion) why the center of mass behaves in such a simple way compared to the other parts of an object. We’re also postponing any discussion of numerical methods for finding an object’s center of mass. Until later in the course, we will only deal with the motion of objects’ centers of mass.

**Center-of-mass motion in one dimension**

In addition to restricting our study of motion to center-of-mass motion, we will begin by considering only cases in which the center of mass moves along a straight line. This will include cases such as objects falling straight down, or a car that speeds up and slows down but does not turn.

Note that even though we are not explicitly studying the more complex aspects of motion, we can still analyze the center-of-mass motion while ignoring other types of motion that might be occurring simultaneously. For instance, if a cat is falling out of a tree and is initially upside-down, it goes through a series of contortions that bring its feet under it. This is definitely not an example of rigid-body motion, but we can still analyze the motion of the cat’s center of mass just as we would for a dropping rock.

**self-check A**

Consider a person running, a person pedaling on a bicycle, a person coasting on a bicycle, and a person coasting on ice skates. In which cases is the center-of-mass motion one-dimensional? Which cases are examples of rigid-body motion? 

▷ Answer, p. 567

**self-check B**

The figure shows a gymnast holding onto the inside of a big wheel. From inside the wheel, how could he make it roll one way or the other? 

▷ Answer, p. 567

### 2.2 Describing distance and time

Center-of-mass motion in one dimension is particularly easy to deal with because all the information about it can be encapsulated in two variables: $x$, the position of the center of mass relative to the origin,
and \( t \), which measures a point in time. For instance, if someone supplied you with a sufficiently detailed table of \( x \) and \( t \) values, you would know pretty much all there was to know about the motion of the object’s center of mass.

**A point in time as opposed to duration**

In ordinary speech, we use the word “time” in two different senses, which are to be distinguished in physics. It can be used, as in “a short time” or “our time here on earth,” to mean a length or duration of time, or it can be used to indicate a clock reading, as in “I didn’t know what time it was,” or “now’s the time.” In symbols, \( t \) is ordinarily used to mean a point in time, while \( \Delta t \) signifies an interval or duration in time. The capital Greek letter delta, \( \Delta \), means “the change in...,” i.e. a duration in time is the change or difference between one clock reading and another. The notation \( \Delta t \) does not signify the product of two numbers, \( \Delta \) and \( t \), but rather one single number, \( \Delta t \). If a matinee begins at a point in time \( t = 1 \) o’clock and ends at \( t = 3 \) o’clock, the duration of the movie was the change in \( t \),

\[
\Delta t = 3 \text{ hours} - 1 \text{ hour} = 2 \text{ hours}.
\]

To avoid the use of negative numbers for \( \Delta t \), we write the clock reading “after” to the left of the minus sign, and the clock reading “before” to the right of the minus sign. A more specific definition of the delta notation is therefore that delta stands for “after minus before.”

Even though our definition of the delta notation guarantees that \( \Delta t \) is positive, there is no reason why \( t \) can’t be negative. If \( t \) could not be negative, what would have happened one second before \( t = 0 \)? That doesn’t mean that time “goes backward” in the sense that adults can shrink into infants and retreat into the womb. It just means that we have to pick a reference point and call it \( t = 0 \), and then times before that are represented by negative values of \( t \).

An example is that a year like 2007 A.D. can be thought of as a positive \( t \) value, while one like 370 B.C. is negative. Similarly, when you hear a countdown for a rocket launch, the phrase “t minus ten seconds” is a way of saying \( t = -10 \text{ s} \), where \( t = 0 \) is the time of blastoff, and \( t > 0 \) refers to times after launch.

Although a point in time can be thought of as a clock reading, it is usually a good idea to avoid doing computations with expressions such as “2:35” that are combinations of hours and minutes. Times can instead be expressed entirely in terms of a single unit, such as hours. Fractions of an hour can be represented by decimals rather than minutes, and similarly if a problem is being worked in terms of minutes, decimals can be used instead of seconds.

**Self-check C**

Of the following phrases, which refer to points in time, which refer to time intervals, and which refer to time in the abstract rather than as a
measurable number?

(1) “The time has come.”

(2) “Time waits for no man.”

(3) “The whole time, he had spit on his chin.”

The Leibniz notation and infinitesimals

Δ is the Greek version of “D,” suggesting that there is a relationship between Δt and the notation dt from calculus. The “d” notation was invented by Leibniz around 1675 to suggest the word “difference.” The idea was that a dt would be like a Δt that was extremely small — smaller than any real number, and yet greater than zero. These infinitesimal numbers were the way the world’s greatest mathematicians thought about calculus for the next two hundred years. For example, dy/dx meant the number you got when you divided dy by dx. The use of infinitesimal numbers was seen as a natural part of the process of generalization that had already seen the invention of fractions and irrational numbers by the ancient Greeks, zero and negative numbers in India, and complex numbers in Renaissance Italy. By the end of the 19th century, mathematicians had begun making formal mathematical descriptions of number systems, and they had succeeded in making nice tidy schemes out of all of these categories except for infinitesimals. Having run into a brick wall, they decided to rebuild calculus using the notion of a limit. Depending on when and where you got your education in calculus, you may have been warned severely that dy and dx were not numbers, and that dy/dx didn’t mean dividing one by another.

But in the 1960’s, the logician Abraham Robinson at Yale proved that infinitesimals could be tamed and domesticated; they were no more self-contradictory than negative numbers or fractions. There is a handy rule for making sure that you don’t come to incorrect conclusions by using infinitesimals. The rule is that you can apply any axiom of the real number system to infinitesimals, and the result will be correct, provided that the axiom can be put in a form like “for any number . . . ,” but not “for any set of numbers . . . .” We carry over the axiom, reinterpreting “number” to mean any member of the enriched number system that includes both the real numbers and the infinitesimals.

Logic and infinitesimals example 2

There is an axiom of the real number system that for any number t, t + 0 = t. This applies to infinitesimals as well, so that dt + 0 = dt.

Position as opposed to change in position

As with time, a distinction should be made between a point in space, symbolized as a coordinate x, and a change in position,
symbolized as $\Delta x$.

As with $t$, $x$ can be negative. If a train is moving down the tracks, not only do you have the freedom to choose any point along the tracks and call it $x = 0$, but it’s also up to you to decide which side of the $x = 0$ point is positive $x$ and which side is negative $x$.

Since we’ve defined the delta notation to mean “after minus before,” it is possible that $\Delta x$ will be negative, unlike $\Delta t$ which is guaranteed to be positive. Suppose we are describing the motion of a train on tracks linking Tucson and Chicago. As shown in the figure, it is entirely up to you to decide which way is positive.

Note that in addition to $x$ and $\Delta x$, there is a third quantity we could define, which would be like an odometer reading, or actual distance traveled. If you drive 10 miles, make a U-turn, and drive back 10 miles, then your $\Delta x$ is zero, but your car’s odometer reading has increased by 20 miles. However important the odometer reading is to car owners and used car dealers, it is not very important in physics, and there is not even a standard name or notation for it. The change in position, $\Delta x$, is more useful because it is so much easier to calculate: to compute $\Delta x$, we only need to know the beginning and ending positions of the object, not all the information about how it got from one position to the other.

**Self-check D**

A ball falls vertically, hits the floor, bounces to a height of one meter, falls, and hits the floor again. Is the $\Delta x$ between the two impacts equal to zero, one, or two meters? $\triangleright$ Answer, p. 567

**Frames of reference**

The example above shows that there are two arbitrary choices you have to make in order to define a position variable, $x$. You have to decide where to put $x = 0$, and also which direction will be positive. This is referred to as choosing a coordinate system or choosing a frame of reference. (The two terms are nearly synonymous, but the first focuses more on the actual $x$ variable, while the second is more of a general way of referring to one’s point of view.) As long as
you are consistent, any frame is equally valid. You just don’t want to change coordinate systems in the middle of a calculation.

Have you ever been sitting in a train in a station when suddenly you notice that the station is moving backward? Most people would describe the situation by saying that you just failed to notice that the train was moving — it only seemed like the station was moving. But this shows that there is yet a third arbitrary choice that goes into choosing a coordinate system: valid frames of reference can differ from each other by moving relative to one another. It might seem strange that anyone would bother with a coordinate system that was moving relative to the earth, but for instance the frame of reference moving along with a train might be far more convenient for describing things happening inside the train.

2.3 Graphs of motion; velocity

Motion with constant velocity

In example o, an object is moving at constant speed in one direction. We can tell this because every two seconds, its position changes by five meters.

In algebra notation, we’d say that the graph of $x$ vs. $t$ shows the same change in position, $\Delta x = 5.0 \text{ m}$, over each interval of $\Delta t = 2.0 \text{ s}$. The object’s velocity or speed is obtained by calculating $v = \Delta x / \Delta t = (5.0 \text{ m})/(2.0 \text{ s}) = 2.5 \text{ m/s}$. In graphical terms, the velocity can be interpreted as the slope of the line. Since the graph is a straight line, it wouldn’t have mattered if we’d taken a longer time interval and calculated $v = \Delta x / \Delta t = (10.0 \text{ m})/(4.0 \text{ s})$. The answer would still have been the same, 2.5 m/s.

Note that when we divide a number that has units of meters by another number that has units of seconds, we get units of meters per second, which can be written m/s. This is another case where we treat units as if they were algebra symbols, even though they’re not.

In example p, the object is moving in the opposite direction: as time progresses, its $x$ coordinate decreases. Recalling the definition of the $\Delta$ notation as “after minus before,” we find that $\Delta t$ is still positive, but $\Delta x$ must be negative. The slope of the line is therefore negative, and we say that the object has a negative velocity, $v = \Delta x / \Delta t = (-5.0 \text{ m})/(2.0 \text{ s}) = -2.5 \text{ m/s}$. We’ve already seen that the plus and minus signs of $\Delta x$ values have the interpretation of telling us which direction the object moved. Since $\Delta t$ is always positive, dividing by $\Delta t$ doesn’t change the plus or minus sign, and the plus and minus signs of velocities are to be interpreted in the same way. In graphical terms, a positive slope characterizes a line that goes up as we go to the right, and a negative slope tells us that the line went down as we went to the right.
Motion with changing velocity

Now what about a graph like figure q? This might be a graph of a car’s motion as the driver cruises down the freeway, then slows down to look at a car crash by the side of the road, and then speeds up again, disappointed that there is nothing dramatic going on such as flames or babies trapped in their car seats. (Note that we are still talking about one-dimensional motion. Just because the graph is curvy doesn’t mean that the car’s path is curvy. The graph is not like a map, and the horizontal direction of the graph represents the passing of time, not distance.)

If we apply the equation \( v = \Delta x / \Delta t \) to this example, we will get the wrong answer, because the \( \Delta x / \Delta t \) gives a single number, but the velocity is clearly changing. This is an example of a good general rule that tells you when you need to use your differential calculus. Any time a rate of change is measured by an expression of the form \( \Delta \ldots / \Delta \ldots \), the result will only be right when the rate of change is constant. When the rate of change is varying, we need to generalize the expression by making it into a derivative. Just as an infinitesimally small \( \Delta t \) is notated \( d\, t \), an infinitesimally small \( \Delta x \) is a \( d\, x \). The velocity is then the derivative \( d\, x / d\, t \).

Units of velocity example 3

\( \Rightarrow \) Verify that the units of \( v = dx / dt \) make sense.

\( \Rightarrow \) We expect the velocity to have units of meters per second, and it does come out to have those units, since \( dx \) has units of meters and \( dt \) seconds. This ability to check the units of derivatives is one of the main reasons that Leibniz designed his notation for derivatives the way he did.

An insect pest example 4

\( \Rightarrow \) An insect pest from the United States is inadvertently released in a village in rural China. The pests spread outward at a rate of \( s \) kilometers per year, forming a widening circle of contagion. Find the number of square kilometers per year that become newly infested. Check that the units of the result make sense. Interpret the result.

\( \Rightarrow \) Let \( t \) be the time, in years, since the pest was introduced. The radius of the circle is \( r = st \), and its area is \( a = \pi r^2 = \pi (st)^2 \). The derivative is

\[
\frac{da}{dt} = (2\pi s^2) t
\]

The units of \( s \) are km/year, so squaring it gives \( \text{km}^2/\text{year}^2 \). The 2 and the \( \pi \) are unitless, and multiplying by \( t \) gives units of \( \text{km}^2/\text{year} \), which is what we expect for \( da / dt \), since it represents the number of square kilometers per year that become infested.

\(^1\) see p. 65
Interpreting the result, we notice a couple of things. First, the rate of infestation isn’t constant; it’s proportional to \( t \), so people might not pay so much attention at first, but later on the effort required to combat the problem will grow more and more quickly. Second, we notice that the result is proportional to \( s^2 \). This suggests that anything that could be done to reduce \( s \) would be very helpful. For instance, a measure that cut \( s \) in half would reduce \( \frac{da}{dt} \) by a factor of four.

2.4 The principle of inertia

Physical effects relate only to a change in velocity

Consider two statements of a kind that was at one time made with the utmost seriousness:

People like Galileo and Copernicus who say the earth is rotating must be crazy. We know the earth can’t be moving. Why, if the earth was really turning once every day, then our whole city would have to be moving hundreds of leagues in an hour. That’s impossible! Buildings would shake on their foundations. Gale-force winds would knock us over. Trees would fall down. The Mediterranean would come sweeping across the east coasts of Spain and Italy. And furthermore, what force would be making the world turn?

All this talk of passenger trains moving at forty miles an hour is sheer hogwash! At that speed, the air in a passenger compartment would all be forced against the back wall. People in the front of the car would suffocate, and people at the back would die because in such concentrated air, they wouldn’t be able to expel a breath.

Some of the effects predicted in the first quote are clearly just based on a lack of experience with rapid motion that is smooth and free of vibration. But there is a deeper principle involved. In each case, the speaker is assuming that the mere fact of motion must have dramatic physical effects. More subtly, they also believe that a force is needed to keep an object in motion: the first person thinks a force would be needed to maintain the earth’s rotation, and the second apparently thinks of the rear wall as pushing on the air to keep it moving.

Common modern knowledge and experience tell us that these people’s predictions must have somehow been based on incorrect reasoning, but it is not immediately obvious where the fundamental flaw lies. It’s one of those things a four-year-old could infuriate you by demanding a clear explanation of. One way of getting at the fundamental principle involved is to consider how the modern concept of the universe differs from the popular conception at the time of the Italian Renaissance. To us, the word “earth” implies
a planet, one of the nine planets of our solar system, a small ball of rock and dirt that is of no significance to anyone in the universe except for members of our species, who happen to live on it. To Galileo’s contemporaries, however, the earth was the biggest, most solid, most important thing in all of creation, not to be compared with the wandering lights in the sky known as planets. To us, the earth is just another object, and when we talk loosely about “how fast” an object such as a car “is going,” we really mean the car-object’s velocity relative to the earth-object.

Motion is relative

According to our modern world-view, it isn’t reasonable to expect that a special force should be required to make the air in the train have a certain velocity relative to our planet. After all, the “moving” air in the “moving” train might just happen to have zero velocity relative to some other planet we don’t even know about. Aristotle claimed that things “naturally” wanted to be at rest, lying on the surface of the earth. But experiment after experiment has shown that there is really nothing so special about being at rest relative to the earth. For instance, if a mattress falls out of the back of a truck on the freeway, the reason it rapidly comes to rest with
respect to the planet is simply because of friction forces exerted by the asphalt, which happens to be attached to the planet.

Galileo’s insights are summarized as follows:

**The principle of inertia**
No force is required to maintain motion with constant velocity in a straight line, and absolute motion does not cause any observable physical effects.

There are many examples of situations that seem to disprove the principle of inertia, but these all result from forgetting that friction is a force. For instance, it seems that a force is needed to keep a sailboat in motion. If the wind stops, the sailboat stops too. But the wind’s force is not the only force on the boat; there is also a frictional force from the water. If the sailboat is cruising and the wind suddenly disappears, the backward frictional force still exists, and since it is no longer being counteracted by the wind’s forward force, the boat stops. To disprove the principle of inertia, we would have to find an example where a moving object slowed down even though no forces whatsoever were acting on it. Over the years since Galileo’s lifetime, physicists have done more and more precise experiments to search for such a counterexample, but the results have always been negative. Three such tests are described on pp. 115, 261, and 153.

**self-check E**
What is incorrect about the following supposed counterexamples to the principle of inertia?

1. When astronauts blast off in a rocket, their huge velocity does cause a physical effect on their bodies — they get pressed back into their seats, the flesh on their faces gets distorted, and they have a hard time lifting their arms.

2. When you’re driving in a convertible with the top down, the wind in your face is an observable physical effect of your absolute motion.

Answer, p. 567

**Solved problem: a bug on a wheel** page 97, problem 13

**Discussion questions**

A A passenger on a cruise ship finds, while the ship is docked, that he can leap off of the upper deck and just barely make it into the pool on the lower deck. If the ship leaves dock and is cruising rapidly, will this adrenaline junkie still be able to make it?

B You are a passenger in the open basket hanging under a helium balloon. The balloon is being carried along by the wind at a constant velocity. If you are holding a flag in your hand, will the flag wave? If so, which way? [Based on a question from PSSC Physics.]

C Aristotle stated that all objects naturally wanted to come to rest, with the unspoken implication that “rest” would be interpreted relative to the
surface of the earth. Suppose we go back in time and transport Aristotle
to the moon. Aristotle knew, as we do, that the moon circles the earth; he
said it didn’t fall down because, like everything else in the heavens, it was
made out of some special substance whose “natural” behavior was to go
in circles around the earth. We land, put him in a space suit, and kick
him out the door. What would he expect his fate to be in this situation? If
intelligent creatures inhabited the moon, and one of them independently
came up with the equivalent of Aristotelian physics, what would they think
about objects coming to rest?

D The glass is sitting on a level table in a train’s dining car, but the
surface of the water is tilted. What can you infer about the motion of the
train?

2.5 Addition of velocities

Addition of velocities to describe relative motion

Since absolute motion cannot be unambiguously measured, the
only way to describe motion unambiguously is to describe the motion
of one object relative to another. Symbolically, we can write \( v_{PQ} \)
for the velocity of object \( P \) relative to object \( Q \).

Velocities measured with respect to different reference points can
be compared by addition. In the figure below, the ball’s velocity
relative to the couch equals the ball’s velocity relative to the truck
plus the truck’s velocity relative to the couch:

\[
v_{BC} = v_{BT} + v_{TC}
\]

\[
= 5 \text{ cm/s} + 10 \text{ cm/s}
\]

\[
= 15 \text{ cm/s}
\]

The same equation can be used for any combination of three
objects, just by substituting the relevant subscripts for B, T, and
C. Just remember to write the equation so that the velocities being
added have the same subscript twice in a row. In this example, if
you read off the subscripts going from left to right, you get BC... = ...
BTTC. The fact that the two “inside” subscripts on the right are
the same means that the equation has been set up correctly. Notice
how subscripts on the left look just like the subscripts on the right,
but with the two T’s eliminated.

Negative velocities in relative motion

My discussion of how to interpret positive and negative signs of
velocity may have left you wondering why we should bother. Why
not just make velocity positive by definition? The original reason
why negative numbers were invented was that bookkeepers decided
it would be convenient to use the negative number concept for pay-
ments to distinguish them from receipts. It was just plain easier than
writing receipts in black and payments in red ink. After adding up
These two highly competent physicists disagree on absolute velocities, but they would agree on relative velocities. Purple Dino considers the couch to be at rest, while Green Dino thinks of the truck as being at rest. They agree, however, that the truck's velocity relative to the couch is $v_{TC} = 10$ cm/s, the ball's velocity relative to the truck is $v_{BT} = 5$ cm/s, and the ball’s velocity relative to the couch is $v_{BC} = v_{BT} + v_{TC} = 15$ cm/s.

your month’s positive receipts and negative payments, you either got a positive number, indicating profit, or a negative number, showing a loss. You could then show that total with a high-tech “+” or “−” sign, instead of looking around for the appropriate bottle of ink.

Nowadays we use positive and negative numbers for all kinds of things, but in every case the point is that it makes sense to add and subtract those things according to the rules you learned in grade school, such as “minus a minus makes a plus, why this is true we need not discuss.” Adding velocities has the significance of comparing relative motion, and with this interpretation negative and positive velocities can be used within a consistent framework. For example, the truck’s velocity relative to the couch equals the truck’s velocity relative to the ball plus the ball’s velocity relative to the couch:

$$v_{TC} = v_{TB} + v_{BC}$$
$$= -5 \text{ cm/s} + 15 \text{ cm/s}$$
$$= 10 \text{ cm/s}$$

If we didn’t have the technology of negative numbers, we would have had to remember a complicated set of rules for adding velocities: (1) if the two objects are both moving forward, you add, (2) if one is moving forward and one is moving backward, you subtract, but (3)
if they’re both moving backward, you add. What a pain that would have been.

\[
\text{\textbf{Solved problem: two dimensions}}
\]

\textbf{Airspeed \hspace{1cm} example 5}

On June 1, 2009, Air France flight 447 disappeared without warning over the Atlantic Ocean. All 232 people aboard were killed. Investigators believe the disaster was triggered because the pilots lost the ability to accurately determine their speed relative to the air. This is done using sensors called Pitot tubes, mounted outside the plane on the wing. Automated radio signals showed that these sensors gave conflicting readings before the crash, possibly because they iced up. For fuel efficiency, modern passenger jets fly at a very high altitude, but in the thin air they can only fly within a very narrow range of speeds. If the speed is too low, the plane stalls, and if it’s too high, it breaks up. If the pilots can’t tell what their airspeed is, they can’t keep it in the safe range.

Many people’s reaction to this story is to wonder why planes don’t just use GPS to measure their speed. One reason is that GPS tells you your speed relative to the ground, not relative to the air. Letting P be the plane, A the air, and G the ground, we have

\[
v_{PG} = v_{PA} + v_{AG},
\]

where \(v_{PG}\) (the “true ground speed”) is what GPS would measure, \(v_{PA}\) (“airspeed”) is what’s critical for stable flight, and \(v_{AG}\) is the velocity of the wind relative to the ground 9000 meters below. Knowing \(v_{PG}\) isn’t enough to determine \(v_{PA}\) unless \(v_{AG}\) is also known.

Discussion questions

A \hspace{1cm} Interpret the general rule \(v_{AB} = -v_{BA}\) in words.

B \hspace{1cm} Wa-Chuen slips away from her father at the mall and walks up the down escalator, so that she stays in one place. Write this in terms of symbols.
2.6  ➤ Relativity

Time is not absolute

So far we’ve been discussing relativity according to Galileo and Newton, but there is also relativity according to Einstein. When Einstein first began to develop the theory of relativity, around 1905, the only real-world observations he could draw on were ambiguous and indirect. Today, the evidence is part of everyday life. For example, every time you use a GPS receiver, you’re using Einstein’s theory of relativity. Somewhere between 1905 and today, technology became good enough to allow conceptually simple experiments that students in the early 20th century could only discuss in terms like “Imagine that we could…” A good jumping-on point is 1971. In that year, J.C. Hafele and R.E. Keating brought atomic clocks aboard commercial airliners, and went around the world, once from east to west and once from west to east. Hafele and Keating observed that there was a discrepancy between the times measured by the traveling clocks and the times measured by similar clocks that stayed home at the U.S. Naval Observatory in Washington. The east-going clock lost time, ending up off by $-59 \pm 10$ nanoseconds, while the west-going one gained $273 \pm 7$ ns.

The correspondence principle

This establishes that time doesn’t work the way Newton believed it did when he wrote that “Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external…” We are used to thinking of time as absolute and universal, so it is disturbing to find that it can flow at a different rate for observers in different frames of reference. Nevertheless, the effects that Hafele and Keating observed were small. This makes sense: Newton’s laws have already been thoroughly tested by experiments under a wide variety of conditions, so a new theory like relativity must agree with Newton’s to a good approximation, within the Newtonian theory’s realm of applicability. This requirement of backward-compatibility is known as the correspondence principle.

Causality

It’s also reassuring that the effects on time were small compared to the three-day lengths of the plane trips. There was therefore no opportunity for paradoxical scenarios such as one in which the east-going experimenter arrived back in Washington before he left and then convinced himself not to take the trip. A theory that maintains this kind of orderly relationship between cause and effect is said to satisfy causality.

Causality is like a water-hungry front-yard lawn in Los Angeles: we know we want it, but it’s not easy to explain why. Even in plain old Newtonian physics, there is no clear distinction between past
and future. In figure 2, number 18 throws the football to number 25, and the ball obeys Newton’s laws of motion. If we took a video of the pass and played it backward, we would see the ball flying from 25 to 18, and Newton’s laws would still be satisfied. Nevertheless, we have a strong psychological impression that there is a forward arrow of time. I can remember what the stock market did last year, but I can’t remember what it will do next year. Joan of Arc’s military victories against England caused the English to burn her at the stake; it’s hard to accept that Newton’s laws provide an equally good description of a process in which her execution in 1431 caused her to win a battle in 1429. There is no consensus at this point among physicists on the origin and significance of time’s arrow, and for our present purposes we don’t need to solve this mystery. Instead, we merely note the empirical fact that, regardless of what causality really means and where it really comes from, its behavior is consistent. Specifically, experiments show that if an observer in a certain frame of reference observes that event A causes event B, then observers in other frames agree that A causes B, not the other way around. This is merely a generalization about a large body of experimental results, not a logically necessary assumption. If Keating had gone around the world and arrived back in Washington before he left, it would have disproved this statement about causality.

**Time distortion arising from motion and gravity**

Hafele and Keating were testing specific quantitative predictions of relativity, and they verified them to within their experiment’s error bars. Let’s work backward instead, and inspect the empirical results for clues as to how time works.

The two traveling clocks experienced effects in opposite directions, and this suggests that the rate at which time flows depends on the motion of the observer. The east-going clock was moving in the same direction as the earth’s rotation, so its velocity relative to the earth’s center was greater than that of the clock that remained in Washington, while the west-going clock’s velocity was correspondingly reduced. The fact that the east-going clock fell behind, and the west-going one got ahead, shows that the effect of motion is to make time go more slowly. This effect of motion on time was predicted by Einstein in his original 1905 paper on relativity, written when he was 26.

If this had been the only effect in the Hafele-Keating experiment, then we would have expected to see effects on the two flying clocks that were equal in size. Making up some simple numbers to keep the arithmetic transparent, suppose that the earth rotates from west to east at 1000 km/hr, and that the planes fly at 300 km/hr. Then the speed of the clock on the ground is 1000 km/hr, the speed of the clock on the east-going plane is 1300 km/hr, and that of the west-going clock 700 km/hr. Since the speeds of 700, 1000, and 1300
km/hr have equal spacing on either side of 1000, we would expect the discrepancies of the moving clocks relative to the one in the lab to be equal in size but opposite in sign.

In fact, the two effects are unequal in size: \(-59\) ns and \(273\) ns. This implies that there is a second effect involved, simply due to the planes’ being up in the air. This was verified more directly in a 1978 experiment by Iijima and Fujiwara, figure ab, in which identical atomic clocks were kept at rest at the top and bottom of a mountain near Tokyo. This experiment, unlike the Hafele-Keating one, isolates one effect on time, the gravitational one: time’s rate of flow increases with height in a gravitational field. Einstein didn’t figure out how to incorporate gravity into relativity until 1915, after much frustration and many false starts. The simpler version of the theory without gravity is known as special relativity, the full version as general relativity. We’ll restrict ourselves to special relativity, and that means that what we want to focus on right now is the distortion of time due to motion, not gravity.

We can now see in more detail how to apply the correspondence principle. The behavior of the three clocks in the Hafele-Keating

![A graph showing the time difference between two atomic clocks.](image)

ab / A graph showing the time difference between two atomic clocks. One clock was kept at Mitaka Observatory, at 58 m above sea level. The other was moved back and forth to a second observatory, Norikura Corona Station, at the peak of the Norikura volcano, 2876 m above sea level. The plateaus on the graph are data from the periods when the clocks were compared side by side at Mitaka. The difference between one plateau and the next shows a gravitational effect on the rate of flow of time, accumulated during the period when the mobile clock was at the top of Norikura.

![A graph showing the time difference between two atomic clocks.](image)

ac / The correspondence principle requires that the relativistic distortion of time become small for small velocities.
experiment shows that the amount of time distortion increases as the speed of the clock's motion increases. Newton lived in an era when the fastest mode of transportation was a galloping horse, and the best pendulum clocks would accumulate errors of perhaps a minute over the course of several days. A horse is much slower than a jet plane, so the distortion of time would have had a relative size of only \( \sim 10^{-15} \) — much smaller than the clocks were capable of detecting. At the speed of a passenger jet, the effect is about \( 10^{-12} \), and state-of-the-art atomic clocks in 1971 were capable of measuring that. A GPS satellite travels much faster than a jet airplane, and the effect on the satellite turns out to be \( \sim 10^{-10} \). The general idea here is that all physical laws are approximations, and approximations aren’t simply right or wrong in different situations. Approximations are better or worse in different situations, and the question is whether a particular approximation is good enough in a given situation to serve a particular purpose. The faster the motion, the worse the Newtonian approximation of absolute time. Whether the approximation is good enough depends on what you’re trying to accomplish. The correspondence principle says that the approximation must have been good enough to explain all the experiments done in the centuries before Einstein came up with relativity.

By the way, don’t get an inflated idea of the importance of the Hafele-Keating experiment. Special relativity had already been confirmed by a vast and varied body of experiments decades before 1971. The only reason I’m giving such a prominent role to this experiment, which was actually more important as a test of general relativity, is that it is conceptually very direct.

**Distortion of space and time**

*The Lorentz transformation*

Relativity says that when two observers are in different frames of reference, each observer considers the other one’s perception of time to be distorted. We’ll also see that something similar happens to their observations of distances, so both space and time are distorted. What exactly is this distortion? How do we even conceptualize it?

The idea isn’t really as radical as it might seem at first. We can visualize the structure of space and time using a graph with position and time on its axes. These graphs are familiar by now, but we’re going to look at them in a slightly different way. Before, we used them to describe the motion of objects. The grid underlying the graph was merely the stage on which the actors played their parts. Now the background comes to the foreground: it’s time and space themselves that we’re studying. We don’t necessarily need to have a line or a curve drawn on top of the grid to represent a particular object. We may, for example, just want to talk about events, depicted as points on the graph as in figure ad. A distortion of the Cartesian grid underlying the graph can arise for perfectly
ordinary reasons that Isaac Newton would have readily accepted. For example, we can simply change the units used to measure time and position, as in figure ae.

We’re going to have quite a few examples of this type, so I’ll adopt the convention shown in figure af for depicting them. Figure af summarizes the relationship between figures ad and ae in a more compact form. The gray rectangle represents the original coordinate grid of figure ad, while the grid of black lines represents the new version from figure ae. Omitting the grid from the gray rectangle makes the diagram easier to decode visually.

Our goal of unraveling the mysteries of special relativity amounts to nothing more than finding out how to draw a diagram like af in the case where the two different sets of coordinates represent measurements of time and space made by two different observers, each in motion relative to the other. Galileo and Newton thought they knew the answer to this question, but their answer turned out to be only approximately right. To avoid repeating the same mistakes, we need to clearly spell out what we think are the basic properties of time and space that will be a reliable foundation for our reasoning. I want to emphasize that there is no purely logical way of deciding on this list of properties. The ones I’ll list are simply a summary of the patterns observed in the results from a large body of experiments. Furthermore, some of them are only approximate. For example, property 1 below is only a good approximation when the gravitational field is weak, so it is a property that applies to special relativity, not to general relativity.

Experiments show that:

1. No point in time or space has properties that make it different from any other point.

2. Likewise, all directions in space have the same properties.

3. Motion is relative, i.e., all inertial frames of reference are equally valid.

4. Causality holds, in the sense described on page 75.

5. Time depends on the state of motion of the observer.

Most of these are not very subversive. Properties 1 and 2 date back to the time when Galileo and Newton started applying the same universal laws of motion to the solar system and to the earth; this contradicted Aristotle, who believed that, for example, a rock would naturally want to move in a certain special direction (down) in order to reach a certain special location (the earth’s surface). Property 3 is the reason that Einstein called his theory “relativity,” but Galileo and Newton believed exactly the same thing to be true,
A Galilean version of the relationship between two frames of reference. As in all such graphs in this chapter, the original coordinates, represented by the gray rectangle, have a time axis that goes to the right, and a position axis that goes straight up.

A transformation that leads to disagreements about whether two events occur at the same time and place. This is not just a matter of opinion. Either the arrow hit the bull’s-eye or it didn’t.

A nonlinear transformation.

If it were not for property 5, we could imagine that figure ag would give the correct transformation between frames of reference in motion relative to one another. Let’s say that observer 1, whose grid coincides with the gray rectangle, is a hitch-hiker standing by the side of a road. Event A is a raindrop hitting his head, and event B is another raindrop hitting his head. He says that A and B occur at the same location in space. Observer 2 is a motorist who drives by without stopping; to him, the passenger compartment of his car is at rest, while the asphalt slides by underneath. He says that A and B occur at different points in space, because during the time between the first raindrop and the second, the hitch-hiker has moved backward. On the other hand, observer 2 says that events A and C occur in the same place, while the hitch-hiker disagrees. The slope of the grid-lines is simply the velocity of the relative motion of each observer relative to the other.

Figure ag has familiar, comforting, and eminently sensible behavior, but it also happens to be wrong, because it violates property 5. The distortion of the coordinate grid has only moved the vertical lines up and down, so both observers agree that events like B and C are simultaneous. If this was really the way things worked, then all observers could synchronize all their clocks with one another for once and for all, and the clocks would never get out of sync. This contradicts the results of the Hafele-Keating experiment, in which all three clocks were initially synchronized in Washington, but later went out of sync because of their different states of motion.

It might seem as though we still had a huge amount of wiggle room available for the correct form of the distortion. It turns out, however, that properties 1-5 are sufficient to prove that there is only one answer, which is the one found by Einstein in 1905. To see why this is, let’s work by a process of elimination.

Figure ah shows a transformation that might seem at first glance to be as good a candidate as any other, but it violates property 3, that motion is relative, for the following reason. In observer 2’s frame of reference, some of the grid lines cross one another. This means that observers 1 and 2 disagree on whether or not certain events are the same. For instance, suppose that event A marks the arrival of an arrow at thebull’s-eye of a target, and event B is the location and time when the bull’s-eye is punctured. Events A and B occur at the same location and at the same time. If one observer says that A and B coincide, but another says that they don’t, we
have a direct contradiction. Since the two frames of reference in figure ah give contradictory results, one of them is right and one is wrong. This violates property 3, because all inertial frames of reference are supposed to be equally valid. To avoid problems like this, we clearly need to make sure that none of the grid lines ever cross one another.

The next type of transformation we want to kill off is shown in figure ai, in which the grid lines curve, but never cross one another. The trouble with this one is that it violates property 1, the uniformity of time and space. The transformation is unusually “twisty” at A, whereas at B it’s much more smooth. This can’t be correct, because the transformation is only supposed to depend on the relative state of motion of the two frames of reference, and that given information doesn’t single out a special role for any particular point in spacetime. If, for example, we had one frame of reference rotating relative to the other, then there would be something special about the axis of rotation. But we’re only talking about inertial frames of reference here, as specified in property 3, so we can’t have rotation; each frame of reference has to be moving in a straight line at constant speed. For frames related in this way, there is nothing that could single out an event like A for special treatment compared to B, so transformation ai violates property 1.

The examples in figures ah and ai show that the transformation we’re looking for must be linear, meaning that it must transform lines into lines, and furthermore that it has to take parallel lines to parallel lines. Einstein wrote in his 1905 paper that “...on account of the property of homogeneity [property 1] which we ascribe to time and space, the [transformation] must be linear.” Applying this to our diagrams, the original gray rectangle, which is a special type of parallelogram containing right angles, must be transformed into another parallelogram. There are three types of transformations, figure aj, that have this property. Case I is the Galilean transformation of figure ag on page 80, which we’ve already ruled out.

Case II can also be discarded. Here every point on the grid rotates counterclockwise. What physical parameter would determine the amount of rotation? The only thing that could be relevant would be \( v \), the relative velocity of the motion of the two frames of reference with respect to one another. But if the angle of rotation was proportional to \( v \), then for large enough velocities the grid would have left and right reversed, and this would violate property 4, causality: one observer would say that event A caused a later event B, but another observer would say that B came first and caused A.

Three types of transformations that preserve parallelism. Their distinguishing feature is what they do to simultaneity, as shown by what happens to the left edge of the original rectangle. In I, the left edge remains vertical, so simultaneous events remain simultaneous. In II, the left edge turns counterclockwise. In III, it turns clockwise.

The only remaining possibility is case III, which I’ve redrawn in figure ak with a couple of changes. This is the one that Einstein predicted in 1905. The transformation is known as the Lorentz transformation, after Hendrik Lorentz (1853-1928), who partially anticipated Einstein’s work, without arriving at the correct interpretation. The distortion is a kind of smooshing and stretching, as suggested by the hands. Also, we’ve already seen in figures ad-af on page 79 that we’re free to stretch or compress everything as much as we like in the horizontal and vertical directions, because this simply corresponds to choosing different units of measurement for time and distance. In figure ak I’ve chosen units that give the whole drawing a convenient symmetry about a 45-degree diagonal line. Ordinarily it wouldn’t make sense to talk about a 45-degree angle on a graph whose axes had different units. But in relativity, the symmetric appearance of the transformation tells us that space and time ought to be treated on the same footing, and measured in the same units.

As in our discussion of the Galilean transformation, slopes are interpreted as velocities, and the slope of the near-horizontal lines in figure al is interpreted as the relative velocity of the two observers. The difference between the Galilean version and the relativistic one is that now there is smooshing happening from the other side as well. Lines that were vertical in the original grid, representing simultaneous events, now slant over to the right. This tells us that, as required by property 5, different observers do not agree on whether events that occur in different places are simultaneous. The Hafele-Keating experiment tells us that this non-simultaneity effect is fairly small, even when the velocity is as big as that of a passenger jet, and this is what we would have anticipated by the correspondence principle. The way that this is expressed in the graph is that if we pick the time unit to be the second, then the distance unit turns out to be hundreds of thousands of miles. In these units, the velocity of a passenger jet is an extremely small number, so the slope $v$ in figure al is extremely small, and the amount of distortion is tiny — it would be much too small to see on this scale.
The only thing left to determine about the Lorentz transformation is the size of the transformed parallelogram relative to the size of the original one. Although the drawing of the hands in figure ak may suggest that the grid deforms like a framework made of rigid coat-hanger wire, that is not the case. If you look carefully at the figure, you’ll see that the edges of the smooshed parallelogram are actually a little longer than the edges of the original rectangle. In fact what stays the same is not lengths but areas, as proved in the caption to figure am.

**Proof that Lorentz transformations don’t change area:** We first subject a square to a transformation with velocity \( v \), and this increases its area by a factor \( R(v) \), which we want to prove equals 1. We chop the resulting parallelogram up into little squares and finally apply a \( -v \) transformation; this changes each little square’s area by a factor \( R(-v) \), so the whole figure’s area is also scaled by \( R(-v) \). The final result is to restore the square to its original shape and area, so \( R(v)R(-v) = 1 \). But \( R(v) = R(-v) \) by property 2 of spacetime on page 79, which states that all directions in space have the same properties, so \( R(v) = 1 \).

**The \( \gamma \) factor**

With a little algebra and geometry (homework problem 18, page 98), one can use the equal-area property to show that the factor \( \gamma \) (Greek letter gamma) defined in figure an is given by the equation

\[
\gamma = \frac{1}{\sqrt{1 - v^2}}.
\]

If you’ve had good training in physics, the first thing you probably think when you look at this equation is that it must be nonsense, because its units don’t make sense. How can we take something with units of velocity squared, and subtract it from a unitless 1? But remember that this is expressed in our special relativistic units, in which the same units are used for distance and time. We refer to these as *natural* units. In this system, velocities are always unitless. This sort of thing happens frequently in physics. For instance, before James Joule discovered conservation of energy, nobody knew that heat and mechanical energy were different forms of the same thing, so instead of measuring them both in units of joules as we would do now, they measured heat in one unit (such as calories)
and mechanical energy in another (such as foot-pounds). In ordinary metric units, we just need an extra conversion factor $c$, and the equation becomes

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$ 

Here’s why we care about $\gamma$. Figure an defines it as the ratio of two times: the time between two events as expressed in one coordinate system, and the time between the same two events as measured in the other one. The interpretation is:

**Time dilation**

A clock runs fastest in the frame of reference of an observer who is at rest relative to the clock. An observer in motion relative to the clock at speed $v$ perceives the clock as running more slowly by a factor of $\gamma$.

As proved in figures ao and ap, lengths are also distorted:

**Length contraction**

A meter-stick appears longest to an observer who is at rest relative to it. An observer moving relative to the meter-stick at $v$ observes the stick to be shortened by a factor of $\gamma$.

**self-check F**

What is $\gamma$ when $v = 0$? What does this mean?  

$\triangleright$ Answer, p. 567
Figure 1 shows the behavior of $\gamma$ as a function of $v$.

Changing an equation from natural units to SI example 6
Often it is easier to do all of our algebra in natural units, which are simpler because $c = 1$, and all factors of $c$ can therefore be omitted. For example, suppose we want to solve for $v$ in terms of $\gamma$. In natural units, we have $\gamma = \frac{1}{\sqrt{1 - v^2}}$, so $\gamma^2 = 1 - v^2$, and $v = \sqrt{1 - \gamma^{-2}}$.

This form of the result might be fine for many purposes, but if we wanted to find a value of $v$ in SI units, we would need to reinsert factors of $c$ in the final result. There is no need to do this throughout the whole derivation. By looking at the final result, we see that there is only one possible way to do this so that the results make sense in SI, which is to write $v = c\sqrt{1 - \gamma^{-2}}$.

Motion of a ray of light example 7
▷ The motion of a certain ray of light is given by the equation $x = -t$. Is this expressed in natural units, or in SI units? Convert to the other system.

▷ The equation is in natural units. It wouldn’t make sense in SI units, because we would have meters on the left and seconds on the right. To convert to SI units, we insert a factor of $c$ in the only possible place that will cause the equation to make sense: $x = -ct$.

An interstellar road trip example 8
Alice stays on earth while her twin Betty heads off in a spaceship for Tau Ceti, a nearby star. Tau Ceti is 12 light-years away, so even though Betty travels at 87% of the speed of light, it will take her a long time to get there: 14 years, according to Alice.

Betty experiences time dilation. At this speed, her $\gamma$ is 2.0, so that the voyage will only seem to her to last 7 years. But there is perfect symmetry between Alice’s and Betty’s frames of reference, so Betty agrees with Alice on their relative speed; Betty sees herself as being at rest, while the sun and Tau Ceti both move backward at 87% of the speed of light. How, then, can she observe Tau Ceti to get to her in only 7 years, when it should take 14 years to travel 12 light-years at this speed?
Muons accelerated to nearly $c$ undergo radioactive decay much more slowly than they would according to an observer at rest with respect to the muons. The first two data-points (unfilled circles) were subject to large systematic errors.

We need to take into account length contraction. Betty sees the distance between the sun and Tau Ceti to be shrunk by a factor of 2. The same thing occurs for Alice, who observes Betty and her spaceship to be foreshortened.

The correspondence principle example 9
The correspondence principle requires that $\gamma$ be close to 1 for the velocities much less than $c$ encountered in everyday life. In natural units, $\gamma = (1 - v^2)^{-1/2}$. For small values of $\epsilon$, the approximation $(1 + \epsilon)^p \approx 1 + p\epsilon$ holds (see p. ??). Applying this approximation, we find $\gamma \approx 1 + v^2/2$.

As expected, this gives approximately 1 when $v$ is small compared to 1 (i.e., compared to $c$, which equals 1 in natural units).

In problem ?? on p. ?? we rewrite this in SI units.

Figure aq on p. 85 shows that the approximation is not valid for large values of $v/c$. In fact, $\gamma$ blows up to infinity as $v$ gets closer and closer to $c$.

Large time dilation example 10
The time dilation effect in the Hafele-Keating experiment was very small. If we want to see a large time dilation effect, we can’t do it with something the size of the atomic clocks they used; the kinetic energy would be greater than the total megatonnage of all the world’s nuclear arsenals. We can, however, accelerate subatomic particles to speeds at which $\gamma$ is large. For experimental particle physicists, relativity is something you do all day before heading home and stopping off at the store for milk. An early, low-precision experiment of this kind was performed by Rossi and Hall in 1941, using naturally occurring cosmic rays. Figure at shows a 1974 experiment of a similar type which verified the time dilation predicted by relativity to a precision of about one part per thousand.

Particles called muons (named after the Greek letter $\mu$, “myoo”) were produced by an accelerator at CERN, near Geneva. A muon is essentially a heavier version of the electron. Muons undergo radioactive decay, lasting an average of only 2.197 $\mu$s before they evaporate into an electron and two neutrinos. The 1974 experiment was actually built in order to measure the magnetic properties of muons, but it produced a high-precision test of time dilation as a byproduct. Because muons have the same electric charge as electrons, they can be trapped using magnetic fields. Muons were injected into the ring shown in figure at, circling around it until they underwent radioactive decay. At the speed at which these muons were traveling, they had $\gamma = 29.33$, so on the average they were

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$^3$Bailey at al., Nucl. Phys. B150(1979) 1
lasted 29.33 times longer than the normal lifetime. In other words, they were like tiny alarm clocks that self-destructed at a randomly selected time. Figure as shows the number of radioactive decays counted, as a function of the time elapsed after a given stream of muons was injected into the storage ring. The two dashed lines show the rates of decay predicted with and without relativity. The relativistic line is the one that agrees with experiment.

Colliding nuclei show relativistic length contraction.

Figure au shows an artist's rendering of the length contraction for the collision of two gold nuclei at relativistic speeds in the RHIC accelerator in Long Island, New York, which went on line in 2000. The gold nuclei would appear nearly spherical (or just slightly lengthened like an American football) in frames moving along with them, but in the laboratory's frame, they both appear drastically foreshortened as they approach the point of collision. The later pictures show the nuclei merging to form a hot soup, in which experimenters hope to observe a new form of matter.
Example 12: In the garage’s frame of reference, the bus is moving, and can fit in the garage due to its length contraction. In the bus’s frame of reference, the garage is moving, and can’t hold the bus due to its length contraction.

The garage paradox example 12

One of the most famous of all the so-called relativity paradoxes has to do with our incorrect feeling that simultaneity is well defined. The idea is that one could take a schoolbus and drive it at relativistic speeds into a garage of ordinary size, in which it normally would not fit. Because of the length contraction, the bus would supposedly fit in the garage. The driver, however, will perceive the garage as being contracted and thus even less able to contain the bus.

The paradox is resolved when we recognize that the concept of fitting the bus in the garage “all at once” contains a hidden assumption, the assumption that it makes sense to ask whether the front and back of the bus can simultaneously be in the garage. Observers in different frames of reference moving at high relative speeds do not necessarily agree on whether things happen simultaneously. As shown in figure av, the person in the garage’s frame can shut the door at an instant B he perceives to be simultaneous with the front bumper’s arrival A at the back wall of the garage, but the driver would not agree about the simultaneity of these two events, and would perceive the door as having shut long after she plowed through the back wall.
The universal speed $c$

Let’s think a little more about the role of the 45-degree diagonal in the Lorentz transformation. Slopes on these graphs are interpreted as velocities. This line has a slope of 1 in relativistic units, but that slope corresponds to $c$ in ordinary metric units. We already know that the relativistic distance unit must be extremely large compared to the relativistic time unit, so $c$ must be extremely large. Now note what happens when we perform a Lorentz transformation: this particular line gets stretched, but the new version of the line lies right on top of the old one, and its slope stays the same. In other words, if one observer says that something has a velocity equal to $c$, every other observer will agree on that velocity as well. (The same thing happens with $-c$.)

$\triangleright$ Velocities don’t simply add and subtract.

This is surprising, since we expect, as in section 2.5.1, that a velocity $c$ in one frame should become $c + v$ in a frame moving at velocity $v$ relative to the first one. But velocities are measured by dividing a distance by a time, and both distance and time are distorted by relativistic effects, so we actually shouldn’t expect the ordinary arithmetic addition of velocities to hold in relativity; it’s an approximation that’s valid at velocities that are small compared to $c$. Problem 22 on p. 100 shows that relativistically, combining velocities $u$ and $v$ gives not $u + v$ but $(u + v)/(1 + uv)$ (in units where $c = 1$).

$\triangleright$ A universal speed limit

For example, suppose Janet takes a trip in a spaceship, and accelerates until she is moving at $0.6c$ relative to the earth. She then launches a space probe in the forward direction at a speed relative to her ship of $0.6c$. We might think that the probe was then moving at a velocity of $1.2c$, but in fact the answer is still less than $c$ (problem 21, page 99). This is an example of a more general fact about relativity, which is that $c$ represents a universal speed limit. This is required by causality, as shown in figure aw.

$\triangleright$ Light travels at $c$.

Now consider a beam of light. We’re used to talking casually about the “speed of light,” but what does that really mean? Motion is relative, so normally if we want to talk about a velocity, we have to specify what it’s measured relative to. A sound wave has a certain speed relative to the air, and a water wave has its own speed relative to the water. If we want to measure the speed of an ocean wave, for example, we should make sure to measure it in a frame of reference at rest relative to the water. But light isn’t a vibration of a physical medium; it can propagate through the near-perfect vacuum of outer space, as when rays of sunlight travel to earth. This seems like a
The Michelson-Morley experiment, shown in photographs, and drawings from the original 1887 paper. 1. A simplified drawing of the apparatus. A beam of light from the source, s, is partially reflected and partially transmitted by the half-silvered mirror $h_1$. The two half-intensity parts of the beam are reflected by the mirrors at a and b, reunited, and observed in the telescope, t. If the earth’s surface was supposed to be moving through the ether, then the times taken by the two light waves to pass through the moving ether would be unequal, and the resulting time lag would be detectable by observing the interference between the waves when they were reunited. 2. In the real apparatus, the light beams were reflected multiple times. The effective length of each arm was increased to 11 meters, which greatly improved its sensitivity to the small expected difference in the speed of light. 3. In an earlier version of the experiment, they had run into problems with its “extreme sensitiveness to vibration,” which was “so great that it was impossible to see the interference fringes except at brief intervals … even at two o’clock in the morning.” They therefore mounted the whole thing on a massive stone floating in a pool of mercury, which also made it possible to rotate it easily. 4. A photo of the apparatus.

paradox: light is supposed to have a specific speed, but there is no way to decide what frame of reference to measure it in. The way out of the paradox is that light must travel at a velocity equal to $c$. Since all observers agree on a velocity of $c$, regardless of their frame of reference, everything is consistent.

The Michelson-Morley experiment

The constancy of the speed of light had in fact already been observed when Einstein was an 8-year-old boy, but because nobody
could figure out how to interpret it, the result was largely ignored. In 1887 Michelson and Morley set up a clever apparatus to measure any difference in the speed of light beams traveling east-west and north-south. The motion of the earth around the sun at 110,000 km/hour (about 0.01% of the speed of light) is to our west during the day. Michelson and Morley believed that light was a vibration of a mysterious medium called the ether, so they expected that the speed of light would be a fixed value relative to the ether. As the earth moved through the ether, they thought they would observe an effect on the velocity of light along an east-west line. For instance, if they released a beam of light in a westward direction during the day, they expected that it would move away from them at less than the normal speed because the earth was chasing it through the ether. They were surprised when they found that the expected 0.01% change in the speed of light did not occur.

1. The ring laser gyroscope example 13
If you’ve flown in a jet plane, you can thank relativity for helping you to avoid crashing into a mountain or an ocean. Figure ay shows a standard piece of navigational equipment called a ring laser gyroscope. A beam of light is split into two parts, sent around the perimeter of the device, and reunited. Since the speed of light is constant, we expect the two parts to come back together at the same time. If they don’t, it’s evidence that the device has been rotating. The plane’s computer senses this and notes how much rotation has accumulated.

1. No frequency-dependence example 14
Relativity has only one universal speed, so it requires that all light waves travel at the same speed, regardless of their frequency and wavelength. Presently the best experimental tests of the invariance of the speed of light with respect to wavelength come from astronomical observations of gamma-ray bursts, which are sudden outpourings of high-frequency light, believed to originate from a supernova explosion in another galaxy. One such observation, in 2009,4 found that the times of arrival of all the different frequencies in the burst differed by no more than 2 seconds out of a total time in flight on the order of ten billion years!

4http://arxiv.org/abs/0908.1832
**Discussion questions**

**A**  A person in a spaceship moving at 99.99999999% of the speed of light relative to Earth shines a flashlight forward through dusty air, so the beam is visible. What does she see? What would it look like to an observer on Earth?

**B**  A question that students often struggle with is whether time and space can really be distorted, or whether it just seems that way. Compare with optical illusions or magic tricks. How could you verify, for instance, that the lines in the figure are actually parallel? Are relativistic effects the same, or not?

**C**  On a spaceship moving at relativistic speeds, would a lecture seem even longer and more boring than normal?

**D**  Mechanical clocks can be affected by motion. For example, it was a significant technological achievement to build a clock that could sail aboard a ship and still keep accurate time, allowing longitude to be determined. How is this similar to or different from relativistic time dilation?

**E**  Figure au from page 87, depicting the collision of two nuclei at the RHIC accelerator, is reproduced below. What would the shapes of the two nuclei look like to a microscopic observer riding on the left-hand nucleus? To an observer riding on the right-hand one? Can they agree on what is happening? If not, why not — after all, shouldn’t they see the same thing if they both compare the two nuclei side-by-side at the same instant in time?

**Discussion question E:** colliding nuclei show relativistic length contraction.

**F**  If you stick a piece of foam rubber out the window of your car while driving down the freeway, the wind may compress it a little. Does it make sense to interpret the relativistic length contraction as a type of strain that pushes an object’s atoms together like this? How does this relate to discussion question E?
The machine-gunner in the figure sends out a spray of bullets. Suppose that the bullets are being shot into outer space, and that the distances traveled are trillions of miles (so that the human figure in the diagram is not to scale). After a long time, the bullets reach the points shown with dots which are all equally far from the gun. Their arrivals at those points are events A through E, which happen at different times. Sketch these events on a position-time graph. The chain of impacts extends across space at a speed greater than $c$. Does this violate special relativity?

Discussion question G.
Summary

Selected vocabulary
- center of mass: the balance point of an object
- velocity: the rate of change of position; the slope of the tangent line on an $x-t$ graph.

Notation
- $x$: a point in space
- $t$: a point in time, a clock reading
- $\Delta$: “change in;” the value of a variable afterwards minus its value before
- $\Delta x$: a distance, or more precisely a change in $x$, which may be less than the distance traveled; its plus or minus sign indicates direction
- $\Delta t$: a duration of time
- $v$: velocity
- $v_{AB}$: the velocity of object A relative to object B

Other terminology and notation
- displacement: a name for the symbol $\Delta x$
- speed: the absolute value of the velocity, i.e., the velocity stripped of any information about its direction

Summary

An object’s center of mass is the point at which it can be balanced. For the time being, we are studying the mathematical description only of the motion of an object’s center of mass in cases restricted to one dimension. The motion of an object’s center of mass is usually far simpler than the motion of any of its other parts.

It is important to distinguish location, $x$, from distance, $\Delta x$, and clock reading, $t$, from time interval $\Delta t$. When an object’s $x-t$ graph is linear, we define its velocity as the slope of the line, $\Delta x/\Delta t$. When the graph is curved, we generalize the definition so that the velocity is the derivative $dx/dt$.

Galileo’s principle of inertia states that no force is required to maintain motion with constant velocity in a straight line, and absolute motion does not cause any observable physical effects. Things typically tend to reduce their velocity relative to the surface of our planet only because they are physically rubbing against the planet (or something attached to the planet), not because there is anything special about being at rest with respect to the earth’s surface. When it seems, for instance, that a force is required to keep a book sliding across a table, in fact the force is only serving to cancel the contrary force of friction.

Absolute motion is not a well-defined concept, and if two observers are not at rest relative to one another they will disagree about the absolute velocities of objects. They will, however, agree...
about relative velocities. If object A is in motion relative to object 
B, and B is in motion relative to C, then A’s velocity relative to C 
is given by \( v_{AC} = v_{AB} + v_{BC} \). Positive and negative signs are used 
to indicate the direction of an object’s motion.

Modern experiments show that space and time only approximately have the properties claimed by Galileo and Newton. Time 
and space as seen by one observer are distorted compared to another 
observer’s perceptions if they are moving relative to each other. This 
distortion is quantified by the factor

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},
\]

where \( v \) is the relative velocity of the two observers, and \( c \) is a 
universal velocity that is the same in all frames of reference. Light 
travels at \( c \). A clock appears to run fastest to an observer who is 
not in motion relative to it, and appears to run too slowly by a 
factor of \( \gamma \) to an observer who has a velocity \( v \) relative to the clock. 
Similarly, a meter-stick appears longest to an observer who sees it 
at rest, and appears shorter to other observers. Time and space are 
relative, not absolute. In particular, there is no well-defined concept 
of simultaneity. Velocities don’t add according to \( u + v \) but rather 
\( (u + v)/(1 + uv) \) (in units where \( c = 1 \)).

All of these strange effects, however, are very small when the rel-

tive velocities are small compared to \( c \). This makes sense, because 
Newton’s laws have already been thoroughly tested by experiments 
at such speeds, so a new theory like relativity must agree with the 
old one in their realm of common applicability. This requirement of 
backwards-compatibility is known as the correspondence principle.
Problems

Key
✓ A computerized answer check is available online.
∫ A problem that requires calculus.
☆ A difficult problem.

1 The graph represents the motion of a ball that rolls up a hill and then back down. When does the ball return to the location it had at $t = 0$? $\Rightarrow$ Solution, p. 552

2 The graph represents the velocity of a bee along a straight line. At $t = 0$, the bee is at the hive. (a) When is the bee farthest from the hive? (b) How far is the bee at its farthest point from the hive? (c) At $t = 13$ s, how far is the bee from the hive?

3 (a) Let $\theta$ be the latitude of a point on the Earth's surface. Derive an algebra equation for the distance, $L$, traveled by that point during one rotation of the Earth about its axis, i.e., over one day, expressed in terms of $\theta$ and $R$, the radius of the earth. Check: Your equation should give $L = 0$ for the North Pole.
(b) At what speed is Fullerton, at latitude $\theta = 34^\circ$, moving with the rotation of the Earth about its axis? Give your answer in units of mi/h. [See the table in the back of the book for the relevant data.]

4 A honeybee’s position as a function of time is given by $x = 10t - t^3$, where $t$ is in seconds and $x$ in meters. What is its velocity at $t = 3.0$ s?

5 Freddi Fish(TM) has a position as a function of time given by $x = a/(b + t^2)$. (a) Infer the units of the constants $a$ and $b$. (b) Find her maximum speed. (c) Check that your answer has the right units.

6 A metal square expands and contracts with temperature, the lengths of its sides varying according to the equation $\ell = (1+\alpha T)\ell_o$. Infer the units of $\alpha$. Find the rate of change of its surface area with respect to temperature. That is, find $dA/dT$. Check that your answer has the right units, as in example 4 on page 68.
Let $t$ be the time that has elapsed since the Big Bang. In that time, one would imagine that light, traveling at speed $c$, has been able to travel a maximum distance $ct$. (In fact the distance is several times more than this, because according to Einstein’s theory of general relativity, space itself has been expanding while the ray of light was in transit.) The portion of the universe that we can observe would then be a sphere of radius $ct$, with volume $v = (4/3)\pi r^3 = (4/3)\pi (ct)^3$. Compute the rate $dV/dt$ at which the volume of the observable universe is increasing, and check that your answer has the right units, as in example 4 on page 68.

(a) Express the chain rule in Leibniz (“d”) notation, and show that it always results in an answer whose units make sense.

(b) An object has a position as a function of time given by $x = A \cos(bt)$, where $A$ and $b$ are constants. Infer the units of $A$ and $b$, and interpret their physical meanings.

(c) Find the velocity of this object, and check that the chain rule has indeed given an answer with the right units.

9  (a) Translate the following information into symbols, using the notation with two subscripts introduced in section 2.5. Eowyn is riding on her horse at a velocity of 11 m/s. She twists around in her saddle and fires an arrow backward. Her bow fires arrows at 25 m/s. (b) Find the velocity of the arrow relative to the ground.

10  Our full discussion of two- and three-dimensional motion is postponed until the second half of the book, but here is a chance to use a little mathematical creativity in anticipation of that generalization. Suppose a ship is sailing east at a certain speed $v$, and a passenger is walking across the deck at the same speed $v$, so that his track across the deck is perpendicular to the ship’s center-line. What is his speed relative to the water, and in what direction is he moving relative to the water?

11  You’re standing in a freight train, and have no way to see out. If you have to lean to stay on your feet, what, if anything, does that tell you about the train’s velocity? Explain.

12  Driving along in your car, you take your foot off the gas, and your speedometer shows a reduction in speed. Describe a frame of reference in which your car was speeding up during that same period of time. (The frame of reference should be defined by an observer who, although perhaps in motion relative to the earth, is not changing her own speed or direction of motion.)

13  The figure shows the motion of a point on the rim of a rolling wheel. (The shape is called a cycloid.) Suppose bug A is riding on the rim of the wheel on a bicycle that is rolling, while bug B is on the spinning wheel of a bike that is sitting upside down on the floor.
Bug A is moving along a cycloid, while bug B is moving in a circle. Both wheels are doing the same number of revolutions per minute. Which bug has a harder time holding on, or do they find it equally difficult? Solution, p. 553

14 Astronauts in three different spaceships are communicating with each other. Those aboard ships A and B agree on the rate at which time is passing, but they disagree with the ones on ship C. (a) Alice is aboard ship A. How does she describe the motion of her own ship, in its frame of reference? (b) Describe the motion of the other two ships according to Alice. (c) Give the description according to Betty, whose frame of reference is ship B. (d) Do the same for Cathy, aboard ship C.

15 What happens in the equation for $\gamma$ when you put in a negative number for $v$? Explain what this means physically, and why it makes sense.

16 The Voyager 1 space probe, launched in 1977, is moving faster relative to the earth than any other human-made object, at 17,000 meters per second. (a) Calculate the probe’s $\gamma$. (b) Over the course of one year on earth, slightly less than one year passes on the probe. How much less? (There are 31 million seconds in a year.)

17 The earth is orbiting the sun, and therefore is contracted relativistically in the direction of its motion. Compute the amount by which its diameter shrinks in this direction.

18 In this homework problem, you’ll fill in the steps of the algebra required in order to find the equation for $\gamma$ on page 83. To keep the algebra simple, let the time $t$ in figure an equal 1, as suggested in the figure accompanying this homework problem. The original square then has an area of 1, and the transformed parallelogram must also have an area of 1. (a) Prove that point P is at $x = vY$, so that its $(t, x)$ coordinates are $(\gamma, vY)$. (b) Find the $(t, x)$ coordinates of point Q. (c) Find the length of the short diagonal connecting P and Q. (d) Average the coordinates of P and Q to find the coordinates of the midpoint C of the parallelogram, and then find distance OC. (e) Find the area of the parallelogram by computing twice the area of triangle PQO. [Hint: You can take PQ to be the base of the triangle.] (f) Set this area equal to 1 and solve for $\gamma$ to prove $\gamma = 1/\sqrt{1 - v^2}$.

19 (a) Show that for $v = (3/5)c$, $\gamma$ comes out to be a simple fraction. (b) Find another value of $v$ for which $\gamma$ is a simple fraction.

20 In the equation for the relativistic addition of velocities $u$ and $v$, consider the limit in which $u$ approaches 1, but $v$ simultaneously approaches $-1$. Give both a physical and a mathematical interpretation.
The figure illustrates a Lorentz transformation using the conventions employed in section 2.6.2. For simplicity, the transformation chosen is one that lengthens one diagonal by a factor of 2. Since Lorentz transformations preserve area, the other diagonal is shortened by a factor of 2. Let the original frame of reference, depicted with the square, be A, and the new one B. (a) By measuring with a ruler on the figure, show that the velocity of frame B relative to frame A is 0.6c. (b) Print out a copy of the page. With a ruler, draw a third parallelogram that represents a second successive Lorentz transformation, one that lengthens the long diagonal by another factor of 2. Call this third frame C. Use measurements with a ruler to determine frame C’s velocity relative to frame A. Does it equal double the velocity found in part a? Explain why it should be expected to turn out the way it does. A general equation for this type of calculation is derived in problem 22.
22 The purpose of this problem is to prove the general result \( w = \frac{(u + v)}{(1 + uv)} \) (in units where \( c = 1 \)) for the kind of combination of velocities found graphically in problem 21. Suppose that we perform two Lorentz transformations, with velocities \( u \) and \( v \), one after the other. Representing these transformations as distortions of parallelograms, we stretch the stretching diagonals by factors \( S(u) \) and \( S(v) \) (and compress the compressing ones by the inverses of these factors), so that the combined result is a stretching by \( S(u)S(v) \). We want to prove that \( S(w) = S(u)S(v) \) gives the expression claimed above for \( w \). One can easily show by fiddling with the result of part c of problem 18 that \( S(x) = \frac{\sqrt{1 + x}}{(1 - x)} \).

(a) Use these facts to write down an equation relating \( u \), \( v \), and \( w \).
(b) Solve for \( w \) in terms of \( u \) and \( v \).
(c) Show that your answer to part b satisfies the correspondence principle.
(d) Show that it is consistent with the constancy of \( c \).

23 Sometimes doors are built with mechanisms that automatically close them after they have been opened. The designer can set both the strength of the spring and the amount of friction. If there is too much friction in relation to the strength of the spring, the door takes too long to close, but if there is too little, the door will oscillate. For an optimal design, we get motion of the form \( x = cte^{-bt} \), where \( x \) is the position of some point on the door, and \( c \) and \( b \) are positive constants. (Similar systems are used for other mechanical devices, such as stereo speakers and the recoil mechanisms of guns.) In this example, the door moves in the positive direction up until a certain time, then stops and settles back in the negative direction, eventually approaching \( x = 0 \). This would be the type of motion we would get if someone flung a door open and the door closer then brought it back closed again.

(a) Infer the units of the constants \( b \) and \( c \).
(b) Find the door’s maximum speed (i.e., the greatest absolute value of its velocity) as it comes back to the closed position.
(c) Show that your answer has units that make sense.

24 At a picnic, someone hands you a can of beer. The ground is uneven, and you don’t want to spill your drink. You reason that it will be more stable if you drink some of it first in order to lower its center of mass. How much should you drink in order to make the center of mass as low as possible? [Based on a problem by Walter van B. Roberts and Martin Gardner.]

25 (a) In a race, you run the first half of the distance at speed \( u \), and the second half at speed \( v \). Find the over-all speed, i.e., the total distance divided by the total time.
(b) Check the units of your equation using the method shown in example 1 on p. 22.
(c) Check that your answer makes sense in the case where \( u = v \).
(d) Show that the dependence of your result on \( u \) and \( v \) makes sense. That is, first check whether making \( u \) bigger makes the result bigger, or smaller. Then compare this with what you expect physically.
[Problem by B. Shotwell.]