Radioactive contamination at Chernobyl example 3

One of the most dangerous radioactive isotopes released by the Chernobyl disaster in 1986 was $^{90}$Sr, whose half-life is 28 years.

(a) How long will it be before the contamination is reduced to one tenth of its original level? (b) If a total of $10^{27}$ atoms was released, about how long would it be before not a single atom was left?

(a) We want to know the amount of time that a $^{90}$Sr nucleus has a probability of 0.1 of surviving. Starting with the exponential decay formula,

$$P_{\text{surv}} = 0.5^{t/t_{1/2}},$$

we want to solve for $t$. Taking natural logarithms of both sides,

$$\ln P = \frac{t}{t_{1/2}} \ln 0.5,$$

so

$$t = t_{1/2} \ln P \ln 0.5.$$ 

Plugging in $P = 0.1$ and $t_{1/2} = 28$ years, we get $t = 93$ years.

(b) This is just like the first part, but $P = 10^{-27}$. The result is about 2500 years.

Rate of decay

If you want to find how many radioactive decays occur within a time interval lasting from time $t$ to $t + \Delta t$, the most straightforward approach is to calculate it like this:

$$\text{(number of decays between } t \text{ and } t + \Delta t) = N(t) - N(t + \Delta t) = N(0) [P_{\text{surv}}(t) - P_{\text{surv}}(t + \Delta t)] = N(0) \left[0.5^{t/t_{1/2}} - 0.5^{(t+\Delta t)/t_{1/2}}\right] = N(0) \left[1 - 0.5^{\Delta t/t_{1/2}}\right] 0.5^{t/t_{1/2}}$$

A problem arises when $\Delta t$ is small compared to $t_{1/2}$. For instance, suppose you have a hunk of $10^{22}$ atoms of $^{235}$U, with a half-life of 700 million years, which is $2.2 \times 10^{16}$ s. You want to know how many decays will occur in $\Delta t = 1$ s. Since we’re specifying the current number of atoms, $t = 0$. As you plug in to the formula above on your calculator, the quantity $0.5^{\Delta t/t_{1/2}}$ comes out on your calculator to equal one, so the final result is zero. That’s incorrect, though. In reality, $0.5^{\Delta t/t_{1/2}}$ should equal $0.999999999999999968$, but your calculator only gives eight digits of precision, so it rounded it off to one. In other words, the probability that a $^{235}$U atom will survive for 1 s is very close to one, but not equal to one. The number of decays in one second is therefore $3.2 \times 10^5$, not zero.
Well, my calculator only does eight digits of precision, just like yours, so how did I know the right answer? The way to do it is to use the following approximation (see p. 1059):

\[ a^b \approx 1 + b \ln a, \quad \text{if } b \ll 1 \]

(The symbol \( \ll \) means “is much less than.”) Using it, we can find the following approximation:

\[
\begin{align*}
\text{(number of decays between } t \text{ and } t + \Delta t) & = N(0) \left[ 1 - 0.5^{t/t_{1/2}} \right] 0.5^{t/t_{1/2}} \\
& \approx N(0) \left[ 1 - \left( 1 + \frac{\Delta t}{t_{1/2}} \ln 0.5 \right) \right] 0.5^{t/t_{1/2}} \\
& \approx (\ln 2) N(0) 0.5^{t/t_{1/2}} \frac{\Delta t}{t_{1/2}}
\end{align*}
\]

This also gives us a way to calculate the rate of decay, i.e., the number of decays per unit time. Dividing by \( \Delta t \) on both sides, we have

\[
\text{(decays per unit time)} \approx \frac{(\ln 2) N(0) 0.5^{t/t_{1/2}}}{t_{1/2}}, \quad \text{if } \Delta t \ll t_{1/2}.
\]

---

**The hot potato example 4**

A nuclear physicist with a demented sense of humor tosses you a cigar box, yelling “hot potato.” The label on the box says “contains \( 10^{20} \) atoms of \((^{17}\text{F})\), half-life of 66 s, produced today in our reactor at 1 p.m.” It takes you two seconds to read the label, after which you toss it behind some lead bricks and run away. The time is 1:40 p.m. Will you die?

The time elapsed since the radioactive fluorine was produced in the reactor was 40 minutes, or 2400 s. The number of elapsed half-lives is therefore \( t/t_{1/2} = 36 \). The initial number of atoms was \( N(0) = 10^{20} \). The number of decays per second is now about \( 10^7 \text{ s}^{-1} \), so it produced about \( 2 \times 10^7 \) high-energy electrons while you held it in your hands. Although twenty million electrons sounds like a lot, it is not really enough to be dangerous.

By the way, none of the equations we’ve derived so far was the actual probability distribution for the time at which a particular radioactive atom will decay. That probability distribution would be found by substituting \( N(0) = 1 \) into the equation for the rate of decay.

If the sheer number of equations is starting to seem formidable, let’s pause and think for a second. The simple equation for \( P_{\text{surv}} \) is
something you can derive easily from the law of independent probabilities any time you need it. From that, you can quickly find the exact equation for the rate of decay. The derivation of the approximate equations for $\Delta t \ll t$ is a little hairier, but note that except for the factors of $\ln 2$, everything in these equations can be found simply from considerations of logic and units. For instance, a longer half-life will obviously lead to a slower rate of decays, so it makes sense that we divide by it. As for the $\ln 2$ factors, they are exactly the kind of thing that one looks up in a book when one needs to know them.

**Discussion questions**

A In the medical procedure involving $^{131}\text{I}$, why is it the gamma rays that are detected, not the electrons or neutrinos that are also emitted?

B For 1 s, Fred holds in his hands 1 kg of radioactive stuff with a half-life of 1000 years. Ginger holds 1 kg of a different substance, with a half-life of 1 min, for the same amount of time. Did they place themselves in equal danger, or not?

C How would you interpret it if you calculated $N(t)$, and found it was less than one?

D Does the half-life depend on how much of the substance you have? Does the expected time until the sample decays completely depend on how much of the substance you have?

### 33.5 Applications of calculus

The area under the probability distribution is of course an integral. If we call the random number $x$ and the probability distribution $D(x)$, then the probability that $x$ lies in a certain range is given by

$$(\text{probability of } a \leq x \leq b) = \int_a^b D(x) \, dx.$$ 

What about averages? If $x$ had a finite number of equally probable values, we would simply add them up and divide by how many we had. If they weren’t equally likely, we’d make the weighted average $x_1 P_1 + x_2 P_2 + \ldots$. But we need to generalize this to a variable $x$ that can take on any of a continuum of values. The continuous version of a sum is an integral, so the average is

$$(\text{average value of } x) = \int x D(x) \, dx,$$

where the integral is over all possible values of $x$. 
Here is a rigorous justification for the statement in section 33.4 that the probability distribution for radioactive decay is found by substituting $N(0) = 1$ into the equation for the rate of decay. We know that the probability distribution must be of the form

$$D(t) = k0.5^{t/t_{1/2}},$$

where $k$ is a constant that we need to determine. The atom is guaranteed to decay eventually, so normalization gives us

$$(	ext{probability of } 0 \leq t < \infty) = 1 = \int_0^\infty D(t) \, dt.$$ 

The integral is most easily evaluated by converting the function into an exponential with $e$ as the base

$$D(t) = k \exp \left[ \ln \left( 0.5^{t/t_{1/2}} \right) \right]$$

$$= k \exp \left[ - \frac{t}{t_{1/2}} \ln 0.5 \right]$$

$$= k \exp \left( - \frac{\ln 2}{t_{1/2}} t \right),$$

which gives an integral of the familiar form $\int e^{cx} \, dx = \frac{1}{c} e^{cx}$. We thus have

$$1 = - \frac{kt_{1/2}}{\ln 2} \exp \left( - \frac{\ln 2}{t_{1/2}} t \right),$$

which gives the desired result:

$$k = \frac{\ln 2}{t_{1/2}}.$$ 

You might think that the half-life would also be the average lifetime of an atom, since half the atoms' lives are shorter and half longer. But the half whose lives are longer include some that survive for many half-lives, and these rare long-lived atoms skew the average. We can calculate the average lifetime as follows:

$$(\text{average lifetime}) = \int_0^\infty t \, D(t) \, dt$$ 

Using the convenient base-$e$ form again, we have

$$(\text{average lifetime}) = \frac{\ln 2}{t_{1/2}} \int_0^\infty t \exp \left( - \frac{\ln 2}{t_{1/2}} t \right) \, dt.$$ 

This integral is of a form that can either be attacked with integration by parts or by looking it up in a table. The result is
\[ \int xe^{cx} \, dx = \frac{1}{c} e^{cx} - \frac{1}{c^2} e^{cx}, \] and the first term can be ignored for our purposes because it equals zero at both limits of integration. We end up with

\[
\text{(average lifetime)} = \frac{\ln 2}{t_{1/2}} \left( \frac{t_{1/2}}{\ln 2} \right)^2
\]

\[ = \frac{t_{1/2}}{\ln 2}
\]

\[ = 1.443 t_{1/2}, \]

which is, as expected, longer than one half-life.
Summary

Selected vocabulary
probability . . . . the likelihood that something will happen, expressed as a number between zero and one
normalization . . the property of probabilities that the sum of the probabilities of all possible outcomes must equal one
independence . . the lack of any relationship between two random events
probability distribution . . . . . . . a curve that specifies the probabilities of various random values of a variable; areas under the curve correspond to probabilities
FWHM . . . . . the full width at half-maximum of a probability distribution; a measure of the width of the distribution
half-life . . . . . the amount of time that a radioactive atom will survive with probability 1/2 without decaying

Notation
$P$ . . . . . . . probability
$t_{1/2}$ . . . . . . half-life
$D$ . . . . . . . a probability distribution (used only in optional section 33.5; not a standardized notation)

Summary
Quantum physics differs from classical physics in many ways, the most dramatic of which is that certain processes at the atomic level, such as radioactive decay, are random rather than deterministic. There is a method to the madness, however: quantum physics still rules out any process that violates conservation laws, and it also offers methods for calculating probabilities numerically.

In this chapter we focused on certain generic methods of working with probabilities, without concerning ourselves with any physical details. Without knowing any of the details of radioactive decay, for example, we were still able to give a fairly complete treatment of the relevant probabilities. The most important of these generic methods is the law of independent probabilities, which states that if two random events are not related in any way, then the probability that they will both occur equals the product of the two probabilities,

\[
\text{probability of A and B} = P_A P_B \quad \text{[if A and B are independent].}
\]

The most important application is to radioactive decay. The time that a radioactive atom has a 50% chance of surviving is called the half-life, $t_{1/2}$. The probability of surviving for two half-lives is
(1/2)(1/2) = 1/4, and so on. In general, the probability of surviving a time $t$ is given by

$$P_{\text{surv}}(t) = 0.5^{t/t_{1/2}}.$$ 

Related quantities such as the rate of decay and probability distribution for the time of decay are given by the same type of exponential function, but multiplied by certain constant factors.
Problems

Key
✓  A computerized answer check is available online.
∫  A problem that requires calculus.
⋆  A difficult problem.

1  If a radioactive substance has a half-life of one year, does this mean that it will be completely decayed after two years? Explain.

2  Many individuals carry the recessive gene for albinism, but they are not albino unless they receive the gene from both their parents. In the U.S., an individual’s probability of receiving the gene from a given parent is about 0.014. What is the probability that a given child will be born albino?

3  Problem 3 has been deleted.

4  Use a calculator to check the approximation that

\[ a^b \approx 1 + b \ln a, \]

if \( b \ll 1 \), using some arbitrary numbers. Then see how good the approximation is for values of \( b \) that are not quite as small compared to one.

5  Make up an example of a numerical problem involving a rate of decay where \( \Delta t \ll t_{1/2} \), but the exact expression for the rate of decay on page 951 can still be evaluated on a calculator without getting something that rounds off to zero. Check that you get approximately the same result using both methods on pp. 951-952 to calculate the number of decays between \( t \) and \( t + \Delta t \). Keep plenty of significant figures in your results, in order to show the difference between them.

6  Devise a method for testing experimentally the hypothesis that a gambler’s chance of winning at craps is independent of her previous record of wins and losses. If you don’t invoke the mathematical definition of statistical independence, then you haven’t proposed a test. This has nothing to do with the details of the rules of craps, or with the fact that it’s a game played using dice.

7  Refer to the probability distribution for people’s heights in figure f on page 946.
   (a) Show that the graph is properly normalized.
   (b) Estimate the fraction of the population having heights between 140 and 150 cm.


Chapter 33 Rules of Randomness
8  (a) A nuclear physicist is studying a nuclear reaction caused in an accelerator experiment, with a beam of ions from the accelerator striking a thin metal foil and causing nuclear reactions when a nucleus from one of the beam ions happens to hit one of the nuclei in the target. After the experiment has been running for a few hours, a few billion radioactive atoms have been produced, embedded in the target. She does not know what nuclei are being produced, but she suspects they are an isotope of some heavy element such as Pb, Bi, Fr or U. Following one such experiment, she takes the target foil out of the accelerator, sticks it in front of a detector, measures the activity every 5 min, and makes a graph (figure). The isotopes she thinks may have been produced are:

<table>
<thead>
<tr>
<th>isotope</th>
<th>half-life (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{211}$Pb</td>
<td>36.1</td>
</tr>
<tr>
<td>$^{214}$Pb</td>
<td>26.8</td>
</tr>
<tr>
<td>$^{214}$Bi</td>
<td>19.7</td>
</tr>
<tr>
<td>$^{223}$Fr</td>
<td>21.8</td>
</tr>
<tr>
<td>$^{239}$U</td>
<td>23.5</td>
</tr>
</tbody>
</table>

Which one is it?

(b) Having decided that the original experimental conditions produced one specific isotope, she now tries using beams of ions traveling at several different speeds, which may cause different reactions. The following table gives the activity of the target 10, 20 and 30 minutes after the end of the experiment, for three different ion speeds.

<table>
<thead>
<tr>
<th>activity (millions of decays/s) after...</th>
<th>10 min</th>
<th>20 min</th>
<th>30 min</th>
</tr>
</thead>
<tbody>
<tr>
<td>first ion speed</td>
<td>1.933</td>
<td>0.832</td>
<td>0.382</td>
</tr>
<tr>
<td>second ion speed</td>
<td>1.200</td>
<td>0.545</td>
<td>0.248</td>
</tr>
<tr>
<td>third ion speed</td>
<td>7.211</td>
<td>1.296</td>
<td>0.248</td>
</tr>
</tbody>
</table>

Since such a large number of decays is being counted, assume that the data are only inaccurate due to rounding off when writing down the table. Which are consistent with the production of a single isotope, and which imply that more than one isotope was being created?
All helium on earth is from the decay of naturally occurring heavy radioactive elements such as uranium. Each alpha particle that is emitted ends up claiming two electrons, which makes it a helium atom. If the original $^{238}\text{U}$ atom is in solid rock (as opposed to the earth’s molten regions), the He atoms are unable to diffuse out of the rock. This problem involves dating a rock using the known decay properties of uranium 238. Suppose a geologist finds a sample of hardened lava, melts it in a furnace, and finds that it contains 1230 mg of uranium and 2.3 mg of helium. $^{238}\text{U}$ decays by alpha emission, with a half-life of $4.5 \times 10^9$ years. The subsequent chain of alpha and electron (beta) decays involves much shorter half-lives, and terminates in the stable nucleus $^{206}\text{Pb}$. (You may want to review alpha and beta decay.) Almost all natural uranium is $^{238}\text{U}$, and the chemical composition of this rock indicates that there were no decay chains involved other than that of $^{238}\text{U}$.

(a) How many alphas are emitted in decay chain of a single $^{238}\text{U}$ atom? [Hint: Use conservation of mass.]
(b) How many electrons are emitted per decay chain? [Hint: Use conservation of charge.]
(c) How long has it been since the lava originally hardened? ✓

Physicists thought for a long time that bismuth-209 was the heaviest stable isotope. (Very heavy elements decay by alpha emission because of the strong electrical repulsion of all their protons.) However, a 2003 paper by Marcillac et al. describes an experiment in which bismuth-209 lost its claim to fame — it actually undergoes alpha decay with a half-life of $1.9 \times 10^{19}$ years.

(a) After the alpha particle is emitted, what is the isotope left over?
(b) Compare the half-life to the age of the universe, which is about 14 billion years.
(c) A tablespoon of Pepto-Bismol contains about $4 \times 10^{20}$ bismuth-209 atoms. Once you’ve swallowed it, how much time will it take, on the average, before the first atomic decay? ✓

A blindfolded person fires a gun at a circular target of radius $b$, and is allowed to continue firing until a shot actually hits it. Any part of the target is equally likely to get hit. We measure the random distance $r$ from the center of the circle to where the bullet went in.

(a) Show that the probability distribution of $r$ must be of the form $D(r) = kr$, where $k$ is some constant. (Of course we have $D(r) = 0$ for $r > b$.)
(b) Determine $k$ by requiring $D$ to be properly normalized. ✓
(c) Find the average value of $r$. ✓
(d) Interpreting your result from part c, how does it compare with $b/2$? Does this make sense? Explain. ✓

We are given some atoms of a certain radioactive isotope,
with half-life $t_{1/2}$. We pick one atom at random, and observe it for one half-life, starting at time zero. If it decays during that one-half-life period, we record the time $t$ at which the decay occurred. If it doesn’t, we reset our clock to zero and keep trying until we get an atom that cooperates. The final result is a time $0 \leq t \leq t_{1/2}$, with a distribution that looks like the usual exponential decay curve, but with its tail chopped off.

(a) Find the distribution $D(t)$, with the proper normalization. √
(b) Find the average value of $t$. √
(c) Interpreting your result from part b, how does it compare with $t_{1/2}/2$? Does this make sense? Explain.

13 The speed, $v$, of an atom in an ideal gas has a probability distribution of the form $D(v) = bve^{-cv^2}$, where $0 \leq v < \infty$, $c$ relates to the temperature, and $b$ is determined by normalization.
(a) Sketch the distribution.
(b) Find $b$ in terms of $c$. √
(c) Find the average speed in terms of $c$, eliminating $b$. (Don’t try to do the indefinite integral, because it can’t be done in closed form. The relevant definite integral can be found in tables or done with computer software.) √

14 Neutrinos interact so weakly with normal matter that, of the neutrinos from the sun that enter the earth from the day side, only about $10^{-10}$ of them fail to reemerge on the night side. From this fact, estimate the thickness of matter, in units of light-years, that would be required in order to block half of them. This “half-distance” is analogous to a half-life for radioactive decay.
In recent decades, a huge hole in the ozone layer has spread out from Antarctica.

Chapter 34
Light as a Particle

The only thing that interferes with my learning is my education.

Albert Einstein

Radioactivity is random, but do the laws of physics exhibit randomness in other contexts besides radioactivity? Yes. Radioactive decay was just a good playpen to get us started with concepts of randomness, because all atoms of a given isotope are identical. By stocking the playpen with an unlimited supply of identical atom-toys, nature helped us to realize that their future behavior could be different regardless of their original identicality. We are now ready to leave the playpen, and see how randomness fits into the structure of physics at the most fundamental level.

The laws of physics describe light and matter, and the quantum revolution rewrote both descriptions. Radioactivity was a good example of matter’s behaving in a way that was inconsistent with classical physics, but if we want to get under the hood and understand how nonclassical things happen, it will be easier to focus on light rather than matter. A radioactive atom such as uranium-235 is after all an extremely complex system, consisting of 92 protons, 143 neutrons, and 92 electrons. Light, however, can be a simple sine wave.

However successful the classical wave theory of light had been — allowing the creation of radio and radar, for example — it still failed
to describe many important phenomena. An example that is currently of great interest is the way the ozone layer protects us from the dangerous short-wavelength ultraviolet part of the sun’s spectrum. In the classical description, light is a wave. When a wave passes into and back out of a medium, its frequency is unchanged, and although its wavelength is altered while it is in the medium, it returns to its original value when the wave reemerges. Luckily for us, this is not at all what ultraviolet light does when it passes through the ozone layer, or the layer would offer no protection at all!

34.1 Evidence for light as a particle

For a long time, physicists tried to explain away the problems with the classical theory of light as arising from an imperfect understanding of atoms and the interaction of light with individual atoms and molecules. The ozone paradox, for example, could have been attributed to the incorrect assumption that the ozone layer was a smooth, continuous substance, when in reality it was made of individual ozone molecules. It wasn’t until 1905 that Albert Einstein threw down the gauntlet, proposing that the problem had nothing to do with the details of light’s interaction with atoms and everything to do with the fundamental nature of light itself.

In those days the data were sketchy, the ideas vague, and the experiments difficult to interpret; it took a genius like Einstein to cut through the thicket of confusion and find a simple solution. Today, however, we can get right to the heart of the matter with a piece of ordinary consumer electronics, the digital camera. Instead of film, a digital camera has a computer chip with its surface divided up into a grid of light-sensitive squares, called “pixels.” Compared to a grain of the silver compound used to make regular photographic film, a digital camera pixel is activated by an amount of light energy orders of magnitude smaller. We can learn something new about light by using a digital camera to detect smaller and smaller amounts of light, as shown in figures a/1 through a/3. Figure 1 is fake, but 2 and 3 are real digital-camera images made by Prof. Lyman Page of Princeton University as a classroom demonstration. Figure 1 is
what we would see if we used the digital camera to take a picture of a fairly dim source of light. In figures 2 and 3, the intensity of the light was drastically reduced by inserting semitransparent absorbers like the tinted plastic used in sunglasses. Going from 1 to 2 to 3, more and more light energy is being thrown away by the absorbers.

The results are dramatically different from what we would expect based on the wave theory of light. If light was a wave and nothing but a wave, b, then the absorbers would simply cut down the wave’s amplitude across the whole wavefront. The digital camera’s entire chip would be illuminated uniformly, and weakening the wave with an absorber would just mean that every pixel would take a long time to soak up enough energy to register a signal.

But figures a/2 and a/3 show that some pixels take strong hits while others pick up no energy at all. Instead of the wave picture, the image that is naturally evoked by the data is something more like a hail of bullets from a machine gun, c. Each “bullet” of light apparently carries only a tiny amount of energy, which is why detecting them individually requires a sensitive digital camera rather than an eye or a piece of film.

Although Einstein was interpreting different observations, this is the conclusion he reached in his 1905 paper: that the pure wave theory of light is an oversimplification, and that the energy of a beam of light comes in finite chunks rather than being spread smoothly throughout a region of space.
We now think of these chunks as particles of light, and call them “photons,” although Einstein avoided the word “particle,” and the word “photon” was invented later. Regardless of words, the trouble was that waves and particles seemed like inconsistent categories. The reaction to Einstein’s paper could be kindly described as vigorously skeptical. Even twenty years later, Einstein wrote, “There are therefore now two theories of light, both indispensable, and — as one must admit today despite twenty years of tremendous effort on the part of theoretical physicists — without any logical connection.” In the remainder of this chapter we will learn how the seeming paradox was eventually resolved.

Discussion questions

A  Suppose someone rebuts the digital camera data in figure a, claiming that the random pattern of dots occurs not because of anything fundamental about the nature of light but simply because the camera’s pixels are not all exactly the same — some are just more sensitive than others. How could we test this interpretation?

B  Discuss how the correspondence principle applies to the observations and concepts discussed in this section.

34.2 How much light is one photon?

The photoelectric effect

We have seen evidence that light energy comes in little chunks, so the next question to be asked is naturally how much energy is in one chunk. The most straightforward experimental avenue for addressing this question is a phenomenon known as the photoelectric effect. The photoelectric effect occurs when a photon strikes the surface of a solid object and knocks out an electron. It occurs continually all around you. It is happening right now at the surface of your skin and on the paper or computer screen from which you are reading these words. It does not ordinarily lead to any observable electrical effect, however, because on the average, free electrons are wandering back in just as frequently as they are being ejected. (If an object did somehow lose a significant number of electrons, its growing net positive charge would begin attracting the electrons back more and more strongly.)

Figure e shows a practical method for detecting the photoelectric effect. Two very clean parallel metal plates (the electrodes of a capacitor) are sealed inside a vacuum tube, and only one plate is exposed to light. Because there is a good vacuum between the plates, any ejected electron that happens to be headed in the right direction will almost certainly reach the other capacitor plate without colliding with any air molecules.

The illuminated (bottom) plate is left with a net positive charge, and the unilluminated (top) plate acquires a negative charge from
the electrons deposited on it. There is thus an electric field between the plates, and it is because of this field that the electrons’ paths are curved, as shown in the diagram. However, since vacuum is a good insulator, any electrons that reach the top plate are prevented from responding to the electrical attraction by jumping back across the gap. Instead they are forced to make their way around the circuit, passing through an ammeter. The ammeter measures the strength of the photoelectric effect.

An unexpected dependence on frequency

The photoelectric effect was discovered serendipitously by Heinrich Hertz in 1887, as he was experimenting with radio waves. He was not particularly interested in the phenomenon, but he did notice that the effect was produced strongly by ultraviolet light and more weakly by lower frequencies. Light whose frequency was lower than a certain critical value did not eject any electrons at all.\(^1\) This dependence on frequency didn’t make any sense in terms of the classical wave theory of light. A light wave consists of electric and magnetic fields. The stronger the fields, i.e., the greater the wave’s amplitude, the greater the forces that would be exerted on electrons that found themselves bathed in the light. It should have been amplitude (brightness) that was relevant, not frequency. The dependence on frequency not only proves that the wave model of light needs modifying, but with the proper interpretation it allows us to determine how much energy is in one photon, and it also leads to a connection between the wave and particle models that we need in order to reconcile them.

To make any progress, we need to consider the physical process by which a photon would eject an electron from the metal electrode. A metal contains electrons that are free to move around. Ordinarily, in the interior of the metal, such an electron feels attractive forces from atoms in every direction around it. The forces cancel out. But if the electron happens to find itself at the surface of the metal, the attraction from the interior side is not balanced out by any attraction from outside. In popping out through the surface the electron therefore loses some amount of energy \(E_s\), which depends on the type of metal used.

Suppose a photon strikes an electron, annihilating itself and giving up all its energy to the electron.\(^2\) The electron will (1) lose kinetic energy through collisions with other electrons as it plows through the metal on its way to the surface; (2) lose an amount of kinetic energy equal to \(E_s\) as it emerges through the surface; and (3) lose more energy on its way across the gap between the plates, due to

\(^1\)In fact this was all prior to Thomson’s discovery of the electron, so Hertz would not have described the effect in terms of electrons — we are discussing everything with the benefit of hindsight.

\(^2\)We now know that this is what always happens in the photoelectric effect, although it had not yet been established in 1905 whether or not the photon was completely annihilated.
the electric field between the plates. Even if the electron happens to be right at the surface of the metal when it absorbs the photon, and even if the electric field between the plates has not yet built up very much, $E_s$ is the bare minimum amount of energy that the electron must receive from the photon if it is to contribute to a measurable current. The reason for using very clean electrodes is to minimize $E_s$ and make it have a definite value characteristic of the metal surface, not a mixture of values due to the various types of dirt and crud that are present in tiny amounts on all surfaces in everyday life.

We can now interpret the frequency dependence of the photoelectric effect in a simple way: apparently the amount of energy possessed by a photon is related to its frequency. A low-frequency red or infrared photon has an energy less than $E_s$, so a beam of them will not produce any current. A high-frequency blue or violet photon, on the other hand, packs enough of a punch to allow an electron to get out of the electrode. At frequencies higher than the minimum, the photoelectric current continues to increase with the frequency of the light because of effects (1) and (3).

### Numerical relationship between energy and frequency

Figure g shows an experiment that is used sometimes in college laboratory courses to probe the relationship between the energy and frequency of a photon. The idea is simply to illuminate one plate of the vacuum tube with light of a single wavelength and monitor the voltage difference between the two plates as they charge up. Since the resistance of a voltmeter is very high (much higher than the resistance of an ammeter), we can assume to a good approximation that electrons reaching the top plate are stuck there permanently, so the voltage will keep on increasing for as long as electrons are making it across the vacuum tube.

At a moment when the voltage difference has reached a value $\Delta V$, the minimum energy required by an electron to make it out of the bottom plate and across the gap to the other plate is $E_s + e\Delta V$. As $\Delta V$ increases, we eventually reach a point at which $E_s + e\Delta V$ equals the energy of one photon. No more electrons can cross the gap, and the reading on the voltmeter stops rising. The quantity $E_s + e\Delta V$ now tells us the energy of one photon. If we determine this energy for a variety of frequencies, $h$, we find the following simple relationship between the energy of a photon and the frequency of the light:

$$E = hf,$$

where $h$ is a constant with a numerical value of $6.63 \times 10^{-34}$ J·s. Note how the equation brings the wave and particle models of light under the same roof: the left side is the energy of one particle of light, while the right side is the frequency of the same light, interpreted as a wave. The constant $h$ is known as Planck’s constant, after theorist Max Planck.
self-check A
How would you extract h from the graph in figure h? What if you didn’t even know $E_s$ in advance, and could only graph $e\Delta V$ versus $f$?  

Answer, p. 1044

Since the energy of a photon is $hf$, a beam of light can only have energies of $hf$, $2hf$, $3hf$, etc. Its energy is quantized — there is no such thing as a fraction of a photon. Quantum physics gets its name from the fact that it quantizes things like energy, momentum, and angular momentum that had previously been thought to be smooth, continuous and infinitely divisible.

**Number of photons emitted by a lightbulb per second example 1**

> Roughly how many photons are emitted by a 100-W lightbulb in 1 second?

> People tend to remember wavelengths rather than frequencies for visible light. The bulb emits photons with a range of frequencies and wavelengths, but let’s take 600 nm as a typical wavelength for purposes of estimation. The energy of a single photon is

$$E_{\text{photon}} = hf = \frac{hc}{\lambda}$$

A power of 100 W means 100 joules per second, so the number of photons is

$$\frac{100 \text{ J}}{E_{\text{photon}}} = \frac{100 \text{ J}}{hc/\lambda} \approx 3 \times 10^{20}.$$  

This hugeness of this number is consistent with the correspondence principle. The experiments that established the classical theory of optics weren’t wrong. They were right, within their domain of applicability, in which the number of photons was so large as to be indistinguishable from a continuous beam.

**Measuring the wave example 2**

When surfers are out on the water waiting for their chance to catch a wave, they’re interested in both the height of the waves and when the waves are going to arrive. In other words, they observe both the amplitude and phase of the waves, and it doesn’t matter to them that the water is granular at the molecular level. The correspondence principle requires that we be able to do the same thing for electromagnetic waves, since the classical theory of electricity and magnetism was all stated and verified experimentally in terms of the fields $E$ and $B$, which are the amplitude of an electromagnetic wave. The phase is also necessary, since the laws of induction predict different results depending on whether an oscillating field is on its way up or on its way back down.
This is a more demanding application of the correspondence principle than the one in example 1, since amplitudes and phases constitute more detailed information than the over-all intensity of a beam of light. Eyeball measurements can't detect this type of information, since the eye is much bigger than a wavelength, but for example an AM radio receiver can do it with radio waves, since the wavelength for a station at 1000 kHz is about 300 meters, which is much larger than the antenna. The correspondence principle demands that we be able to explain this in terms of the photon theory, and this requires not just that we have a large number of photons emitted by the transmitter per second, as in example 1, but that even by the time they spread out and reach the receiving antenna, there should be many photons overlapping each other within a space of one cubic wavelength. Problem 13 on p. 984 verifies that the number is in fact extremely large.

\[\text{Momentum of a photon example 3}\]

\[\Delta\] According to the theory of relativity, the momentum of a beam of light is given by \(p = E/c\) (see homework problem 12 on page 810). Apply this to find the momentum of a single photon in terms of its frequency, and in terms of its wavelength.

\[\Delta\] Combining the equations \(p = E/c\) and \(E = hf\), we find

\[p = \frac{E}{c} = \frac{hf}{c}\.

To reexpress this in terms of wavelength, we use \(c = f\lambda\):

\[p = \frac{hf}{f\lambda} = \frac{h}{\lambda}.

The second form turns out to be simpler.

\textbf{Discussion questions}

\textbf{A} The photoelectric effect only ever ejects a very tiny percentage of the electrons available near the surface of an object. How well does this agree with the wave model of light, and how well with the particle model? Consider the two different distance scales involved: the wavelength of the light, and the size of an atom, which is on the order of \(10^{-10}\) or \(10^{-9}\) m.

\textbf{B} What is the significance of the fact that Planck's constant is numerically very small? How would our everyday experience of light be different if it was not so small?

\textbf{C} How would the experiments described above be affected if a single electron was likely to get hit by more than one photon?

\textbf{D} Draw some representative trajectories of electrons for \(\Delta V = 0\), \(\Delta V\) less than the maximum value, and \(\Delta V\) greater than the maximum value.
Does $E = hf$ imply that a photon changes its energy when it passes from one transparent material into another substance with a different index of refraction?

### 34.3 Wave-particle duality

How can light be both a particle and a wave? We are now ready to resolve this seeming contradiction. Often in science when something seems paradoxical, it’s because we (1) don’t define our terms carefully, or (2) don’t test our ideas against any specific real-world situation. Let’s define particles and waves as follows:

- Waves exhibit superposition, and specifically interference phenomena.
- Particles can only exist in whole numbers, not fractions.

As a real-world check on our philosophizing, there is one particular experiment that works perfectly. We set up a double-slit interference experiment that we know will produce a diffraction pattern if light is an honest-to-goodness wave, but we detect the light with a detector that is capable of sensing individual photons, e.g., a digital camera. To make it possible to pick out individual dots due to individual photons, we must use filters to cut down the intensity of the light to a very low level, just as in the photos by Prof. Page on p. 964. The whole thing is sealed inside a light-tight box. The results are shown in figure i. (In fact, the similar figures in on page 964 are simply cutouts from these figures.)

Neither the pure wave theory nor the pure particle theory can explain the results. If light was only a particle and not a wave, there would be no interference effect. The result of the experiment would
be like firing a hail of bullets through a double slit, j. Only two spots directly behind the slits would be hit.

If, on the other hand, light was only a wave and not a particle, we would get the same kind of diffraction pattern that would happen with a water wave, k. There would be no discrete dots in the photo, only a diffraction pattern that shaded smoothly between light and dark.

Applying the definitions to this experiment, light must be both a particle and a wave. It is a wave because it exhibits interference effects. At the same time, the fact that the photographs contain discrete dots is a direct demonstration that light refuses to be split into units of less than a single photon. There can only be whole numbers of photons: four photons in figure i/3, for example.

A wrong interpretation: photons interfering with each other

One possible interpretation of wave-particle duality that occurred to physicists early in the game was that perhaps the interference effects came from photons interacting with each other. By analogy, a water wave consists of moving water molecules, and interference of water waves results ultimately from all the mutual pushes and pulls of the molecules. This interpretation has been conclusively disproved by forming interference patterns with light so dim that no more than one photon is in flight at a time. In figure i/3, for example, the intensity of the light has been cut down so much by the absorbers that if it was in the open, the average separation between photons would be on the order of a kilometer! Although most light sources tend to emit photons in bunches, experiments have been done with light sources that really do emit single photons at wide time intervals, and the same type of interference pattern is observed, showing that a single photon can interfere with itself.

The concept of a photon’s path is undefined.

If a single photon can demonstrate double-slit interference, then which slit did it pass through? The unavoidable answer must be that it passes through both! This might not seem so strange if we think of the photon as a wave, but it is highly counterintuitive if we try to visualize it as a particle. The moral is that we should not think in terms of the path of a photon. Like the fully human and fully divine Jesus of Christian theology, a photon is supposed to be 100% wave and 100% particle. If a photon had a well defined path, then it would not demonstrate wave superposition and interference effects, contradicting its wave nature. (In sec. 35.4 we will discuss the Heisenberg uncertainty principle, which gives a numerical way of approaching this issue.)
The probability interpretation

The correct interpretation of wave-particle duality is suggested by the random nature of the experiment we’ve been discussing: even though every photon wave/particle is prepared and released in the same way, the location at which it is eventually detected by the digital camera is different every time. The idea of the probability interpretation of wave-particle duality is that the location of the photon-particle is random, but the probability that it is in a certain location is higher where the photon-wave’s amplitude is greater.

More specifically, the probability distribution of the particle must be proportional to the square of the wave’s amplitude,

\[(\text{probability distribution}) \propto (\text{amplitude})^2.\]

This follows from the correspondence principle and from the fact that a wave’s energy density is proportional to the square of its amplitude. If we run the double-slit experiment for a long enough time, the pattern of dots fills in and becomes very smooth as would have been expected in classical physics. To preserve the correspondence between classical and quantum physics, the amount of energy deposited in a given region of the picture over the long run must be proportional to the square of the wave’s amplitude. The amount of energy deposited in a certain area depends on the number of photons picked up, which is proportional to the probability of finding any given photon there.

\[\text{A microwave oven example 4}\]

The figure shows two-dimensional (top) and one-dimensional (bottom) representations of the standing wave inside a microwave oven. Gray represents zero field, and white and black signify the strongest fields, with white being a field that is in the opposite direction compared to black. Compare the probabilities of detecting a microwave photon at points A, B, and C.

A and C are both extremes of the wave, so the probabilities of detecting a photon at A and C are equal. It doesn’t matter that we have represented C as negative and A as positive, because it is the square of the amplitude that is relevant. The amplitude at B is about 1/2 as much as the others, so the probability of detecting a photon there is about 1/4 as much.

Discussion questions

A Referring back to the example of the carrot in the microwave oven, show that it would be nonsensical to have probability be proportional to the field itself, rather than the square of the field.

B Einstein did not try to reconcile the wave and particle theories of light, and did not say much about their apparent inconsistency. Einstein basically visualized a beam of light as a stream of bullets coming from a machine gun. In the photoelectric effect, a photon “bullet” would only hit one atom, just as a real bullet would only hit one person. Suppose
someone reading his 1905 paper wanted to interpret it by saying that Einstein’s so-called particles of light are simply short wave-trains that only occupy a small region of space. Comparing the wavelength of visible light (a few hundred nm) to the size of an atom (on the order of 0.1 nm), explain why this poses a difficulty for reconciling the particle and wave theories.

C. Can a white photon exist?

D. In double-slit diffraction of photons, would you get the same pattern of dots on the digital camera image if you covered one slit? Why should it matter whether you give the photon two choices or only one?

34.4 Nonlocality and entanglement

Nonlocality

People sometimes say that quantum mechanics is the set of rules for describing the world of the very small, but this is a false generalization, like saying that terriers are untrainable. How do we define our measure of how small is small? The only distance scales we’ve discussed have been wavelengths, and there is no upper limit on wavelengths. The wavelength of an FM radio photon is bigger than my terrier, who is very obedient to Newton’s laws. The only scale built in to the structure of quantum mechanics is Planck’s constant, and Planck’s constant has units of joules per hertz, not meters, so it can’t be converted into a distance. Quantum mechanics is, as far as we can tell, a valid tool for describing systems at scales from quarks to galaxies.

So quantum behavior can occur at any scale, even large ones. For an example that may be a little disturbing, consider the arrangement shown in figure n. A single photon comes in from the left and encounters a diagonal piece of glass. The glass reflects half the light and transmits half of it. The photon is a wave, and this is expected wave behavior. But the photon is also a particle, and we can’t have half a particle. Therefore either camera A will detect a whole photon and B will see none, or it will be the other way around. If we repeat the experiment many times times, we might come up with a list of results like this:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>yes</td>
<td>no</td>
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<td>no</td>
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<tr>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

An instant before the moment of detection, the photon is a wave pattern that just happens to consist of two widely separated pieces,
each carrying half the energy. The situation seems perfectly symmetric, but then a moment later we find that B has detected the photon and A hasn’t. If B’s detection of the photon is random, then how does the information get to A that it had better not detect it? This seems as though there is some sort of conspiracy being carried out over arbitrarily large distances and with no time delay. It’s as though the two parts of the wave are a pair of criminal suspects who would like to line up their stories but are being kept in separate jail cells so that they can’t communicate. If the part of the wave at B is going to be detected (at full strength, carrying 100% of the energy $E = hf$), how does the part at A get the message that it should fade away like the Cheshire cat? This coordination would have to occur over very large distances — real-world experiments of this type have been done over distances of a thousand kilometers, with the photons traveling either through outer space or through fiber-optic cables. Einstein derisively referred to this apparent coordination as “spooky action at a distance.”

Niels Bohr and two collaborators proposed in 1924 the seemingly reasonable solution that there can’t be any such coordination. Then the random detection of the photon by camera A and camera B would be independent. Independent probabilities multiply, so there would be a probability of $(1/2)(1/2) = 1/4$ that both cameras would see photons. This would violate conservation of energy, since the original energy $E = hf$ would have been detected twice, and the universe would have gained $1hf$ worth of total energy. But Bohr pointed out that there would also be the same probability that neither camera would detect a photon, in which case the change in the universe’s energy would be $-1hf$. On the average, energy would be conserved. According to Bohr’s theory, conservation of energy and momentum would not be absolute laws of physics but only rules that would be true on the average.

The experimentalists Geiger and Bothe immediately set out to test this prediction. They performed an experiment analogous to the one in figure n, but with x-rays rather than visible light. Their results, published in 1926, showed that if one detector saw the x-ray photon, the other did not, so that energy was always conserved at the microscopic level, not just on the average. We never observe an outcome in which both A and B detect a photon, or one in which neither detects it. That is, the occurrence of event A (camera A sees a photon) and event B (camera B sees one) are both random, but they are not independent.

**Entanglement**

At a 1927 conference in Brussels, Einstein protested that this was a problem, because the two detectors could in principle make their observations simultaneously, and it would then seem that some influence or communication was being transmitted between them faster...
than the speed of light. “It seems to me,” he complained, “that this
difficulty cannot be overcome unless the description of the process
in terms of the . . . wave is supplemented by some detailed specifi-
cation of the [trajectory of the particle]. . . . If one works only with
. . . waves, the interpretation . . ., I think, contradicts the postulate
of relativity.”

The experimental fact ends up being that the spooky action at a
distance exists, and it does go faster than light. In 2012, Guerreiro
et al.\textsuperscript{3} carried out a very direct and conceptually simple enactment
of exactly the experiment in figure n, with electronic timing precise
enough to prove that the detection events at A and B were sepa-
rated from each other by too great a distance to have been linked
by any influence traveling at \( \leq c \). These findings are summarized by
saying that quantum mechanics is nonlocal. A single wave-particle
can be spread out over an arbitrarily large region of space, but its
interactions that transfer energy and momentum are always corre-
lated over these distances in such a way that the conservation laws
are maintained.

What Einstein had not originally appreciated was that these correla-
tions do not violate relativity because they do not actually transport
any energy, or even any information, between A and B. For exam-
ple, if Alice is at detector A, and Bob is at B, a million kilometers
away, Alice can detect the photon and know immediately that Bob
did not detect it. She learns something seemingly instantaneously
about Bob — Bob is probably sad and disappointed right now. But
because Bob does not have any control over the result, he cannot
use this fact to send a message to Alice, so there is no transmission
of information. Alice and Bob’s states are said to be entangled.

Entanglement is like finding
that you only have your left glove,
so that you must have left your
right glove at home. There is a
gain in information, but no sudden
transmission of information from
the dog to you.

By analogy, suppose that you head off to work on a winter day in
New York. As you step out of the subway station into the cold air,
you reach into your pockets for your gloves, but you find that you
only have your left glove. Oh, you must have dropped your right

\textsuperscript{3}arxiv.org/abs/1204.1712. The paper is very readable.
glove on the floor while you were petting your adorable terrier on the way out the door. The presence of your left glove tells you that your right glove must be at home. But there has been no spooky action at a distance. You have simply recovered some information about a region of space that lies at some distance from you.

Einstein and Bohr had strong physical intuitions that led them to incorrect predictions about experiments, and these predictions were the fruits of inappropriate mental pictures of what was going on. If we take the principles of quantum mechanics seriously, then the correct picture is the following. Before the photon in figure n hits the glass diagonal, the state of things is the following.

\[
A \text{ photon is headed to the right.}
\]

Our photon is then partially reflected and partially transmitted. Now we have a superposition of two wave patterns:

\[
\begin{align*}
&c \text{ The photon has been reflected upward.} + c' \text{ The photon has continue to the right.}
\end{align*}
\]

where the amplitudes \(c\) and \(c'\) are equal in absolute value.\(^4\)

Let’s say that the cameras are at equal distances from the glass diagonal, so that their chances to detect the photon occur simultaneously.\(^5\) After detection, we have this:

\[
\begin{align*}
&c \text{ Camera A detected a photon and B didn’t.} + c' \text{ B detected a photon and A didn’t.}
\end{align*}
\]

Here we have made the nontrivial assumption that material objects like cameras obey the same wave-superposition rules as photons. This turns out to be true. Cameras are made out of things like electrons, and as we’ll see in chapter 35, things like electrons are also wave-particles, and they obey all the same wave-particle rules as photons. The states of the two cameras are now entangled.

You can see where this is going. Alice had been standing by camera A, watching anxiously, while Bob, a million kilometers away, was breathlessly observing camera B.

\[
\begin{align*}
&c \text{ Alice saw a photon and Bob didn’t. They consider this result to have been random.} + c' \text{ Bob saw a photon and Alice didn’t. They consider this result to have been random.}
\end{align*}
\]

\(^4\)Conservation of energy requires \(c^2 = 1/2\) and \(c'^2 = 1/2\), even in classical physics. We could have, for example, \(c = 1/\sqrt{2}\) and \(c' = -1/\sqrt{2}\). Such a possible difference in signs wouldn’t concern us in this example. It would only be relevant if there were some later opportunity for the two parts of the wave to recombine and superimpose on one another, producing interference effects.

\(^5\)According to special relativity, this simultaneity holds only in one frame of reference, say the lab frame. But if simultaneity does hold in one frame, then we can also say that in \(all\) frames, the distance between the two events is “spacelike,” i.e., they are too far apart to have been connected by any causal influence propagating at \(\leq c\).
It doesn’t *seem* to Alice and Bob as though their brains are in a superposition of two states. They *feel* as though they have only experienced the one possibility that actually happened, not a mixture of both at the same time. And yet this picture of the physics explains very nicely how the deterministic laws of physics produce a result that *seems* to them to have been random.

If Alice and Bob have been split into two ghostlike halves of themselves, then conceivably these half-selves could undergo interference, as in the double-slit experiment. But there are practical reasons why we cannot actually detect such interference effects. For one thing, Alice and Bob are macroscopic objects, with energies $E$ on the order of many joules. Because Planck’s constant is small, their wave frequencies $f = E/h$ are extremely high, and their wavelengths incredibly short (on the order of $10^{-34}$ m!). We have seen that diffraction becomes undetectable when wavelengths are too short. Furthermore, there is a phenomenon called decoherence, in which interactions with the environment tend to rapidly randomize the wave-phases of large objects. When phases are randomized, interference and diffraction effects become undetectable.

Historically, it seemed absurd to the originators of quantum mechanics to imagine a macroscopic object in a superposition of states. The most celebrated example is called the Schrödinger’s cat experiment. Luckily for the cat, there probably was no actual experiment — it was simply a “thought experiment” that the German theorist Schrödinger discussed with his colleagues. Schrödinger wrote:

One can even construct quite burlesque cases. A cat is shut up in a steel container, together with the following diabolical apparatus (which one must keep out of the direct clutches of the cat): In a Geiger tube [radiation detector] there is a tiny mass of radioactive substance, so little that in the course of an hour perhaps one atom of it disintegrates, but also with equal probability not even one; if it does happen, the counter [detector] responds and ... activates a hammer that shatters a little flask of prussic acid [filling the chamber with poison gas]. If one has left this entire system to itself for an hour, then one will say to himself that the cat is still living, if in that time no atom has disintegrated. The first atomic disintegration would have poisoned it.

It seemed ridiculous to Schrödinger that at the end of the hour, “The uncertainty originally restricted to the atomic domain has been transformed into a macroscopic uncertainty...,” and the cat would be in a superposed state.

In modern language, people like Einstein and Schrödinger didn’t feel comfortable with nonlocality, or with entanglement of subatomic
particles, and they felt even less comfortable with applying these concepts to macroscopic objects. Today, entanglement has been demonstrated using objects that clearly deserve to be called macroscopic. For example, in 2012, K.C. Lee et al. created a version of the experiment in figure n in which the cameras were replaced by small diamonds, about 1 mm in size. They were separated by 15 cm, which is a macroscopic distance. When a photon hit one of the diamonds, it produced a vibration in the crystal lattice. This vibration was localized to a relatively small region within the diamond, but this region was still large enough that one has to admit that it qualifies as macroscopic. Its atoms had a total weight of about 0.1 nanograms, which is a quantity big enough to weigh on a state-of-the-art balance, and the region was about 0.01 mm in size, which would make it visible with a magnifying glass.

The quantum states of the two diamonds became entangled: if one had detected the photon, the other hadn’t. This entangled state was maintained for only about 7 picoseconds before decoherence destroyed the phase relationship between one diamond and the other. But Lee was able to use additional photons to “read out” the quantum states in only 0.5 ps, before decoherence occurred, and verify that there were wave interference effects in which one diamond’s quantum-mechanical wave had a definite phase relationship with the other’s. Although these experiments are difficult, they suggest that there is no obstruction in principle to observing quantum-mechanical effects such as superposition in arbitrarily large objects.

34.5 Photons in three dimensions

Up until now I’ve been sneaky and avoided a full discussion of the three-dimensional aspects of the probability interpretation. The example of the carrot in the microwave oven, for example, reduced to a one-dimensional situation because we were considering three points along the same line and because we were only comparing ratios of probabilities.

A typical example of a probability distribution in section 33.3 was the distribution of heights of human beings. The thing that varied randomly, height, \( h \), had units of meters, and the probability distribution was a graph of a function \( D(h) \). The units of the probability distribution had to be \( \text{m}^{-1} \) (inverse meters) so that areas under the curve, interpreted as probabilities, would be unitless: \( \text{(area)} = \text{(height)} \times \text{(width)} = \text{m}^{-1} \cdot \text{m} \).

Now suppose we have a two-dimensional problem, e.g., the probability distribution for the place on the surface of a digital camera chip where a photon will be detected. The point where it is detected would be described with two variables, \( x \) and \( y \), each having units of meters. The probability distribution will be a function of both
variables, $D(x, y)$. A probability is now visualized as the volume under the surface described by the function $D(x, y)$, as shown in figure q. The units of $D$ must be m$^{-2}$ so that probabilities will be unitless: (probability) = (depth)(length)(width) = m$^{-2}$·m·m.

Generalizing finally to three dimensions, we find by analogy that the probability distribution will be a function of all three coordinates, $D(x, y, z)$, and will have units of m$^{-3}$. It is, unfortunately, impossible to visualize the graph unless you are a mutant with a natural feel for life in four dimensions. If the probability distribution is nearly constant within a certain volume of space $v$, the probability that the photon is in that volume is simply $vD$. If you know enough calculus, it should be clear that this can be generalized to $P = \int D \, dx \, dy \, dz$ if $D$ is not constant.
Summary

Selected vocabulary
- photon . . . . . . . . . . . a particle of light
- photoelectric effect . . . the ejection, by a photon, of an electron from the surface of an object
- wave-particle duality . . the idea that light is both a wave and a particle

Summary

Around the turn of the twentieth century, experiments began to show problems with the classical wave theory of light. In any experiment sensitive enough to detect very small amounts of light energy, it becomes clear that light energy cannot be divided into chunks smaller than a certain amount. Measurements involving the photoelectric effect demonstrate that this smallest unit of light energy equals $hf$, where $f$ is the frequency of the light and $h$ is a number known as Planck’s constant. We say that light energy is quantized in units of $hf$, and we interpret this quantization as evidence that light has particle properties as well as wave properties. Particles of light are called photons.

The only method of reconciling the wave and particle natures of light that has stood the test of experiment is the probability interpretation: the probability that the particle is at a given location is proportional to the square of the amplitude of the wave at that location.

One important consequence of wave-particle duality is that we must abandon the concept of the path the particle takes through space. To hold on to this concept, we would have to contradict the well established wave nature of light, since a wave can spread out in every direction simultaneously.
Problems

Key
✓ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

For some of these homework problems, you may find it convenient to refer to the diagram of the electromagnetic spectrum shown on p. 711.

1 Give a numerical comparison of the number of photons per second emitted by a hundred-watt FM radio transmitter and a hundred-watt lightbulb. (√)

2 Two different flashes of light each have the same energy. One consists of photons with a wavelength of 600 nm, the other 400 nm. If the number of photons in the 600-nm flash is $3.0 \times 10^{18}$, how many photons are in the 400-nm flash? (√)

3 When light is reflected from a mirror, perhaps only 80% of the energy comes back. The rest is converted to heat. One could try to explain this in two different ways: (1) 80% of the photons are reflected, or (2) all the photons are reflected, but each loses 20% of its energy. Based on your everyday knowledge about mirrors, how can you tell which interpretation is correct? [Based on a problem from PSSC Physics.]

4 Suppose we want to build an electronic light sensor using an apparatus like the one described in section 34.2 on p. 966. How would its ability to detect different parts of the spectrum depend on the type of metal used in the capacitor plates?

5 The photoelectric effect can occur not just for metal cathodes but for any substance, including living tissue. Ionization of DNA molecules can cause cancer or birth defects. If the energy required to ionize DNA is on the same order of magnitude as the energy required to produce the photoelectric effect in a metal, which of these types of electromagnetic waves might pose such a hazard? Explain.

60 Hz waves from power lines
100 MHz FM radio
1900 MHz radio waves from a cellular phone
2450 MHz microwaves from a microwave oven
visible light
ultraviolet light
x-rays

6 The beam of a 100-W overhead projector covers an area of $1 \text{ m} \times 1 \text{ m}$ when it hits the screen 3 m away. Estimate the number...
of photons that are in flight at any given time. (Since this is only an estimate, we can ignore the fact that the beam is not parallel.)

7 The two diffraction patterns were made by sending a flash of light through the same double slit. Give a numerical comparison of the amounts of energy in the two flashes.

8 Three of the four graphs are properly normalized to represent single photons. Which one isn’t? Explain.

9 Photon Fred has a greater energy than photon Ginger. For each of the following quantities, explain whether Fred’s value of that quantity is greater than Ginger’s, less than Ginger’s, or equal to Ginger’s. If there is no way to tell, explain why.

- frequency
- speed
- wavelength
- period
- electric field strength
- magnetic field strength

10 Give experimental evidence to disprove the following interpretation of wave-particle duality: A photon is really a particle, but it travels along a wavy path, like a zigzag with rounded corners. Cite a specific, real experiment.

11 In the photoelectric effect, electrons are observed with virtually no time delay (~ 10 ns), even when the light source is very weak. (A weak light source does however only produce a small number of ejected electrons.) The purpose of this problem is to show that the lack of a significant time delay contradicted the classical wave theory of light, so throughout this problem you should put yourself in the shoes of a classical physicist and pretend you don’t know about photons at all. At that time, it was thought that the electron might have a radius on the order of 10\(^{-15}\) m. (Recent experiments have shown that if the electron has any finite size at all, it is far smaller.)

(a) Estimate the power that would be soaked up by a single electron in a beam of light with an intensity of 1 mW/m\(^2\).

(b) The energy, \(E_s\), required for the electron to escape through the surface of the cathode is on the order of 10\(^{-19}\) J. Find how long it would take the electron to absorb this amount of energy, and explain why your result constitutes strong evidence that there is something wrong with the classical theory.
12 Many radio antennas are designed so that they preferentially emit or receive electromagnetic waves in a certain direction. However, no antenna is perfectly directional. The wave shown in the figure represents a single photon being emitted by an antenna at the center. The antenna is directional, so there is a stronger wave on the right than on the left. What is the probability that the photon will be observed on the right?

13 (a) A radio transmitter radiates power $P$ in all directions, so that the energy spreads out spherically. Find the energy density at a distance $r$. 
(b) Let the wavelength be $\lambda$. As described in example 2 on p. 969, find the number of photons in a volume $\lambda^3$ at this distance $r$. 
(c) For a 1000 kHz AM radio transmitting station, assuming reasonable values of $P$ and $r$, verify, as claimed in the example, that the result from part b is very large.
Chapter 35
Matter as a Wave

[In] a few minutes I shall be all melted... I have been wicked in my day, but I never thought a little girl like you would ever be able to melt me and end my wicked deeds. Look out — here I go!

The Wicked Witch of the West

As the Wicked Witch learned the hard way, losing molecular cohesion can be unpleasant. That’s why we should be very grateful that the concepts of quantum physics apply to matter as well as light. If matter obeyed the laws of classical physics, molecules wouldn’t exist.

Consider, for example, the simplest atom, hydrogen. Why does one hydrogen atom form a chemical bond with another hydrogen atom? Roughly speaking, we’d expect a neighboring pair of hydrogen atoms, A and B, to exert no force on each other at all, attractive or repulsive: there are two repulsive interactions (proton A with proton B and electron A with electron B) and two attractive interactions (proton A with electron B and electron A with proton B). Thinking a little more precisely, we should even expect that once the two atoms got close enough, the interaction would be repulsive. For instance, if you squeezed them so close together that the two protons were almost on top of each other, there would be a tremendously strong repulsion between them due to the $1/r^2$ nature of the
electrical force. The repulsion between the electrons would not be as strong, because each electron ranges over a large area, and is not likely to be found right on top of the other electron. This was only a rough argument based on averages, but the conclusion is validated by a more complete classical analysis: hydrogen molecules should not exist according to classical physics.

Quantum physics to the rescue! As we’ll see shortly, the whole problem is solved by applying the same quantum concepts to electrons that we have already used for photons.

### 35.1 Electrons as waves

We started our journey into quantum physics by studying the random behavior of *matter* in radioactive decay, and then asked how randomness could be linked to the basic laws of nature governing *light*. The probability interpretation of wave-particle duality was strange and hard to accept, but it provided such a link. It is now natural to ask whether the same explanation could be applied to matter. If the fundamental building block of light, the photon, is a particle as well as a wave, is it possible that the basic units of matter, such as electrons, are waves as well as particles?

A young French aristocrat studying physics, Louis de Broglie (pronounced “broylee”), made exactly this suggestion in his 1923 Ph.D. thesis. His idea had seemed so farfetched that there was serious doubt about whether to grant him the degree. Einstein was asked for his opinion, and with his strong support, de Broglie got his degree.

Only two years later, American physicists C.J. Davisson and L. Germer confirmed de Broglie’s idea by accident. They had been studying the scattering of electrons from the surface of a sample of nickel, made of many small crystals. (One can often see such a crystalline pattern on a brass doorknob that has been polished by repeated handling.) An accidental explosion occurred, and when they put their apparatus back together they observed something entirely different: the scattered electrons were now creating an interference pattern! This dramatic proof of the wave nature of matter came about because the nickel sample had been melted by the explosion and then resolidified as a single crystal. The nickel atoms, now nicely arranged in the regular rows and columns of a crystalline lattice, were acting as the lines of a diffraction grating. The new crystal was analogous to the type of ordinary diffraction grating in which the lines are etched on the surface of a mirror (a reflection grating) rather than the kind in which the light passes through the transparent gaps between the lines (a transmission grating).

Although we will concentrate on the wave-particle duality of electrons because it is important in chemistry and the physics of atoms,
all the other “particles” of matter you’ve learned about show wave properties as well. Figure a, for instance, shows a wave interference pattern of neutrons.

It might seem as though all our work was already done for us, and there would be nothing new to understand about electrons: they have the same kind of funny wave-particle duality as photons. That’s almost true, but not quite. There are some important ways in which electrons differ significantly from photons:

1. Electrons have mass, and photons don’t.
2. Photons always move at the speed of light, but electrons can move at any speed less than \( c \).
3. Photons don’t have electric charge, but electrons do, so electric forces can act on them. The most important example is the atom, in which the electrons are held by the electric force of the nucleus.
4. Electrons cannot be absorbed or emitted as photons are. Destroying an electron, or creating one out of nothing, would violate conservation of charge.

(In chapter 36 we will learn of one more fundamental way in which electrons differ from photons, for a total of five.)

Because electrons are different from photons, it is not immediately obvious which of the photon equations from chapter 34 can be applied to electrons as well. A particle property, the energy of one photon, is related to its wave properties via \( E = hf \) or, equivalently, \( E = hc/\lambda \). The momentum of a photon was given by \( p = hf/c \) or
\[ p = \frac{h}{\lambda} \] (example 3 on page 970). Ultimately it was a matter of experiment to determine which of these equations, if any, would work for electrons, but we can make a quick and dirty guess simply by noting that some of the equations involve \( c \), the speed of light, and some do not. Since \( c \) is irrelevant in the case of an electron, we might guess that the equations of general validity are those that do not have \( c \) in them:

\[
E = hf \\
p = \frac{h}{\lambda}
\]

This is essentially the reasoning that de Broglie went through, and experiments have confirmed these two equations for all the fundamental building blocks of light and matter, not just for photons and electrons.

The second equation, which I soft-pedaled in chapter 34, takes on a greater importance for electrons. This is first of all because the momentum of matter is more likely to be significant than the momentum of light under ordinary conditions, and also because force is the transfer of momentum, and electrons are affected by electrical forces.

\textbf{*The wavelength of an elephant*}  
\textbf{example 1}

\( \triangleright \) What is the wavelength of a trotting elephant?

\( \triangleright \) One may doubt whether the equation should be applied to an elephant, which is not just a single particle but a rather large collection of them. Throwing caution to the wind, however, we estimate the elephant’s mass at \( 10^3 \text{ kg} \) and its trotting speed at 10 m/s. Its wavelength is therefore roughly

\[
\lambda = \frac{h}{p} = \frac{h}{mv} \\
= \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(10^3 \text{ kg})(10 \text{ m/s})} \\
\approx 10^{-37} \text{ (kg} \cdot \text{m}^2/\text{s}^2) \cdot \text{s/kg} \cdot \text{m/s} \\
= 10^{-37} \text{ m.}
\]

The wavelength found in this example is so fantastically small that we can be sure we will never observe any measurable wave phenomena with elephants. The result is numerically small because Planck’s constant is so small, and as in some examples encountered previously, this smallness is in accord with the correspondence principle.
Although a smaller mass in the equation $\lambda = h/mv$ does result in a longer wavelength, the wavelength is still quite short even for individual electrons under typical conditions, as shown in the following example.

**The typical wavelength of an electron example 2**

Electrons in circuits and in atoms are typically moving through voltage differences on the order of 1 V, so that a typical energy is $(e)(1 \text{ V})$, which is on the order of $10^{-19}$ J. What is the wavelength of an electron with this amount of kinetic energy?

- This energy is nonrelativistic, since it is much less than $mc^2$. Momentum and energy are therefore related by the nonrelativistic equation $KE = p^2/2m$. Solving for $p$ and substituting in to the equation for the wavelength, we find

$$\lambda = \frac{h}{\sqrt{2m \cdot KE}} = 1.6 \times 10^{-9} \text{ m}.$$  

This is on the same order of magnitude as the size of an atom, which is no accident: as we will discuss in the next chapter in more detail, an electron in an atom can be interpreted as a standing wave. The smallness of the wavelength of a typical electron also helps to explain why the wave nature of electrons wasn’t discovered until a hundred years after the wave nature of light. To scale the usual wave-optics devices such as diffraction gratings down to the size needed to work with electrons at ordinary energies, we need to make them so small that their parts are comparable in size to individual atoms. This is essentially what Davisson and Germer did with their nickel crystal.

**self-check A**

These remarks about the inconvenient smallness of electron wavelengths apply only under the assumption that the electrons have typical energies. What kind of energy would an electron have to have in order to have a longer wavelength that might be more convenient to work with?

- Answer, p. 1044

**What kind of wave is it?**

If a sound wave is a vibration of matter, and a photon is a vibration of electric and magnetic fields, what kind of a wave is an electron made of? The disconcerting answer is that there is no experimental “observable,” i.e., directly measurable quantity, to correspond to the electron wave itself. In other words, there are devices like microphones that detect the oscillations of air pressure in a sound wave, and devices such as radio receivers that measure the oscillation of the electric and magnetic fields in a light wave, but nobody has ever found any way to measure an electron wave directly.

We can of course detect the energy (or momentum) possessed by an electron just as we could detect the energy of a photon using a digital
These two electron waves are not distinguishable by any measuring device. (In fact I’d imagine that an unmodified digital camera chip placed in a vacuum chamber would detect electrons just as handily as photons.) But this only allows us to determine where the wave carries high probability and where it carries low probability. Probability is proportional to the square of the wave’s amplitude, but measuring its square is not the same as measuring the wave itself. In particular, we get the same result by squaring either a positive number or its negative, so there is no way to determine the positive or negative sign of an electron wave.

Most physicists tend toward the school of philosophy known as operationalism, which says that a concept is only meaningful if we can define some set of operations for observing, measuring, or testing it. According to a strict operationalist, then, the electron wave itself is a meaningless concept. Nevertheless, it turns out to be one of those concepts like love or humor that is impossible to measure and yet very useful to have around. We therefore give it a symbol, \( \Psi \) (the capital Greek letter psi), and a special name, the electron wave-function (because it is a function of the coordinates \( x, y, \) and \( z \) that specify where you are in space). It would be impossible, for example, to calculate the shape of the electron wave in a hydrogen atom without having some symbol for the wave. But when the calculation produces a result that can be compared directly to experiment, the final algebraic result will turn out to involve only \( \Psi^2 \), which is what is observable, not \( \Psi \) itself.

Since \( \Psi \), unlike \( E \) and \( B \), is not directly measurable, we are free to make the probability equations have a simple form: instead of having the probability distribution equal to some funny constant multiplied by \( \Psi^2 \), we simply define \( \Psi \) so that the constant of proportionality is one:

\[
\text{(probability distribution)} = \Psi^2.
\]

Since the probability distribution has units of \( m^{-3} \), the units of \( \Psi \) must be \( m^{-3/2} \).

**Discussion question**

A Frequency is oscillations per second, whereas wavelength is meters per oscillation. How could the equations \( E = hf \) and \( p = h/\lambda \) be made to look more alike by using quantities that were more closely analogous? (This more symmetric treatment makes it easier to incorporate relativity into quantum mechanics, since relativity says that space and time are not entirely separate.)
35.2 \( \int \star \) Dispersive waves

A colleague of mine who teaches chemistry loves to tell the story about an exceptionally bright student who, when told of the equation \( p = \frac{h}{\lambda} \), protested, “But when I derived it, it had a factor of 2!” The issue that’s involved is a real one, albeit one that could be glossed over (and is, in most textbooks) without raising any alarms in the mind of the average student. The present optional section addresses this point; it is intended for the student who wishes to delve a little deeper.

Here’s how the now-legendary student was presumably reasoning. We start with the equation \( v = f\lambda \), which is valid for any sine wave, whether it’s quantum or classical. Let’s assume we already know \( E = hf \), and are trying to derive the relationship between wavelength and momentum:

\[
\lambda = \frac{v}{f} = \frac{vh}{E} = \frac{vh}{\frac{1}{2}mv^2} = \frac{2h}{mv} = \frac{2h}{p}
\]

The reasoning seems valid, but the result does contradict the accepted one, which is after all solidly based on experiment.

The mistaken assumption is that we can figure everything out in terms of pure sine waves. Mathematically, the only wave that has a perfectly well defined wavelength and frequency is a sine wave, and not just any sine wave but an infinitely long one, \( c \). The unphysical thing about such a wave is that it has no leading or trailing edge, so it can never be said to enter or leave any particular region of space. Our derivation made use of the velocity, \( v \), and if velocity is to be a meaningful concept, it must tell us how quickly stuff (mass, energy, momentum,...) is transported from one region of space to another. Since an infinitely long sine wave doesn’t remove any stuff from one region and take it to another, the “velocity of its stuff” is not a well defined concept.

Of course the individual wave peaks do travel through space, and one might think that it would make sense to associate their speed with the “speed of stuff,” but as we will see, the two velocities are in general unequal when a wave’s velocity depends on wavelength. Such a wave is called a dispersive wave, because a wave pulse consisting of a superposition of waves of different wavelengths will separate (disperse) into its separate wavelengths as the waves move through

\( c / \) Part of an infinite sine wave.
space at different speeds. Nearly all the waves we have encountered have been nondispersive. For instance, sound waves and light waves (in a vacuum) have speeds independent of wavelength. A water wave is one good example of a dispersive wave. Long-wavelength water waves travel faster, so a ship at sea that encounters a storm typically sees the long-wavelength parts of the wave first. When dealing with dispersive waves, we need symbols and words to distinguish the two speeds. The speed at which wave peaks move is called the phase velocity, \( v_p \), and the speed at which “stuff” moves is called the group velocity, \( v_g \).

An infinite sine wave can only tell us about the phase velocity, not the group velocity, which is really what we would be talking about when we referred to the speed of an electron. If an infinite sine wave is the simplest possible wave, what’s the next best thing? We might think the runner up in simplicity would be a wave train consisting of a chopped-off segment of a sine wave, d. However, this kind of wave has kinks in it at the end. A simple wave should be one that we can build by superposing a small number of infinite sine waves, but a kink can never be produced by superposing any number of infinitely long sine waves.

Actually the simplest wave that transports stuff from place to place is the pattern shown in figure e. Called a beat pattern, it is formed by superposing two sine waves whose wavelengths are similar but not quite the same. If you have ever heard the pulsating howling sound of musicians in the process of tuning their instruments to each other, you have heard a beat pattern. The beat pattern gets stronger and weaker as the two sine waves go in and out of phase with each other. The beat pattern has more “stuff” (energy, for example) in the areas where constructive interference occurs, and less in the regions of cancellation. As the whole pattern moves through space, stuff is transported from some regions and into other ones.

If the frequency of the two sine waves differs by 10%, for instance, then ten periods will be occur between times when they are in phase. Another way of saying it is that the sinusoidal “envelope” (the dashed lines in figure e) has a frequency equal to the difference in frequency between the two waves. For instance, if the waves had frequencies of 100 Hz and 110 Hz, the frequency of the envelope would be 10 Hz.

To apply similar reasoning to the wavelength, we must define a quantity \( z = 1/\lambda \) that relates to wavelength in the same way that frequency relates to period. In terms of this new variable, the \( z \) of the envelope equals the difference between the \( z \)'s of the two sine waves.

The group velocity is the speed at which the envelope moves through space. Let \( \Delta f \) and \( \Delta z \) be the differences between the frequencies and \( z \)'s of the two sine waves, which means that they equal the frequency
and \( z \) of the envelope. The group velocity is \( v_g = \frac{\Delta f}{\Delta z} \). If \( \Delta f \) and \( \Delta z \) are sufficiently small, we can approximate this expression as a derivative,

\[
v_g = \frac{df}{dz}.
\]

This expression is usually taken as the definition of the group velocity for wave patterns that consist of a superposition of sine waves having a narrow range of frequencies and wavelengths. In quantum mechanics, with \( f = \frac{E}{\hbar} \) and \( z = \frac{p}{\hbar} \), we have \( v_g = \frac{dE}{dp} \). In the case of a nonrelativistic electron the relationship between energy and momentum is \( E = \frac{p^2}{2m} \), so the group velocity is \( \frac{dE}{dp} = \frac{p}{m} = v \), exactly what it should be. It is only the phase velocity that differs by a factor of two from what we would have expected, but the phase velocity is not the physically important thing.

### 35.3 Bound states

Electrons are at their most interesting when they’re in atoms, that is, when they are bound within a small region of space. We can understand a great deal about atoms and molecules based on simple arguments about such bound states, without going into any of the realistic details of atom. The simplest model of a bound state is known as the particle in a box: like a ball on a pool table, the electron feels zero force while in the interior, but when it reaches an edge it encounters a wall that pushes back inward on it with a large force. In particle language, we would describe the electron as bouncing off of the wall, but this incorrectly assumes that the electron has a certain path through space. It is more correct to describe the electron as a wave that undergoes 100% reflection at the boundaries of the box.

Like a generation of physics students before me, I rolled my eyes when initially introduced to the unrealistic idea of putting a particle in a box. It seemed completely impractical, an artificial textbook invention. Today, however, it has become routine to study electrons in rectangular boxes in actual laboratory experiments. The “box” is actually just an empty cavity within a solid piece of silicon, amounting in volume to a few hundred atoms. The methods for creating these electron-in-a-box setups (known as “quantum dots”) were a by-product of the development of technologies for fabricating computer chips.

For simplicity let’s imagine a one-dimensional electron in a box, i.e., we assume that the electron is only free to move along a line. The resulting standing wave patterns, of which the first three are shown in figure f, are just like some of the patterns we encountered with sound waves in musical instruments. The wave patterns must be zero at the ends of the box, because we are assuming the walls...
The spectrum of the light from the star Sirius. Photograph by the author.

are impenetrable, and there should therefore be zero probability of finding the electron outside the box. Each wave pattern is labeled according to $n$, the number of peaks and valleys it has. In quantum physics, these wave patterns are referred to as “states” of the particle-in-the-box system.

The following seemingly innocuous observations about the particle in the box lead us directly to the solutions to some of the most vexing failures of classical physics:

The particle’s energy is quantized (can only have certain values). Each wavelength corresponds to a certain momentum, and a given momentum implies a definite kinetic energy, $E = p^2 / 2m$. (This is the second type of energy quantization we have encountered. The type we studied previously had to do with restricting the number of particles to a whole number, while assuming some specific wavelength and energy for each particle. This type of quantization refers to the energies that a single particle can have. Both photons and matter particles demonstrate both types of quantization under the appropriate circumstances.)

The particle has a minimum kinetic energy. Long wavelengths correspond to low momenta and low energies. There can be no state with an energy lower than that of the $n = 1$ state, called the ground state.

The smaller the space in which the particle is confined, the higher its kinetic energy must be. Again, this is because long wavelengths give lower energies.

Spectra of thin gases example 3

A fact that was inexplicable by classical physics was that thin gases absorb and emit light only at certain wavelengths. This was observed both in earthbound laboratories and in the spectra of stars. Figure g shows the example of the spectrum of the star Sirius, in which there are “gap teeth” at certain wavelengths. Taking this spectrum as an example, we can give a straightforward explanation using quantum physics.

Energy is released in the dense interior of the star, but the outer layers of the star are thin, so the atoms are far apart and electrons are confined within individual atoms. Although their standing-wave patterns are not as simple as those of the particle in the box, their energies are quantized.

When a photon is on its way out through the outer layers, it can be absorbed by an electron in an atom, but only if the amount of energy it carries happens to be the right amount to kick the electron from one of the allowed energy levels to one of the higher levels. The photon energies that are missing from the spectrum are the ones that equal the difference in energy between two electron energy levels. (The most prominent of the absorption lines in
Sirius’s spectrum are absorption lines of the hydrogen atom.)

The stability of atoms example 4
In many Star Trek episodes the Enterprise, in orbit around a planet, suddenly lost engine power and began spiraling down toward the planet’s surface. This was utter nonsense, of course, due to conservation of energy: the ship had no way of getting rid of energy, so it did not need the engines to replenish it.

Consider, however, the electron in an atom as it orbits the nucleus. The electron does have a way to release energy: it has an acceleration due to its continuously changing direction of motion, and according to classical physics, any accelerating charged particle emits electromagnetic waves. According to classical physics, atoms should collapse!

The solution lies in the observation that a bound state has a minimum energy. An electron in one of the higher-energy atomic states can and does emit photons and hop down step by step in energy. But once it is in the ground state, it cannot emit a photon because there is no lower-energy state for it to go to.

Chemical bonds in hydrogen molecules example 5
I began this chapter with a classical argument that chemical bonds, as in an H₂ molecule, should not exist. Quantum physics explains why this type of bonding does in fact occur. When the atoms are next to each other, the electrons are shared between them. The “box” is about twice as wide, and a larger box allows a smaller energy. Energy is required in order to separate the atoms. (A qualitatively different type of bonding is discussed in on page 1021.)

Discussion questions

A Neutrons attract each other via the strong nuclear force, so according to classical physics it should be possible to form nuclei out of clusters of two or more neutrons, with no protons at all. Experimental searches, however, have failed to turn up evidence of a stable two-neutron system (dineutron) or larger stable clusters. These systems are apparently not just unstable in the sense of being able to beta decay but unstable in the sense that they don’t hold together at all. Explain based on quantum physics why a dineutron might spontaneously fly apart.

B The following table shows the energy gap between the ground state and the first excited state for four nuclei, in units of picojoules. (The nuclei were chosen to be ones that have similar structures, e.g., they are all spherical in shape.)

<table>
<thead>
<tr>
<th>nucleus</th>
<th>energy gap (picojoules)</th>
</tr>
</thead>
<tbody>
<tr>
<td>^4He</td>
<td>3.234</td>
</tr>
<tr>
<td>^16O</td>
<td>0.968</td>
</tr>
<tr>
<td>^40Ca</td>
<td>0.536</td>
</tr>
<tr>
<td>^208Pb</td>
<td>0.418</td>
</tr>
</tbody>
</table>

Explain the trend in the data.
35.4 The uncertainty principle

Eliminating randomness through measurement?

A common reaction to quantum physics, among both early-twentieth-century physicists and modern students, is that we should be able to get rid of randomness through accurate measurement. If I say, for example, that it is meaningless to discuss the path of a photon or an electron, you might suggest that we simply measure the particle’s position and velocity many times in a row. This series of snapshots would amount to a description of its path.

A practical objection to this plan is that the process of measurement will have an effect on the thing we are trying to measure. This may not be of much concern, for example, when a traffic cop measures your car’s motion with a radar gun, because the energy and momentum of the radar pulses aren’t enough to change the car’s motion significantly. But on the subatomic scale it is a very real problem. Making a videotape of an electron orbiting a nucleus is not just difficult, it is theoretically impossible, even with the video camera hooked up to the best imaginable microscope. The video camera makes pictures of things using light that has bounced off them and come into the camera. If even a single photon of the right wavelength was to bounce off of the electron we were trying to study, the electron’s recoil would be enough to change its behavior significantly (see homework problem 4).

The Heisenberg uncertainty principle

This insight, that measurement changes the thing being measured, is the kind of idea that clove-cigarette-smoking intellectuals outside of the physical sciences like to claim they knew all along. If only, they say, the physicists had made more of a habit of reading literary journals, they could have saved a lot of work. The anthropologist Margaret Mead has recently been accused of inadvertently encouraging her teenaged Samoan informants to exaggerate the freedom of youthful sexual experimentation in their society. If this is considered a damning critique of her work, it is because she could have done better: other anthropologists claim to have been able to eliminate the observer-as-participant problem and collect untainted data.

The German physicist Werner Heisenberg, however, showed that in quantum physics, any measuring technique runs into a brick wall when we try to improve its accuracy beyond a certain point. Heisenberg showed that the limitation is a question of what there is to be known, even in principle, about the system itself, not of the inability of a particular measuring device to ferret out information that is knowable.

Suppose, for example, that we have constructed an electron in a box (quantum dot) setup in our laboratory, and we are able to adjust
the length $L$ of the box as desired. All the standing wave patterns pretty much fill the box, so our knowledge of the electron’s position is of limited accuracy. If we write $\Delta x$ for the range of uncertainty in our knowledge of its position, then $\Delta x$ is roughly the same as the length of the box:

$$\Delta x \approx L$$

If we wish to know its position more accurately, we can certainly squeeze it into a smaller space by reducing $L$, but this has an unintended side-effect. A standing wave is really a superposition of two traveling waves going in opposite directions. The equation $p = h/\lambda$ only gives the magnitude of the momentum vector, not its direction, so we should really interpret the wave as a 50/50 mixture of a right-going wave with momentum $p = h/\lambda$ and a left-going one with momentum $p = -h/\lambda$. The uncertainty in our knowledge of the electron’s momentum is $\Delta p = 2h/\lambda$, covering the range between these two values. Even if we make sure the electron is in the ground state, whose wavelength $\lambda = 2L$ is the longest possible, we have an uncertainty in momentum of $\Delta p = h/L$. In general, we find

$$\Delta p \gtrsim h/L,$$

with equality for the ground state and inequality for the higher-energy states. Thus if we reduce $L$ to improve our knowledge of the electron’s position, we do so at the cost of knowing less about its momentum. This trade-off is neatly summarized by multiplying the two equations to give

$$\Delta p \Delta x \gtrsim h.$$

Although we have derived this in the special case of a particle in a box, it is an example of a principle of more general validity:

**the Heisenberg uncertainty principle**

It is not possible, even in principle, to know the momentum and the position of a particle simultaneously and with perfect accuracy. The uncertainties in these two quantities are always such that

$$\Delta p \Delta x \gtrsim h.$$  

(This approximation can be made into a strict inequality, $\Delta p \Delta x > h/4\pi$, but only with more careful definitions, which we will not bother with.)

Note that although I encouraged you to think of this derivation in terms of a specific real-world system, the quantum dot, I never made any reference to specific measuring equipment. The argument is simply that we cannot *know* the particle’s position very accurately.

---

1See homework problems 6 and 7.
unless it *has* a very well defined position, it cannot have a very well
defined position unless its wave-pattern covers only a very small
amount of space, and its wave-pattern cannot be thus compressed
without giving it a short wavelength and a correspondingly uncer-
tain momentum. The uncertainty principle is therefore a restriction
on how much there is to know about a particle, not just on what we
can know about it with a certain technique.

**An estimate for electrons in atoms**

▷ A typical energy for an electron in an atom is on the order of
(1 volt)·e, which corresponds to a speed of about 1% of the speed
of light. If a typical atom has a size on the order of 0.1 nm, how
close are the electrons to the limit imposed by the uncertainty
principle?

▷ If we assume the electron moves in all directions with equal
probability, the uncertainty in its momentum is roughly twice its
typical momentum. This only an order-of-magnitude estimate, so
we take \( \Delta p \) to be the same as a typical momentum:

\[
\Delta p \Delta x = p_{\text{typical}} \Delta x \\
= (m_{\text{electron}}) (0.01c) (0.1 \times 10^{-9} \text{ m}) \\
= 3 \times 10^{-34} \text{ J} \cdot \text{s}
\]

This is on the same order of magnitude as Planck's constant, so
evidently the electron is “right up against the wall.” (The fact that
it is somewhat less than \( h \) is of no concern since this was only an
estimate, and we have not stated the uncertainty principle in its
most exact form.)

**self-check B**

If we were to apply the uncertainty principle to human-scale objects,
what would be the significance of the small numerical value of Planck's
constant?

▷ Answer, p. 1044

**self-check C**

Suppose rain is falling on your roof, and there is a tiny hole that lets
raindrops into your living room now and then. All these drops hit the
same spot on the floor, so they have the same value of \( x \). Not only
that, but if the rain is falling straight down, they all have zero horizontal
momentum. Thus it seems that the raindrops have \( \Delta p = 0, \Delta x = 0, \)
and \( \Delta p \Delta x = 0 \), violating the uncertainty principle. To look for the hole in
this argument, consider how it would be acted out on the microscopic
scale: an electron wave comes along and hits a narrow slit. What really
happens?

▷ Answer, p. 1045

**Discussion questions**

**A** Compare \( \Delta p \) and \( \Delta x \) for the two lowest energy levels of the one-
dimensional particle in a box, and discuss how this relates to the uncer-
tainty principle.

**B** On a graph of \( \Delta p \) versus \( \Delta x \), sketch the regions that are allowed and
forbidden by the Heisenberg uncertainty principle. Interpret the graph: Where does an atom lie on it? An elephant? Can either $p$ or $x$ be measured with perfect accuracy if we don’t care about the other?

### 35.5 Electrons in electric fields

So far the only electron wave patterns we’ve considered have been simple sine waves, but whenever an electron finds itself in an electric field, it must have a more complicated wave pattern. Let’s consider the example of an electron being accelerated by the electron gun at the back of a TV tube. The electron is moving from a region of low voltage into a region of higher voltage. Since its charge is negative, it loses PE by moving to a higher voltage, so its KE increases. As its potential energy goes down, its kinetic energy goes up by an equal amount, keeping the total energy constant. Increasing kinetic energy implies a growing momentum, and therefore a shortening wavelength, $j$.

The wavefunction as a whole does not have a single well-defined wavelength, but the wave changes so gradually that if you only look at a small part of it you can still pick out a wavelength and relate it to the momentum and energy. (The picture actually exaggerates by many orders of magnitude the rate at which the wavelength changes.)

But what if the electric field was stronger? The electric field in a TV is only $\sim 10^5 \text{ N/C}$, but the electric field within an atom is more like $10^{12} \text{ N/C}$. In figure k, the wavelength changes so rapidly that there is nothing that looks like a sine wave at all. We could get a general idea of the wavelength in a given region by measuring the distance between two peaks, but that would only be a rough approximation. Suppose we want to know the wavelength at point P. The trick is to construct a sine wave, like the one shown with the dashed line, which matches the curvature of the actual wavefunction as closely as possible near P. The sine wave that matches as well as possible is called the “osculating” curve, from a Latin word meaning “to kiss.”

The wavelength of the osculating curve is the wavelength that will relate correctly to conservation of energy.

#### Tunneling

We implicitly assumed that the particle-in-a-box wavefunction would cut off abruptly at the sides of the box, l/1, but that would be unphysical. A kink has infinite curvature, and curvature is related to energy, so it can’t be infinite. A physically realistic wavefunction must always “tail off” gradually, l/2. In classical physics, a particle can never enter a region in which its potential energy would be greater than the amount of energy it has available. But in quantum...
in physics the wavefunction will always have a tail that reaches into the classically forbidden region. If it was not for this effect, called tunneling, the fusion reactions that power the sun would not occur due to the high potential energy that nuclei need in order to get close together! Tunneling is discussed in more detail in the next section.

35.6 ∫ The Schrödinger equation

In section 35.5 we were able to apply conservation of energy to an electron’s wavefunction, but only by using the clumsy graphical technique of osculating sine waves as a measure of the wave’s curvature. You have learned a more convenient measure of curvature in calculus: the second derivative. To relate the two approaches, we take the second derivative of a sine wave:

\[
\frac{d^2}{dx^2} \sin \left( \frac{2\pi x}{\lambda} \right) = \frac{d}{dx} \left( \frac{2\pi}{\lambda} \cos \frac{2\pi x}{\lambda} \right) = -\left( \frac{2\pi}{\lambda} \right)^2 \sin \frac{2\pi x}{\lambda}
\]

Taking the second derivative gives us back the same function, but with a minus sign and a constant out in front that is related to the wavelength. We can thus relate the second derivative to the osculating wavelength:

\[ \frac{d^2 \Psi}{dx^2} = -\left( \frac{2\pi}{\lambda} \right)^2 \Psi \]

This could be solved for \( \lambda \) in terms of \( \Psi \), but it will turn out to be more convenient to leave it in this form.

Using conservation of energy, we have

\[
E = KE + PE
\]

\[
= \frac{p^2}{2m} + PE
\]

\[
= \left( \frac{h}{\lambda} \right)^2 \frac{1}{2m} + PE
\]