its pressure before the molecules were split?
Problem 9.

9 The figure shows a demonstration performed by Otto von Guericke for Emperor Ferdinand III, in which two teams of horses failed to pull apart a pair of hemispheres from which the air had been evacuated. (a) What object makes the force that holds the hemispheres together? (b) The hemispheres are in a museum in Berlin, and have a diameter of 65 cm. What is the amount of force holding them together? (Hint: The answer would be the same if they were cylinders or pie plates rather than hemispheres.)

10 Even when resting, the human body needs to do a certain amount of mechanical work to keep the heart beating. This quantity is difficult to define and measure with high precision, and also depends on the individual and her level of activity, but it's estimated to be about 1 to 5 watts. Suppose we consider the human body as nothing more than a pump. A person who is just lying in bed all day needs about 1000 kcal/day worth of food to stay alive. (a) Estimate the person's thermodynamic efficiency as a pump, and (b) compare with the maximum possible efficiency imposed by the laws of thermodynamics for a heat engine operating across the difference between a body temperature of 37°C and an ambient temperature of 22°C. (c) Interpret your answer.  

11 (a) Atmospheric pressure at sea level is 101 kPa. The deepest spot in the world’s oceans is a valley called the Challenger Deep, in the Marianas Trench, with a depth of 11.0 km. Find the pressure at this depth, in units of atmospheres. Although water under this amount of pressure does compress by a few percent, assume for the purposes of this problem that it is incompressible. (b) Suppose that an air bubble is formed at this depth and then rises to the surface. Estimate the change in its volume and radius.
Our sun is powered by nuclear fusion reactions, and as a first step in these reactions, one proton must approach another proton to within a short enough range \( r \). This is difficult to achieve, because the protons have electric charge \( +e \) and therefore repel one another electrically. (It’s a good thing that it’s so difficult, because otherwise the sun would use up all of its fuel very rapidly and explode.) To make fusion possible, the protons must be moving fast enough to come within the required range. Even at the high temperatures present in the core of our sun, almost none of the protons are moving fast enough.

(a) For comparison, the early universe, soon after the Big Bang, had extremely high temperatures. Estimate the temperature \( T \) that would have been required so that protons with average energies could fuse. State your result in terms of \( r \), the mass \( m \) of the proton, and universal constants.

(b) Show that the units of your answer to part a make sense.

(c) Evaluate your result from part a numerically, using \( r = 10^{-15} \) m and \( m = 1.7 \times 10^{-27} \) kg. As a check, you should find that this is much hotter than the sun’s core temperature of \( \sim 10^7 \) K.

13 Object A is a brick. Object B is half of a similar brick. If A is heated, we have \( \Delta S = Q/T \). Show that if this equation is valid for A, then it is also valid for B.

14 Typically the atmosphere gets colder with increasing altitude. However, sometimes there is an inversion layer, in which this trend is reversed, e.g., because a less dense mass of warm air moves into a certain area, and rises above the denser colder air that was already present. Suppose that this causes the pressure \( P \) as a function of height \( y \) to be given by a function of the form \( P = P_0 e^{-ky}(1 + by) \), where constant temperature would give \( b = 0 \) and an inversion layer would give \( b > 0 \). (a) Infer the units of the constants \( P_0 \), \( k \), and \( b \).

(b) Find the density of the air as a function of \( y \), of the constants, and of the acceleration of gravity \( g \).

(c) Check that the units of your answer to part b make sense.

15 You use a spoon at room temperature, 22°C, to mix your coffee, which is at 80°C. During this brief period of thermal contact, 1.3 J of heat is transferred from the coffee to the spoon. Find the total change in the entropy of the universe.
Vibrations and Waves
The vibrations of this electric bass string are converted to electrical vibrations, then to sound vibrations, and finally to vibrations of our eardrums.

Chapter 17
Vibrations

Dandelion. Cello. Read those two words, and your brain instantly conjures a stream of associations, the most prominent of which have to do with vibrations. Our mental category of “dandelion-ness” is strongly linked to the color of light waves that vibrate about half a million billion times a second: yellow. The velvety throb of a cello has as its most obvious characteristic a relatively low musical pitch — the note you are spontaneously imagining right now might be one whose sound vibrations repeat at a rate of a hundred times a second.

Evolution has designed our two most important senses around the assumption that not only will our environment be drenched with information-bearing vibrations, but in addition those vibrations will often be repetitive, so that we can judge colors and pitches by the rate of repetition. Granting that we do sometimes encounter non-repeating waves such as the consonant “sh,” which has no recognizable pitch, why was Nature’s assumption of repetition nevertheless so right in general?

Repeating phenomena occur throughout nature, from the orbits of electrons in atoms to the reappearance of Halley’s Comet every 75 years. Ancient cultures tended to attribute repetitious phenomena
like the seasons to the cyclical nature of time itself, but we now have a less mystical explanation. Suppose that instead of Halley’s Comet’s true, repeating elliptical orbit that closes seamlessly upon itself with each revolution, we decide to take a pen and draw a whimsical alternative path that never repeats. We will not be able to draw for very long without having the path cross itself. But at such a crossing point, the comet has returned to a place it visited once before, and since its potential energy is the same as it was on the last visit, conservation of energy proves that it must again have the same kinetic energy and therefore the same speed. Not only that, but the comet’s direction of motion cannot be randomly chosen, because angular momentum must be conserved as well. Although this falls short of being an ironclad proof that the comet’s orbit must repeat, it no longer seems surprising that it does.

Conservation laws, then, provide us with a good reason why repetitive motion is so prevalent in the universe. But it goes deeper than that. Up to this point in your study of physics, I have been indoctrinating you with a mechanistic vision of the universe as a giant piece of clockwork. Breaking the clockwork down into smaller and smaller bits, we end up at the atomic level, where the electrons circling the nucleus resemble — well, little clocks! From this point of view, particles of matter are the fundamental building blocks of everything, and vibrations and waves are just a couple of the tricks that groups of particles can do. But at the beginning of the 20th century, the tables were turned. A chain of discoveries initiated by Albert Einstein led to the realization that the so-called subatomic “particles” were in fact waves. In this new world-view, it is vibrations and waves that are fundamental, and the formation of matter is just one of the tricks that waves can do.

### 17.1 Period, frequency, and amplitude

Figure b shows our most basic example of a vibration. With no forces on it, the spring assumes its equilibrium length, b/1. It can be stretched, 2, or compressed, 3. We attach the spring to a wall on the left and to a mass on the right. If we now hit the mass with a hammer, 4, it oscillates as shown in the series of snapshots, 4-13. If we assume that the mass slides back and forth without friction and that the motion is one-dimensional, then conservation of energy proves that the motion must be repetitive. When the block comes back to its initial position again, 7, its potential energy is the same again, so it must have the same kinetic energy again. The motion is in the opposite direction, however. Finally, at 10, it returns to its initial position with the same kinetic energy and the same direction of motion. The motion has gone through one complete cycle, and will now repeat forever in the absence of friction.

The usual physics terminology for motion that repeats itself over
and over is periodic motion, and the time required for one repetition is called the period, \( T \). (The symbol \( P \) is not used because of the possible confusion with momentum.) One complete repetition of the motion is called a cycle.

We are used to referring to short-period sound vibrations as “high” in pitch, and it sounds odd to have to say that high pitches have low periods. It is therefore more common to discuss the rapidity of a vibration in terms of the number of vibrations per second, a quantity called the frequency, \( f \). Since the period is the number of seconds per cycle and the frequency is the number of cycles per second, they are reciprocals of each other,

\[
f = 1/T.
\]

\[\text{A carnival game example 1}\]

In the carnival game shown in figure d, the rube is supposed to push the bowling ball on the track just hard enough so that it goes over the hump and into the valley, but does not come back out again. If the only types of energy involved are kinetic and potential, this is impossible. Suppose you expect the ball to come back to a point such as the one shown with the dashed outline, then stop and turn around. It would already have passed through this point once before, going to the left on its way into the valley. It was moving then, so conservation of energy tells us that it cannot be at rest when it comes back to the same point. The motion that the customer hopes for is physically impossible. There is a physically possible periodic motion in which the ball rolls back and forth, staying confined within the valley, but there is no way to get the ball into that motion beginning from the place where we start. There is a way to beat the game, though. If you put enough spin on the ball, you can create enough kinetic friction so that a significant amount of heat is generated. Conservation of energy then allows the ball to be at rest when it comes back to a point like the outlined one, because kinetic energy has been converted into heat.

\[\text{Period and frequency of a fly’s wing-beats example 2}\]

A Victorian parlor trick was to listen to the pitch of a fly’s buzz, reproduce the musical note on the piano, and announce how many times the fly’s wings had flapped in one second. If the fly’s wings flap, say, 200 times in one second, then the frequency of their motion is \( f = 200/1 \text{ s} = 200 \text{ s}^{-1} \). The period is one 200th of a second, \( T = 1/f = (1/200) \text{ s} = 0.005 \text{ s} \).
Units of inverse second, $s^{-1}$, are awkward in speech, so an abbreviation has been created. One Hertz, named in honor of a pioneer of radio technology, is one cycle per second. In abbreviated form, $1 \text{ Hz} = 1 \text{ s}^{-1}$. This is the familiar unit used for the frequencies on the radio dial.

**Frequency of a radio station**

Example 3

KKJZ’s frequency is 88.1 MHz. What does this mean, and what period does this correspond to?

The metric prefix M- is mega-, i.e., millions. The radio waves emitted by KKJZ’s transmitting antenna vibrate 88.1 million times per second. This corresponds to a period of

$$T = \frac{1}{f} = 1.14 \times 10^{-8} \text{ s}.$$ 

This example shows a second reason why we normally speak in terms of frequency rather than period: it would be painful to have to refer to such small time intervals routinely. I could abbreviate by telling people that KKJZ’s period was 11.4 nanoseconds, but most people are more familiar with the big metric prefixes than with the small ones.

Units of frequency are also commonly used to specify the speeds of computers. The idea is that all the little circuits on a computer chip are synchronized by the very fast ticks of an electronic clock, so that the circuits can all cooperate on a task without getting ahead or behind. Adding two numbers might require, say, 30 clock cycles. Microcomputers these days operate at clock frequencies of about a gigahertz.

We have discussed how to measure how fast something vibrates, but not how big the vibrations are. The general term for this is amplitude, $A$. The definition of amplitude depends on the system being discussed, and two people discussing the same system may not even use the same definition. In the example of the block on the end of the spring, e/1, the amplitude will be measured in distance units such as cm. One could work in terms of the distance traveled by the block from the extreme left to the extreme right, but it would be somewhat more common in physics to use the distance from the center to one extreme. The former is usually referred to as the peak-to-peak amplitude, since the extremes of the motion look like mountain peaks or upside-down mountain peaks on a graph of position versus time.

In other situations we would not even use the same units for amplitude. The amplitude of a child on a swing, or a pendulum, e/2, would most conveniently be measured as an angle, not a distance, since her feet will move a greater distance than her head. The electrical vibrations in a radio receiver would be measured in electrical units such as volts or amperes.
17.2 Simple harmonic motion

Why are sine-wave vibrations so common?

If we actually construct the mass-on-a-spring system discussed in the previous section and measure its motion accurately, we will find that its $x-t$ graph is nearly a perfect sine-wave shape, as shown in figure f/1. (We call it a “sine wave” or “sinusoidal” even if it is a cosine, or a sine or cosine shifted by some arbitrary horizontal amount.) It may not be surprising that it is a wiggle of this general sort, but why is it a specific mathematically perfect shape? Why is it not a sawtooth shape like 2 or some other shape like 3? The mystery deepens as we find that a vast number of apparently unrelated vibrating systems show the same mathematical feature. A tuning fork, a sapling pulled to one side and released, a car bouncing on its shock absorbers, all these systems will exhibit sine-wave motion under one condition: the amplitude of the motion must be small.

It is not hard to see intuitively why extremes of amplitude would act differently. For example, a car that is bouncing lightly on its shock absorbers may behave smoothly, but if we try to double the amplitude of the vibrations the bottom of the car may begin hitting the ground, f/4. (Although we are assuming for simplicity in this chapter that energy is never dissipated, this is clearly not a very realistic assumption in this example. Each time the car hits the ground it will convert quite a bit of its potential and kinetic energy into heat and sound, so the vibrations would actually die out quite quickly, rather than repeating for many cycles as shown in the figure.)

The key to understanding how an object vibrates is to know how the force on the object depends on the object’s position. If an object is vibrating to the right and left, then it must have a leftward force on it when it is on the right side, and a rightward force when it is on the left side. In one dimension, we can represent the direction of the force using a positive or negative sign, and since the force changes from positive to negative there must be a point in the middle where the force is zero. This is the equilibrium point, where the object would stay at rest if it was released at rest. For convenience of notation throughout this chapter, we will define the origin of our coordinate system so that $x$ equals zero at equilibrium.

The simplest example is the mass on a spring, for which the force on the mass is given by Hooke’s law,

$$F = -kx.$$

We can visualize the behavior of this force using a graph of $F$ versus $x$, as shown in figure g. The graph is a line, and the spring constant, $k$, is equal to minus its slope. A stiffer spring has a larger value of $k$ and a steeper slope. Hooke’s law is only an approximation, but
it works very well for most springs in real life, as long as the spring isn’t compressed or stretched so much that it is permanently bent or damaged.

The following important theorem, whose proof is given in optional section 17.3, relates the motion graph to the force graph.

**Theorem:** A linear force graph makes a sinusoidal motion graph.

If the total force on a vibrating object depends only on the object’s position, and is related to the objects displacement from equilibrium by an equation of the form \( F = -kx \), then the object’s motion displays a sinusoidal graph with period

\[
T = 2\pi \sqrt{\frac{m}{k}}.
\]

Even if you do not read the proof, it is not too hard to understand why the equation for the period makes sense. A greater mass causes a greater period, since the force will not be able to whip a massive object back and forth very rapidly. A larger value of \( k \) causes a shorter period, because a stronger force can whip the object back and forth more rapidly.

This may seem like only an obscure theorem about the mass-on-a-spring system, but figure h shows it to be far more general than that. Figure h/1 depicts a force curve that is not a straight line. A system with this \( F - x \) curve would have large-amplitude vibrations that were complex and not sinusoidal. But the same system would exhibit sinusoidal small-amplitude vibrations. This is because any curve looks linear from very close up. If we magnify the \( F - x \) graph as shown in figure h/2, it becomes very difficult to tell that the graph is not a straight line. If the vibrations were confined to the region shown in h/2, they would be very nearly sinusoidal. This is the reason why sinusoidal vibrations are a universal feature of all vibrating systems, if we restrict ourselves to small amplitudes. The theorem is therefore of great general significance. It applies throughout the universe, to objects ranging from vibrating stars to vibrating nuclei. A sinusoidal vibration is known as simple harmonic motion.

**Period is approximately independent of amplitude, if the amplitude is small.**

Until now we have not even mentioned the most counterintuitive aspect of the equation \( T = 2\pi \sqrt{m/k} \): it does not depend on amplitude at all. Intuitively, most people would expect the mass-on-a-spring system to take longer to complete a cycle if the amplitude was larger. (We are comparing amplitudes that are different from each other, but both small enough that the theorem applies.) In fact the larger-amplitude vibrations take the same amount of time as the small-amplitude ones. This is because at large amplitudes, the force is greater, and therefore accelerates the object to higher
speeds.

Legend has it that this fact was first noticed by Galileo during what was apparently a less than enthralling church service. A gust of wind would now and then start one of the chandeliers in the cathedral swaying back and forth, and he noticed that regardless of the amplitude of the vibrations, the period of oscillation seemed to be the same. Up until that time, he had been carrying out his physics experiments with such crude time-measuring techniques as feeling his own pulse or singing a tune to keep a musical beat. But after going home and testing a pendulum, he convinced himself that he had found a superior method of measuring time. Even without a fancy system of pulleys to keep the pendulum’s vibrations from dying down, he could get very accurate time measurements, because the gradual decrease in amplitude due to friction would have no effect on the pendulum’s period. (Galileo never produced a modern-style pendulum clock with pulleys, a minute hand, and a second hand, but within a generation the device had taken on the form that persisted for hundreds of years after.)

The pendulum example 4
▷ Compare the periods of pendula having bobs with different masses.
▷ From the equation \( T = 2\pi\sqrt{m/k} \), we might expect that a larger mass would lead to a longer period. However, increasing the mass also increases the forces that act on the pendulum: gravity and the tension in the string. This increases \( k \) as well as \( m \), so the period of a pendulum is independent of \( m \).

Discussion questions

A Suppose that a pendulum has a rigid arm mounted on a bearing, rather than a string tied at its top with a knot. The bob can then oscillate with center-to-side amplitudes greater than 90°. For the maximum amplitude of 180°, what can you say about the period?

B In the language of calculus, Newton’s second law for a simple harmonic oscillator can be written in the form \( \frac{d^2 x}{dt^2} = -(\ldots) x \), where “\( \ldots \)” refers to a constant, and the minus sign says that if we pull the object away from equilibrium, a restoring force tries to bring it back to equilibrium, which is the opposite direction. This is why we get motion that looks like a sine or cosine function: these are functions that, when differentiated twice, give back the original function but with an opposite sign. Now consider the example described in discussion question A, where a pendulum is upright or nearly upright. How does the analysis play out differently?
17.3 * Proofs

In this section we prove (1) that a linear $F - x$ graph gives sinusoidal motion, (2) that the period of the motion is $2\pi \sqrt{m/k}$, and (3) that the period is independent of the amplitude. You may omit this section without losing the continuity of the chapter.

The basic idea of the proof can be understood by imagining that you are watching a child on a merry-go-round from far away. Because you are in the same horizontal plane as her motion, she appears to be moving from side to side along a line. Circular motion viewed edge-on doesn’t just look like any kind of back-and-forth motion, it looks like motion with a sinusoidal $x - t$ graph, because the sine and cosine functions can be defined as the $x$ and $y$ coordinates of a point at angle $\theta$ on the unit circle. The idea of the proof, then, is to show that an object acted on by a force that varies as $F = -kx$ has motion that is identical to circular motion projected down to one dimension. The $v^2/r$ expression will also fall out at the end.

The moons of Jupiter

Before moving on to the proof, we illustrate the concept using the moons of Jupiter. Their discovery by Galileo was an epochal event in astronomy, because it proved that not everything in the universe had to revolve around the earth as had been believed. Galileo’s telescope was of poor quality by modern standards, but figure j shows a simulation of how Jupiter and its moons might appear at intervals of three hours through a large present-day instrument. Because we see the moons’ circular orbits edge-on, they appear to perform sinusoidal vibrations. Over this time period, the innermost moon, Io, completes half a cycle.
For an object performing uniform circular motion, we have

$$|a| = \frac{v^2}{r}.$$ 

The $x$ component of the acceleration is therefore

$$a_x = \frac{v^2}{r} \cos \theta,$$

where $\theta$ is the angle measured counterclockwise from the $x$ axis. Applying Newton’s second law,

$$\frac{F_x}{m} = -\frac{v^2}{r} \cos \theta,$$

so

$$F_x = -m \frac{v^2}{r} \cos \theta.$$ 

Since our goal is an equation involving the period, it is natural to eliminate the variable $v = \text{circumference}/T = 2\pi r/T$, giving

$$F_x = -\frac{4\pi^2 mr}{T^2} \cos \theta.$$ 

The quantity $r \cos \theta$ is the same as $x$, so we have

$$F_x = -\frac{4\pi^2 m}{T^2} x.$$ 

Since everything is constant in this equation except for $x$, we have proved that motion with force proportional to $x$ is the same as circular motion projected onto a line, and therefore that a force proportional to $x$ gives sinusoidal motion. Finally, we identify the constant factor of $4\pi^2 m/T^2$ with $k$, and solving for $T$ gives the desired equation for the period,

$$T = 2\pi \sqrt{\frac{m}{k}}.$$ 

Since this equation is independent of $r$, $T$ is independent of the amplitude, subject to the initial assumption of perfect $F = -kx$ behavior, which in reality will only hold approximately for small $x$. 

Section 17.3  ⋆ Proofs 465
Summary

Selected vocabulary
- periodic motion: motion that repeats itself over and over
- period: the time required for one cycle of a periodic motion
- frequency: the number of cycles per second, the inverse of the period
- amplitude: the amount of vibration, often measured from the center to one side; may have different units depending on the nature of the vibration
- simple harmonic motion: motion whose $x - t$ graph is a sine wave

Notation
- $T$: period
- $f$: frequency
- $A$: amplitude
- $k$: the slope of the graph of $F$ versus $x$, where $F$ is the total force acting on an object and $x$ is the object’s position; for a spring, this is known as the spring constant.

Other terminology and notation
- $\nu$: The Greek letter $\nu$, nu, is used in many books for frequency.
- $\omega$: The Greek letter $\omega$, omega, is often used as an abbreviation for $2\pi f$.

Summary

Periodic motion is common in the world around us because of conservation laws. An important example is one-dimensional motion in which the only two forms of energy involved are potential and kinetic; in such a situation, conservation of energy requires that an object repeat its motion, because otherwise when it came back to the same point, it would have to have a different kinetic energy and therefore a different total energy.

Not only are periodic vibrations very common, but small-amplitude vibrations are always sinusoidal as well. That is, the $x - t$ graph is a sine wave. This is because the graph of force versus position will always look like a straight line on a sufficiently small scale. This type of vibration is called simple harmonic motion. In simple harmonic motion, the period is independent of the amplitude, and is given by

$$T = 2\pi\sqrt{m/k}.$$
Problems

Key
√  A computerized answer check is available online.
∫  A problem that requires calculus.
⋆  A difficult problem.

1  Find an equation for the frequency of simple harmonic motion in terms of $k$ and $m$.

2  Many single-celled organisms propel themselves through water with long tails, which they wiggle back and forth. (The most obvious example is the sperm cell.) The frequency of the tail’s vibration is typically about 10-15 Hz. To what range of periods does this range of frequencies correspond?

3  (a) Pendulum 2 has a string twice as long as pendulum 1. If we define $x$ as the distance traveled by the bob along a circle away from the bottom, how does the $k$ of pendulum 2 compare with the $k$ of pendulum 1? Give a numerical ratio. [Hint: the total force on the bob is the same if the angles away from the bottom are the same, but equal angles do not correspond to equal values of $x$.]

(b) Based on your answer from part (a), how does the period of pendulum 2 compare with the period of pendulum 1? Give a numerical ratio.

4  A pneumatic spring consists of a piston riding on top of the air in a cylinder. The upward force of the air on the piston is given by $F_{air} = ax^{-1.4}$, where $a$ is a constant with funny units of $N \cdot m^{1.4}$. For simplicity, assume the air only supports the weight, $F_W$, of the piston itself, although in practice this device is used to support some other object. The equilibrium position, $x_0$, is where $F_W$ equals $-F_{air}$. (Note that in the main text I have assumed the equilibrium position to be at $x = 0$, but that is not the natural choice here.) Assume friction is negligible, and consider a case where the amplitude of the vibrations is very small. Let $a = 1.0 \, N \cdot m^{1.4}$, $x_0 = 1.00 \, m$, and $F_W = -1.00 \, N$. The piston is released from $x = 1.01 \, m$. Draw a neat, accurate graph of the total force, $F$, as a function of $x$, on graph paper, covering the range from $x = 0.98 \, m$ to $1.02 \, m$. Over this small range, you will find that the force is very nearly proportional to $x - x_0$. Approximate the curve with a straight line, find its slope, and derive the approximate period of oscillation.

5  Consider the same pneumatic piston described in problem 4, but now imagine that the oscillations are not small. Sketch a graph of the total force on the piston as it would appear over this wider range of motion. For a wider range of motion, explain why the vibration of the piston about equilibrium is not simple harmonic motion, and sketch a graph of $x$ vs $t$, showing roughly how the curve is different from a sine wave. [Hint: Acceleration corresponds to the
Problem 7. curvature of the $x - t$ graph, so if the force is greater, the graph should curve around more quickly.]

6 Archimedes’ principle states that an object partly or wholly immersed in fluid experiences a buoyant force equal to the weight of the fluid it displaces. For instance, if a boat is floating in water, the upward pressure of the water (vector sum of all the forces of the water pressing inward and upward on every square inch of its hull) must be equal to the weight of the water displaced, because if the boat was instantly removed and the hole in the water filled back in, the force of the surrounding water would be just the right amount to hold up this new “chunk” of water. (a) Show that a cube of mass $m$ with edges of length $b$ floating upright (not tilted) in a fluid of density $\rho$ will have a draft (depth to which it sinks below the waterline) $h$ given at equilibrium by $h_0 = \frac{m}{b^2\rho}$. (b) Find the total force on the cube when its draft is $h$, and verify that plugging in $h - h_0$ gives a total force of zero. (c) Find the cube’s period of oscillation as it bobs up and down in the water, and show that this can be expressed in terms of $m$ and $g$ only.

7 The figure shows a see-saw with two springs at Codornices Park in Berkeley, California. Each spring has spring constant $k$, and a kid of mass $m$ sits on each seat. (a) Find the period of vibration in terms of the variables $k$, $m$, $a$, and $b$. (b) Discuss the special case where $a = b$, rather than $a > b$ as in the real see-saw. (c) Show that your answer to part a also makes sense in the case of $b = 0$. *

8 Show that the equation $T = 2\pi\sqrt{m/k}$ has units that make sense.

9 A hot scientific question of the 18th century was the shape of the earth: whether its radius was greater at the equator than at the poles, or the other way around. One method used to attack this question was to measure gravity accurately in different locations on the earth using pendula. If the highest and lowest latitudes accessible to explorers were 0 and 70 degrees, then the strength of gravity would in reality be observed to vary over a range from about 9.780 to 9.826 m/s$^2$. This change, amounting to 0.046 m/s$^2$, is greater than the 0.022 m/s$^2$ expected if the earth had been spherical. The greater effect occurs because the equator feels a reduction due not just to the acceleration of the spinning earth out from under it, but also to the greater radius of the earth at the equator. What is the accuracy with which the period of a one-second pendulum would have to be measured in order to prove that the earth was not a sphere, and that it bulged at the equator? √
Exercise 17: Vibrations

Equipment:
- air track and carts of two different masses
- springs
- spring scales

Place the cart on the air track and attach springs so that it can vibrate.

1. Test whether the period of vibration depends on amplitude. Try at least one moderate amplitude, for which the springs do not go slack, at least one amplitude that is large enough so that they do go slack, and one amplitude that’s the very smallest you can possibly observe.

2. Try a cart with a different mass. Does the period change by the expected factor, based on the equation $T = 2\pi \sqrt{m/k}$?

3. Use a spring scale to pull the cart away from equilibrium, and make a graph of force versus position. Is it linear? If so, what is its slope?

4. Test the equation $T = 2\pi \sqrt{m/k}$ numerically.
Chapter 18
Resonance

Soon after the mile-long Tacoma Narrows Bridge opened in July 1940, motorists began to notice its tendency to vibrate frighteningly in even a moderate wind. Nicknamed “Galloping Gertie,” the bridge collapsed in a steady 42-mile-per-hour wind on November 7 of the same year. The following is an eyewitness report from a newspaper editor who found himself on the bridge as the vibrations approached the breaking point.

“Just as I drove past the towers, the bridge began to sway violently from side to side. Before I realized it, the tilt became so violent that I lost control of the car... I jammed on the brakes and...”
got out, only to be thrown onto my face against the curb.

“Around me I could hear concrete cracking. I started to get my
dog Tubby, but was thrown again before I could reach the car. The
car itself began to slide from side to side of the roadway.

“On hands and knees most of the time, I crawled 500 yards or
more to the towers... My breath was coming in gasps; my knees
were raw and bleeding, my hands bruised and swollen from gripping
the concrete curb... Toward the last, I risked rising to my feet and
running a few yards at a time... Safely back at the toll plaza, I
saw the bridge in its final collapse and saw my car plunge into the
Narrows.”

The ruins of the bridge formed an artificial reef, one of the
world’s largest. It was not replaced for ten years. The reason for its
collapse was not substandard materials or construction, nor was the
bridge under-designed: the piers were hundred-foot blocks of con-
crete, the girders massive and made of carbon steel. The bridge was
destroyed because the bridge absorbed energy efficiently from the
wind, but didn’t dissipate it efficiently into heat. The replacement
bridge, which has lasted half a century so far, was built smarter, not
stronger. The engineers learned their lesson and simply included
some slight modifications to avoid the phenomenon that spelled the
doom of the first one.

18.1 Energy in vibrations

One way of describing the collapse of the bridge is that the bridge
kept taking energy from the steadily blowing wind and building up
more and more energetic vibrations. In this section, we discuss the
energy contained in a vibration, and in the subsequent sections we
will move on to the loss of energy and the adding of energy to a
vibrating system, all with the goal of understanding the important
phenomenon of resonance.

Going back to our standard example of a mass on a spring,
we find that there are two forms of energy involved: the potential
energy stored in the spring and the kinetic energy of the moving
mass. We may start the system in motion either by hitting the
mass to put in kinetic energy or by pulling it to one side to put in
potential energy. Either way, the subsequent behavior of the system
is identical. It trades energy back and forth between kinetic and
potential energy. (We are still assuming there is no friction, so that
no energy is converted to heat, and the system never runs down.)

The most important thing to understand about the energy con-
tent of vibrations is that the total energy is proportional to the
square of the amplitude. Although the total energy is constant, it
is instructive to consider two specific moments in the motion of the mass on a spring as examples. When the mass is all the way to one side, at rest and ready to reverse directions, all its energy is potential. We have already seen that the potential energy stored in a spring equals \((1/2)kx^2\), so the energy is proportional to the square of the amplitude. Now consider the moment when the mass is passing through the equilibrium point at \(x = 0\). At this point it has no potential energy, but it does have kinetic energy. The velocity is proportional to the amplitude of the motion, and the kinetic energy, \((1/2)mv^2\), is proportional to the square of the velocity, so again we find that the energy is proportional to the square of the amplitude. The reason for singling out these two points is merely instructive; proving that energy is proportional to \(A^2\) at any point would suffice to prove that energy is proportional to \(A^2\) in general, since the energy is constant.

Are these conclusions restricted to the mass-on-a-spring example? No. We have already seen that \(F = -kx\) is a valid approximation for any vibrating object, as long as the amplitude is small. We are thus left with a very general conclusion: the energy of any vibration is approximately proportional to the square of the amplitude, provided that the amplitude is small.

**Water in a U-tube example 1**

If water is poured into a U-shaped tube as shown in the figure, it can undergo vibrations about equilibrium. The energy of such a vibration is most easily calculated by considering the “turnaround point” when the water has stopped and is about to reverse directions. At this point, it has only potential energy and no kinetic energy, so by calculating its potential energy we can find the energy of the vibration. This potential energy is the same as the work that would have to be done to take the water out of the right-hand side down to a depth \(A\) below the equilibrium level, raise it through a height \(A\), and place it in the left-hand side. The weight of this chunk of water is proportional to \(A\), and so is the height through which it must be lifted, so the energy is proportional to \(A^2\).

**The range of energies of sound waves example 2**

\[ \text{The amplitude of vibration of your eardrum at the threshold of pain is about } 10^6 \text{ times greater than the amplitude with which it vibrates in response to the softest sound you can hear. How many times greater is the energy with which your ear has to cope for the painfully loud sound, compared to the soft sound?} \]

\[ \text{The amplitude is } 10^6 \text{ times greater, and energy is proportional to the square of the amplitude, so the energy is greater by a factor of } 10^{12}. \text{ This is a phenomenally large factor!} \]
We are only studying vibrations right now, not waves, so we are not yet concerned with how a sound wave works, or how the energy gets to us through the air. Note that because of the huge range of energies that our ear can sense, it would not be reasonable to have a sense of loudness that was additive. Consider, for instance, the following three levels of sound:

- barely audible wind
- quiet conversation \ldots \ldots \quad 10^5 \text{ times more energy than the wind}
- heavy metal concert \ldots \quad 10^{12} \text{ times more energy than the wind}

In terms of addition and subtraction, the difference between the wind and the quiet conversation is nothing compared to the difference between the quiet conversation and the heavy metal concert. Evolution wanted our sense of hearing to be able to encompass all these sounds without collapsing the bottom of the scale so that anything softer than the crack of doom would sound the same. So rather than making our sense of loudness additive, mother nature made it multiplicative. We sense the difference between the wind and the quiet conversation as spanning a range of about $5/12$ as much as the whole range from the wind to the heavy metal concert. Although a detailed discussion of the decibel scale is not relevant here, the basic point to note about the decibel scale is that it is logarithmic. The zero of the decibel scale is close to the lower limit of human hearing, and adding 1 unit to the decibel measurement corresponds to multiplying the energy level (or actually the power per unit area) by a certain factor.

### 18.2 Energy lost from vibrations

Until now, we have been making the relatively unrealistic assumption that a vibration would never die out. For a realistic mass on a spring, there will be friction, and the kinetic and potential energy of the vibrations will therefore be gradually converted into heat. Similarly, a guitar string will slowly convert its kinetic and potential energy into sound. In all cases, the effect is to “pinch” the sinusoidal $x - t$ graph more and more with passing time. Friction is not necessarily bad in this context — a musical instrument that never got rid of any of its energy would be completely silent! The dissipation of the energy in a vibration is known as damping.

**self-check A**

Most people who try to draw graphs like those shown on the left will tend to shrink their wiggles horizontally as well as vertically. Why is this wrong?  
\[ \text{Answer, p. 568} \]

In the graphs in figure b, I have not shown any point at which the damped vibration finally stops completely. Is this realistic? Yes
and no. If energy is being lost due to friction between two solid surfaces, then we expect the force of friction to be nearly independent of velocity. This constant friction force puts an upper limit on the total distance that the vibrating object can ever travel without replenishing its energy, since work equals force times distance, and the object must stop doing work when its energy is all converted into heat. (The friction force does reverse directions when the object turns around, but reversing the direction of the motion at the same time that we reverse the direction of the force makes it certain that the object is always doing positive work, not negative work.)

Damping due to a constant friction force is not the only possibility however, or even the most common one. A pendulum may be damped mainly by air friction, which is approximately proportional to $v^2$, while other systems may exhibit friction forces that are proportional to $v$. It turns out that friction proportional to $v$ is the simplest case to analyze mathematically, and anyhow all the important physical insights can be gained by studying this case.

If the friction force is proportional to $v$, then as the vibrations die down, the frictional forces get weaker due to the lower speeds. The less energy is left in the system, the more miserly the system becomes with giving away any more energy. Under these conditions, the vibrations theoretically never die out completely, and mathematically, the loss of energy from the system is exponential: the system loses a fixed percentage of its energy per cycle. This is referred to as exponential decay.

A non-rigorous proof is as follows. The force of friction is proportional to $v$, and $v$ is proportional to how far the object travels in one cycle, so the frictional force is proportional to amplitude. The amount of work done by friction is proportional to the force and to the distance traveled, so the work done in one cycle is proportional to the square of the amplitude. Since both the work and the energy are proportional to $A^2$, the amount of energy taken away by friction in one cycle is a fixed percentage of the amount of energy the system has.

**self-check B**

Figure c shows an x-t graph for a strongly damped vibration, which loses half of its amplitude with every cycle. What fraction of the energy is lost in each cycle?  

It is customary to describe the amount of damping with a quantity called the quality factor, $Q$, defined as the number of cycles required for the energy to fall off by a factor of 535. (The origin of this obscure numerical factor is $e^{2\pi}$, where $e = 2.71828\ldots$ is the base of natural logarithms. Choosing this particular number causes some of our later equations to come out nice and simple.) The terminology arises from the fact that friction is often considered a bad thing, so a mechanical device that can vibrate for many oscillations...
before it loses a significant fraction of its energy would be considered a high-quality device.

**Exponential decay in a trumpet**  
Example 3

The vibrations of the air column inside a trumpet have a $Q$ of about 10. This means that even after the trumpet player stops blowing, the note will keep sounding for a short time. If the player suddenly stops blowing, how will the sound intensity 20 cycles later compare with the sound intensity while she was still blowing?

The trumpet’s $Q$ is 10, so after 10 cycles the energy will have fallen off by a factor of $5^{35}$. After another 10 cycles we lose another factor of $5^{35}$, so the sound intensity is reduced by a factor of $5^{35} \times 5^{35} = 2.9 \times 10^5$.

The decay of a musical sound is part of what gives it its character, and a good musical instrument should have the right $Q$, but the $Q$ that is considered desirable is different for different instruments. A guitar is meant to keep on sounding for a long time after a string has been plucked, and might have a $Q$ of 1000 or 10000. One of the reasons why a cheap synthesizer sounds so bad is that the sound suddenly cuts off after a key is released.

**$Q$ of a stereo speaker**  
Example 4

Stereo speakers are not supposed to reverberate or “ring” after an electrical signal that stops suddenly. After all, the recorded music was made by musicians who knew how to shape the decays of their notes correctly. Adding a longer “tail” on every note would make it sound wrong. We therefore expect that stereo speaker will have a very low $Q$, and indeed, most speakers are designed with a $Q$ of about 1. (Low-quality speakers with larger $Q$ values are referred to as “boomy.”)

We will see later in the chapter that there are other reasons why a speaker should not have a high $Q$.

### 18.3 Putting energy into vibrations

When pushing a child on a swing, you cannot just apply a constant force. A constant force will move the swing out to a certain angle, but will not allow the swing to start swinging. Nor can you give short pushes at randomly chosen times. That type of random pushing would increase the child’s kinetic energy whenever you happened to be pushing in the same direction as her motion, but it would reduce her energy when your pushing happened to be in the opposite direction compared to her motion. To make her build up her energy, you need to make your pushes rhythmic, pushing at the same point in each cycle. In other words, your force needs to form a repeating pattern with the same frequency as the normal frequency of vibration of the swing. Graph d/1 shows what the child’s $x - t$
The amplitude approaches a maximum. A graph of your force versus time would probably look something like graph 2. It turns out, however, that it is much simpler mathematically to consider a vibration with energy being pumped into it by a driving force that is itself a sine-wave. A good example of this is your eardrum being driven by the force of a sound wave.

Now we know realistically that the child on the swing will not keep increasing her energy forever, nor does your eardrum end up exploding because a continuing sound wave keeps pumping more and more energy into it. In any realistic system, there is energy going out as well as in. As the vibrations increase in amplitude, there is an increase in the amount of energy taken away by damping with each cycle. This occurs for two reasons. Work equals force times distance (or, more accurately, the area under the force-distance curve). As the amplitude of the vibrations increases, the damping force is being applied over a longer distance. Furthermore, the damping force usually increases with velocity (we usually assume for simplicity that it is proportional to velocity), and this also serves to increase the rate at which damping forces remove energy as the amplitude increases. Eventually (and small children and our eardrums are thankful for this!), the amplitude approaches a maximum value, e, at which energy is removed by the damping force just as quickly as it is being put in by the driving force.

This process of approaching a maximum amplitude happens extremely quickly in many cases, e.g., the ear or a radio receiver, and we don’t even notice that it took a millisecond or a microsecond for the vibrations to “build up steam.” We are therefore mainly interested in predicting the behavior of the system once it has had enough time to reach essentially its maximum amplitude. This is known as the steady-state behavior of a vibrating system.

Now comes the interesting part: what happens if the frequency of the driving force is mismatched to the frequency at which the system would naturally vibrate on its own? We all know that a radio station doesn’t have to be tuned in exactly, although there is only a small range over which a given station can be received. The designers of the radio had to make the range fairly small to make it possible to eliminate unwanted stations that happened to be nearby in frequency, but it couldn’t be too small or you wouldn’t be able to adjust the knob accurately enough. (Even a digital radio can be tuned to 88.0 MHz and still bring in a station at 88.1 MHz.) The ear also has some natural frequency of vibration, but in this case the range of frequencies to which it can respond is quite broad. Evolution has made the ear’s frequency response as broad as possible because it was to our ancestors’ advantage to be able to hear everything from a low roar to a high-pitched shriek.
The remainder of this section develops four important facts about the response of a system to a driving force whose frequency is not necessarily the same as the system’s natural frequency of vibration. The style is approximate and intuitive, but proofs are given in section 18.4.

First, although we know the ear has a frequency — about 4000 Hz — at which it would vibrate naturally, it does not vibrate at 4000 Hz in response to a low-pitched 200 Hz tone. It always responds at the frequency at which it is driven. Otherwise all pitches would sound like 4000 Hz to us. This is a general fact about driven vibrations:

1. The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system’s own natural frequency of vibration.

Now let’s think about the amplitude of the steady-state response. Imagine that a child on a swing has a natural frequency of vibration of 1 Hz, but we are going to try to make her swing back and forth at 3 Hz. We intuitively realize that quite a large force would be needed to achieve an amplitude of even 30 cm, i.e., the amplitude is less in proportion to the force. When we push at the natural frequency of 1 Hz, we are essentially just pumping energy back into the system to compensate for the loss of energy due to the damping (friction) force. At 3 Hz, however, we are not just counteracting friction. We are also providing an extra force to make the child’s momentum reverse itself more rapidly than it would if gravity and the tension in the chain were the only forces acting. It is as if we are artificially increasing the \( k \) of the swing, but this is wasted effort because we spend just as much time decelerating the child (taking energy out of the system) as accelerating her (putting energy in).

Now imagine the case in which we drive the child at a very low frequency, say 0.02 Hz or about one vibration per minute. We essentially just holding the child in position while very slowly walking back and forth. Again we intuitively recognize that the amplitude will be very small in proportion to our driving force. Imagine how hard it would be to hold the child at our own head-level when she is at the end of her swing! As in the too-fast 3 Hz case, we are spending most of our effort in artificially changing the \( k \) of the swing, but now rather than reinforcing the gravity and tension forces we are working against them, effectively reducing \( k \). Only a very small part of our force goes into counteracting friction, and the rest is used in repetitively putting potential energy in on the upswing and taking it back out on the downswing, without any long-term gain.

We can now generalize to make the following statement, which
is true for all driven vibrations:

(2) A vibrating system resonates at its own natural frequency. That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

An opera singer breaking a wine glass

In order to break a wineglass by singing, an opera singer must first tap the glass to find its natural frequency of vibration, and then sing the same note back.

Collapse of the Nimitz Freeway in an earthquake

I led off the chapter with the dramatic collapse of the Tacoma Narrows Bridge, mainly because it was well documented by a local physics professor, and an unknown person made a movie of the collapse. The collapse of a section of the Nimitz Freeway in Oakland, CA, during a 1989 earthquake is however a simpler example to analyze.

An earthquake consists of many low-frequency vibrations that occur simultaneously, which is why it sounds like a rumble of indeterminate pitch rather than a low hum. The frequencies that we can hear are not even the strongest ones; most of the energy is in the form of vibrations in the range of frequencies from about 1 Hz to 10 Hz.

Now all the structures we build are resting on geological layers of dirt, mud, sand, or rock. When an earthquake wave comes along, the topmost layer acts like a system with a certain natural frequency of vibration, sort of like a cube of jello on a plate being shaken from side to side. The resonant frequency of the layer depends on how stiff it is and also on how deep it is. The ill-fated section of the Nimitz freeway was built on a layer of mud, and analysis by geologist Susan E. Hough of the U.S. Geological Survey shows that the mud layer's resonance was centered on about 2.5 Hz, and had a width covering a range from about 1 Hz to 4 Hz.

When the earthquake wave came along with its mixture of frequencies, the mud responded strongly to those that were close to its own natural 2.5 Hz frequency. Unfortunately, an engineering analysis after the quake showed that the overpass itself had a resonant frequency of 2.5 Hz as well! The mud responded strongly to the earthquake waves with frequencies close to 2.5 Hz, and the bridge responded strongly to the 2.5 Hz vibrations of the mud, causing sections of it to collapse.

Collapse of the Tacoma Narrows Bridge

Let’s now examine the more conceptually difficult case of the...
Tacoma Narrows Bridge. The surprise here is that the wind was steady. If the wind was blowing at constant velocity, why did it shake the bridge back and forth? The answer is a little complicated. Based on film footage and after-the-fact wind tunnel experiments, it appears that two different mechanisms were involved.

The first mechanism was the one responsible for the initial, relatively weak vibrations, and it involved resonance. As the wind moved over the bridge, it began acting like a kite or an airplane wing. As shown in the figure, it established swirling patterns of air flow around itself, of the kind that you can see in a moving cloud of smoke. As one of these swirls moved off of the bridge, there was an abrupt change in air pressure, which resulted in an up or down force on the bridge. We see something similar when a flag flaps in the wind, except that the flag's surface is usually vertical. This back-and-forth sequence of forces is exactly the kind of periodic driving force that would excite a resonance. The faster the wind, the more quickly the swirls would get across the bridge, and the higher the frequency of the driving force would be. At just the right velocity, the frequency would be the right one to excite the resonance. The wind-tunnel models, however, show that the pattern of vibration of the bridge excited by this mechanism would have been a different one than the one that finally destroyed the bridge.

The bridge was probably destroyed by a different mechanism, in which its vibrations at its own natural frequency of 0.2 Hz set up an alternating pattern of wind gusts in the air immediately around it, which then increased the amplitude of the bridge's vibrations. This vicious cycle fed upon itself, increasing the amplitude of the vibrations until the bridge finally collapsed.

As long as we’re on the subject of collapsing bridges, it is worth bringing up the reports of bridges falling down when soldiers marching over them happened to step in rhythm with the bridge’s natural frequency of oscillation. This is supposed to have happened in 1831 in Manchester, England, and again in 1849 in Anjou, France. Many modern engineers and scientists, however, are suspicious of the analysis of these reports. It is possible that the collapses had more to do with poor construction and overloading than with resonance. The Nimitz Freeway and Tacoma Narrows Bridge are far better documented, and occurred in an era when engineers’ abilities to analyze the vibrations of a complex structure were much more advanced.

In a very thin gas, the atoms are sufficiently far apart that they can act as individual vibrating systems. Although the vibrations are of a very strange and abstract type described by the theory of quantum mechanics, they nevertheless obey the same basic rules as ordinary mechanical vibrations. When a thin gas made of a cer-
tain element is heated, it emits light waves with certain specific frequencies, which are like a fingerprint of that element. As with all other vibrations, these atomic vibrations respond most strongly to a driving force that matches their own natural frequency. Thus if we have a relatively cold gas with light waves of various frequencies passing through it, the gas will absorb light at precisely those frequencies at which it would emit light if heated.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to \( Q \).

This is fairly intuitive. The steady-state behavior is an equilibrium between energy input from the driving force and energy loss due to damping. A low-\( Q \) oscillator, i.e., one with strong damping, dumps its energy faster, resulting in lower-amplitude steady-state motion.

**self-check C**

If an opera singer is shopping for a wine glass that she can impress her friends by breaking, what should she look for?  

▷ Answer, p. 568

Piano strings ringing in sympathy with a sung note  

▷ A sufficiently loud musical note sung near a piano with the lid raised can cause the corresponding strings in the piano to vibrate. (A piano has a set of three strings for each note, all struck by the same hammer.) Why would this trick be unlikely to work with a violin?

▷ If you have heard the sound of a violin being plucked (the pizzicato effect), you know that the note dies away very quickly. In other words, a violin’s \( Q \) is much lower than a piano’s. This means that its resonances are much weaker in amplitude.

Our fourth and final fact about resonance is perhaps the most surprising. It gives us a way to determine numerically how wide a range of driving frequencies will produce a strong response. As shown in the graph, resonances do not suddenly fall off to zero outside a certain frequency range. It is usual to describe the width of a resonance by its full width at half-maximum (FWHM) as illustrated in figure \( g \).

(4) The FWHM of a resonance is related to its \( Q \) and its resonant frequency \( f_{res} \) by the equation

\[
\text{FWHM} = \frac{f_{res}}{Q}.
\]

(This equation is only a good approximation when \( Q \) is large.)
Why? It is not immediately obvious that there should be any logical relationship between $Q$ and the FWHM. Here’s the idea. As we have seen already, the reason why the response of an oscillator is smaller away from resonance is that much of the driving force is being used to make the system act as if it had a different $k$. Roughly speaking, the half-maximum points on the graph correspond to the places where the amount of the driving force being wasted in this way is the same as the amount of driving force being used productively to replace the energy being dumped out by the damping force. If the damping force is strong, then a large amount of force is needed to counteract it, and we can waste quite a bit of driving force on changing $k$ before it becomes comparable to the damping force. If, on the other hand, the damping force is weak, then even a small amount of force being wasted on changing $k$ will become significant in proportion, and we cannot get very far from the resonant frequency before the two are comparable.

**Changing the pitch of a wind instrument** example 10

A saxophone player normally selects which note to play by choosing a certain fingering, which gives the saxophone a certain resonant frequency. The musician can also, however, change the pitch significantly by altering the tightness of her lips. This corresponds to driving the horn slightly off of resonance. If the pitch can be altered by about 5% up or down (about one musical half-step) without too much effort, roughly what is the $Q$ of a saxophone?

Five percent is the width on one side of the resonance, so the full width is about 10%, $\text{FWHM} / f_{res} = 0.1$. This implies a $Q$ of about 10, i.e., once the musician stops blowing, the horn will continue sounding for about 10 cycles before its energy falls off by a factor of 535. (Blues and jazz saxophone players will typically choose a mouthpiece that has a low $Q$, so that they can produce the bluesy pitch-slides typical of their style. “Legit,” i.e., classically oriented players, use a higher-$Q$ setup because their style only calls for enough pitch variation to produce a vibrato.)

**Decay of a saxophone tone** example 11

If a typical saxophone setup has a $Q$ of about 10, how long will it take for a 100-Hz tone played on a baritone saxophone to die down by a factor of 535 in energy, after the player suddenly stops blowing?

A $Q$ of 10 means that it takes 10 cycles for the vibrations to die down in energy by a factor of 535. Ten cycles at a frequency of 100 Hz would correspond to a time of 0.1 seconds, which is not very long. This is why a saxophone note doesn’t “ring” like a note played on a piano or an electric guitar.

**$Q$ of a radio receiver** example 12

A radio receiver used in the FM band needs to be tuned in to
within about 0.1 MHz for signals at about 100 MHz. What is its $Q$?

$Q = Q_{\text{res}} / \text{FWHM} = 1000$. This is an extremely high $Q$ compared to most mechanical systems.

1. **A compass needle vibrates about the equilibrium position under the influence of the earth's magnetic forces.**

2. **The orientation of a proton's spin vibrates around its equilibrium direction under the influence of the magnetic forces coming from the surrounding electrons and nuclei.**

Essentially the same physics lies behind the technique called Nuclear Magnetic Resonance (NMR). NMR is a technique used to deduce the molecular structure of unknown chemical substances, and it is also used for making medical images of the inside of people's bodies. If you ever have an NMR scan, they will actually tell you you are undergoing “magnetic resonance imaging” or “MRI,” because people are scared of the word “nuclear.” In fact, the nuclei being referred to are simply the non-radioactive nuclei of atoms found naturally in your body.

Here’s how NMR works. Your body contains large numbers of hydrogen atoms, each consisting of a small, lightweight electron orbiting around a large, heavy proton. That is, the nucleus of a hydrogen atom is just one proton. A proton is always spinning on its own axis, and the combination of its spin and its electrical charge causes it to behave like a tiny magnet. The principle is identical to that of an electromagnet, which consists of a coil of wire through which electrical charges pass; the circling motion of the charges in the coil of wire makes it magnetic, and in the same way, the circling motion of the proton’s charge makes it magnetic.

Now a proton in one of your body’s hydrogen atoms finds itself surrounded by many other whirling, electrically charged particles: its own electron, plus the electrons and nuclei of the other nearby atoms. These neighbors act like magnets, and exert magnetic forces on the proton, $\kappa/2$. The $k$ of the vibrating proton is simply a measure of the total strength of these magnetic forces. Depending on the structure of the molecule in which the hydrogen atom finds itself, there will be a particular set of magnetic forces acting on the proton and a particular value of $k$. The NMR apparatus
bombards the sample with radio waves, and if the frequency of
the radio waves matches the resonant frequency of the proton,
the proton will absorb radio-wave energy strongly and oscillate
wildly. Its vibrations are damped not by friction, because there is
no friction inside an atom, but by the reemission of radio waves.

By working backward through this chain of reasoning, one can de-
termine the geometric arrangement of the hydrogen atom’s neigh-
boring atoms. It is also possible to locate atoms in space, allowing
medical images to be made.

Finally, it should be noted that the behavior of the proton cannot
be described entirely correctly by Newtonian physics. Its vibra-
tions are of the strange and spooky kind described by the laws of
quantum mechanics. It is impressive, however, that the few sim-
ple ideas we have learned about resonance can still be applied
successfully to describe many aspects of this exotic system.

Discussion question

A Nikola Tesla, one of the inventors of radio and an archetypical mad
scientist, told a credulous reporter in 1912 the following story about an
application of resonance. He built an electric vibrator that fit in his pocket,
and attached it to one of the steel beams of a building that was under
construction in New York. Although the article in which he was quoted
didn’t say so, he presumably claimed to have tuned it to the resonant fre-
quency of the building. “In a few minutes, I could feel the beam trembling.
Gradually the trembling increased in intensity and extended throughout
the whole great mass of steel. Finally, the structure began to creak and
weave, and the steelworkers came to the ground panic-stricken, believ-
ing that there had been an earthquake. ... [If] I had kept on ten minutes
more, I could have laid that building flat in the street.” Is this physically
plausible?

18.4 * Proofs

Our first goal is to predict the amplitude of the steady-state vibra-
tions as a function of the frequency of the driving force and the
amplitude of the driving force. With that equation in hand, we will
then prove statements 2, 3, and 4 from section 18.3. We assume
without proof statement 1, that the steady-state motion occurs at
the same frequency as the driving force.

As with the proof in chapter 17, we make use of the fact that
a sinusoidal vibration is the same as the projection of circular mo-
ton onto a line. We visualize the system shown in figures k-m,
in which the mass swings in a circle on the end of a spring. The
spring does not actually change its length at all, but it appears to
from the flattened perspective of a person viewing the system edge-
on. The radius of the circle is the amplitude, \( A \), of the vibrations
as seen edge-on. The damping force can be imagined as a back-
ward drag force supplied by some fluid through which the mass is
moving. As usual, we assume that the damping is proportional to velocity, and we use the symbol $b$ for the proportionality constant, $|F_d| = bv$. The driving force, represented by a hand towing the mass with a string, has a tangential component $|F_t|$ which counteracts the damping force, $|F_t| = |F_d|$, and a radial component $F_r$ which works either with or against the spring’s force, depending on whether we are driving the system above or below its resonant frequency.

The speed of the rotating mass is the circumference of the circle divided by the period, $v = 2\pi A/T$, its acceleration (which is directly inward) is $a = v^2/r$, and Newton’s second law gives $a = F/m = (kA + F_r)/m$. We write $f_o$ for $\frac{1}{2\pi}\sqrt{k/m}$. Straightforward algebra yields

$$F_r \over F_t = \frac{2\pi m}{b f} \left( f^2 - f_o^2 \right).$$

This is the ratio of the wasted force to the useful force, and we see that it becomes zero when the system is driven at resonance.

The amplitude of the vibrations can be found by attacking the equation $|F_t| = bv = 2\pi b A f$, which gives

$$A = \frac{|F_t|}{2\pi b f}.$$

However, we wish to know the amplitude in terms of $|F|$, not $|F_t|$. From now on, let’s drop the cumbersome magnitude symbols. With the Pythagorean theorem, it is easily proved that

$$F_t = \frac{F}{\sqrt{1 + \left( \frac{F_r}{F_t} \right)^2}},$$

and equations 1-3 can then be combined to give the final result

$$A = \frac{F}{2\pi \sqrt{4\pi^2 m^2 (f^2 - f_o^2)^2 + b^2 f^2}}.$$

**Statement 2: maximum amplitude at resonance**

Equation [4] makes it plausible that the amplitude is maximized when the system is driven at close to its resonant frequency. At $f = f_o$, the first term inside the square root vanishes, and this makes the denominator as small as possible, causing the amplitude to be as big as possible. (Actually this is only approximately true, because it is possible to make $A$ a little bigger by decreasing $f$ a little below $f_o$, which makes the second term smaller. This technical issue is addressed in homework problem 3 on page 489.)

**Statement 3: amplitude at resonance proportional to $Q$**

Equation [4] shows that the amplitude at resonance is proportional to $1/b$, and the $Q$ of the system is inversely proportional to $b$, so the amplitude at resonance is proportional to $Q$. 

Section 18.4   * Proofs   485
Statement 4: FWHM related to $Q$

We will satisfy ourselves by proving only the proportionality $FWHM \propto f_o/Q$, not the actual equation $FWHM = f_o/Q$. The energy is proportional to $A^2$, i.e., to the inverse of the quantity inside the square root in equation [4]. At resonance, the first term inside the square root vanishes, and the half-maximum points occur at frequencies for which the whole quantity inside the square root is double its value at resonance, i.e., when the two terms are equal. At the half-maximum points, we have

$$f^2 - f_o^2 = \left( f_o \pm \frac{FWHM}{2} \right)^2 - f_o^2$$

$$= \pm f_o \cdot FWHM + \frac{1}{4} FWHM^2$$

If we assume that the width of the resonance is small compared to the resonant frequency, then the $FWHM^2$ term is negligible compared to the $f_o \cdot FWHM$ term, and setting the terms in equation 4 equal to each other gives

$$4\pi^2 m^2 (f_o FWHM)^2 = b^2 f^2.$$ 

We are assuming that the width of the resonance is small compared to the resonant frequency, so $f$ and $f_o$ can be taken as synonyms. Thus,

$$FWHM = \frac{b}{2\pi m}.$$ 

We wish to connect this to $Q$, which can be interpreted as the energy of the free (undriven) vibrations divided by the work done by damping in one cycle. The former equals $kA^2/2$, and the latter is proportional to the force, $bu \propto bAf_o$, multiplied by the distance traveled, $A$. (This is only a proportionality, not an equation, since the force is not constant.) We therefore find that $Q$ is proportional to $k/bf_o$. The equation for the FWHM can then be restated as a proportionality $FWHM \propto k/Qf_o m \propto f_o/Q$. 

486  Chapter 18  Resonance
Summary

Selected vocabulary

- **damping**: the dissipation of a vibration’s energy into heat energy, or the frictional force that causes the loss of energy.
- **quality factor**: the number of oscillations required for a system’s energy to fall off by a factor of 5/3 due to damping.
- **driving force**: an external force that pumps energy into a vibrating system.
- **resonance**: the tendency of a vibrating system to respond most strongly to a driving force whose frequency is close to its own natural frequency of vibration.
- **steady state**: the behavior of a vibrating system after it has had plenty of time to settle into a steady response to a driving force.

Notation

- \( Q \): the quality factor.
- \( f_0 \): the natural (resonant) frequency of a vibrating system, i.e., the frequency at which it would vibrate if it was simply kicked and left alone.
- \( f \): the frequency at which the system actually vibrates, which in the case of a driven system is equal to the frequency of the driving force, not the natural frequency.

Summary

The energy of a vibration is always proportional to the square of the amplitude, assuming the amplitude is small. Energy is lost from a vibrating system for various reasons such as the conversion to heat via friction or the emission of sound. This effect, called damping, will cause the vibrations to decay exponentially unless energy is pumped into the system to replace the loss. A driving force that pumps energy into the system may drive the system at its own natural frequency or at some other frequency. When a vibrating system is driven by an external force, we are usually interested in its steady-state behavior, i.e., its behavior after it has had time to settle into a steady response to a driving force. In the steady state, the same amount of energy is pumped into the system during each cycle as is lost to damping during the same period.

The following are four important facts about a vibrating system being driven by an external force:

1. The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system’s own natural frequency of vibration.
(2) A vibrating system resonates at its own natural frequency. That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to $Q$.

(4) The FWHM of a resonance is related to its $Q$ and its resonant frequency $f_o$ by the equation

$$\text{FWHM} = \frac{f_o}{Q}.$$  

(This equation is only a good approximation when $Q$ is large.)
Problems

Key
√  A computerized answer check is available online.
∫  A problem that requires calculus.
⋆  A difficult problem.

1  If one stereo system is capable of producing 20 watts of sound power and another can put out 50 watts, how many times greater is the amplitude of the sound wave that can be created by the more powerful system? (Assume they are playing the same music.)

2  Many fish have an organ known as a swim bladder, an air-filled cavity whose main purpose is to control the fish’s buoyancy and allow it to keep from rising or sinking without having to use its muscles. In some fish, however, the swim bladder (or a small extension of it) is linked to the ear and serves the additional purpose of amplifying sound waves. For a typical fish having such an anatomy, the bladder has a resonant frequency of 300 Hz, the bladder’s $Q$ is 3, and the maximum amplification is about a factor of 100 in energy. Over what range of frequencies would the amplification be at least a factor of 50?

3  As noted in section 18.4, it is only approximately true that the amplitude has its maximum at the natural frequency $(1/2\pi)\sqrt{k/m}$. Being more careful, we should actually define two different symbols, $f_o = (1/2\pi)\sqrt{k/m}$ and $f_{\text{res}}$ for the slightly different frequency at which the amplitude is a maximum, i.e., the actual resonant frequency. In this notation, the amplitude as a function of frequency is

$$A = \frac{F}{2\pi \sqrt{4\pi^2 m^2 (f^2 - f_0^2)^2 + b^2 f^2}}.$$ 

Show that the maximum occurs not at $f_o$ but rather at

$$f_{\text{res}} = \sqrt{f_0^2 - \frac{b^2}{8\pi^2 m^2}} = \sqrt{f_0^2 - \frac{1}{2}\text{FWHM}^2}.$$ 

Hint: Finding the frequency that minimizes the quantity inside the square root is equivalent to, but much easier than, finding the frequency that maximizes the amplitude.
4 (a) Let \( W \) be the amount of work done by friction in the first cycle of oscillation, i.e., the amount of energy lost to heat. Find the fraction of the original energy \( E \) that remains in the oscillations after \( n \) cycles of motion.  

(b) From this, prove the equation  
\[
\left(1 - \frac{W}{E}\right)^Q = e^{-2\pi}
\]  
(recalling that the number 535 in the definition of \( Q \) is \( e^{2\pi} \)).

(c) Use this to prove the approximation \( 1/Q \approx (1/2\pi)W/E \). (Hint: Use the approximation \( \ln(1 + x) \approx x \), which is valid for small values of \( x \), as shown on p. 1061.)

5 The goal of this problem is to refine the proportionality \( \text{FWHM} \propto f\text{res}/Q \) into the equation \( \text{FWHM} = f\text{res}/Q \), i.e., to prove that the constant of proportionality equals 1.

(a) Show that the work done by a damping force \( F = -bv \) over one cycle of steady-state motion equals \( W_{\text{damp}} = -2\pi^2bfA^2 \). Hint: It is less confusing to calculate the work done over half a cycle, from \( x = -A \) to \( x = +A \), and then double it.

(b) Show that the fraction of the undriven oscillator’s energy lost to damping over one cycle is \( |W_{\text{damp}}|/E = 4\pi^2bf/k \).

(c) Use the previous result, combined with the result of problem 4, to prove that \( Q \) equals \( k/2\pi bf \).

(d) Combine the preceding result for \( Q \) with the equation \( \text{FWHM} = b/2\pi m \) from section 18.4 to prove the equation \( \text{FWHM} = f\text{res}/Q \).

6 (a) We observe that the amplitude of a certain free oscillation decreases from \( A_0 \) to \( A_0/Z \) after \( n \) oscillations. Find its \( Q \).

(b) The figure is from Shape memory in Spider druglines, Emile, Le Floch, and Vollrath, Nature 440:621 (2006). Panel 1 shows an electron microscope’s image of a thread of spider silk. In 2, a spider is hanging from such a thread. From an evolutionary point of view, it’s probably a bad thing for the spider if it twists back and forth while hanging like this. (We’re referring to a back-and-forth rotation about the axis of the thread, not a swinging motion like a pendulum.) The authors speculate that such a vibration could make the spider easier for predators to see, and it also seems to me that it would be a bad thing just because the spider wouldn’t be able to control its orientation and do what it was trying to do. Panel 3 shows a graph of such an oscillation, which the authors measured using a video camera and a computer, with a 0.1 g mass hung from it in place of a spider. Compared to human-made fibers such as kevlar or copper wire, the spider thread has an unusual set of properties:
Problem 6.

1. It has a low $Q$, so the vibrations damp out quickly.

2. It doesn’t become brittle with repeated twisting as a copper wire would.

3. When twisted, it tends to settle into a new equilibrium angle, rather than insisting on returning to its original angle. You can see this in panel 2, because although the experimenters initially twisted the wire by 35 degrees, the thread only performed oscillations with an amplitude much smaller than ±35 degrees, settling down to a new equilibrium at 27 degrees.

4. Over much longer time scales (hours), the thread eventually resets itself to its original equilibrium angle (shown as zero degrees on the graph). (The graph reproduced here only shows the motion over a much shorter time scale.) Some human-made materials have this “memory” property as well, but they typically need to be heated in order to make them go back to their original shapes.

Focusing on property number 1, estimate the $Q$ of spider silk from the graph.
Exercise 18: Resonance

1. Compare the oscillator’s energies at A, B, C, and D.

2. Compare the Q values of the two oscillators.

3. Match the x-t graphs in #2 with the amplitude-frequency graphs below.
Chapter 19
Free Waves

Your vocal cords or a saxophone reed can vibrate, but being able to vibrate wouldn’t be of much use unless the vibrations could be transmitted to the listener’s ear by sound waves. What are waves and why do they exist? Put your fingertip in the middle of a cup of water and then remove it suddenly. You will have noticed two results that are surprising to most people. First, the flat surface of the water does not simply sink uniformly to fill in the volume vacated by your finger. Instead, ripples spread out, and the process of flattening out occurs over a long period of time, during which the water at the center vibrates above and below the normal water level. This type of wave motion is the topic of the present chapter. Second, you have found that the ripples bounce off of the walls of the cup, in much the same way that a ball would bounce off of a wall. In the next chapter we discuss what happens to waves that have a boundary around them. Until then, we confine ourselves to wave phenomena that can be analyzed as if the medium (e.g., the water) was infinite and the same everywhere.

It isn’t hard to understand why removing your fingertip creates ripples rather than simply allowing the water to sink back down uniformly. The initial crater, (a), left behind by your finger has sloping sides, and the water next to the crater flows downhill to fill in the hole. The water far away, on the other hand, initially has
no way of knowing what has happened, because there is no slope for it to flow down. As the hole fills up, the rising water at the center gains upward momentum, and overshoots, creating a little hill where there had been a hole originally. The area just outside of this region has been robbed of some of its water in order to build the hill, so a depressed “moat” is formed, (b). This effect cascades outward, producing ripples.
The two circular patterns of ripples pass through each other. Unlike material objects, wave patterns can overlap in space, and when this happens they combine by addition.

1. Superposition

The most profound difference is that waves do not display anything analogous to the normal forces between objects that come in contact. Two wave patterns can therefore overlap in the same region of space, as shown in figure b. Where the two waves coincide, they add together. For instance, suppose that at a certain location in at a certain moment in time, each wave would have had a crest 3 cm above the normal water level. The waves combine at this point to make a 6-cm crest. We use negative numbers to represent depressions in the water. If both waves would have had a trough measuring -3 cm, then they combine to make an extra-deep -6 cm trough. A +3 cm crest and a -3 cm trough result in a height of zero, i.e., the waves momentarily cancel each other out at that point. This additive rule is referred to as the principle of superposition, “superposition” being merely a fancy word for “adding.”

Superposition can occur not just with sinusoidal waves like the ones in the figure above but with waves of any shape. The figures on the following page show superposition of wave pulses. A pulse is simply a wave of very short duration. These pulses consist only of a single hump or trough. If you hit a clothesline sharply, you will observe pulses heading off in both directions. This is analogous to
the way ripples spread out in all directions when you make a disturbance at one point on water. The same occurs when the hammer on a piano comes up and hits a string.

Experiments to date have not shown any deviation from the principle of superposition in the case of light waves. For other types of waves, it is typically a very good approximation for low-energy waves.

Discussion question

A In figure c/3, the fifth frame shows the spring just about perfectly flat. If the two pulses have essentially canceled each other out perfectly, then why does the motion pick up again? Why doesn’t the spring just stay flat?

1. A pulse travels to the left. 2. Superposition of two colliding positive pulses. 3. Superposition of two colliding pulses, one positive and one negative.
As the wave pulse goes by, the ribbon tied to the spring is not carried along. The motion of the wave pattern is to the right, but the medium (spring) is moving up and down, not to the right.

2. The medium is not transported with the wave.

Figure d shows a series of water waves before it has reached a rubber duck (left), having just passed the duck (middle) and having progressed about a meter beyond the duck (right). The duck bobs around its initial position, but is not carried along with the wave. This shows that the water itself does not flow outward with the wave. If it did, we could empty one end of a swimming pool simply by kicking up waves! We must distinguish between the motion of the medium (water in this case) and the motion of the wave pattern through the medium. The medium vibrates; the wave progresses through space.

self-check A
In figure e, you can detect the side-to-side motion of the spring because the spring appears blurry. At a certain instant, represented by a single photo, how would you describe the motion of the different parts of the spring? Other than the flat parts, do any parts of the spring have zero velocity?

A worm example 1
The worm in the figure is moving to the right. The wave pattern, a pulse consisting of a compressed area of its body, moves to the left. In other words, the motion of the wave pattern is in the opposite direction compared to the motion of the medium.

As the wave pulse goes by, the ribbon tied to the spring is not carried along. The motion of the wave pattern is to the right, but the medium (spring) is moving up and down, not to the right.
Example 2. The surfer is dragging his hand in the water.

Example 3: a breaking wave.

Example 4. The boat has run up against a limit on its speed because it can't climb over its own wave. Dolphins get around the problem by leaping out of the water.

**Surfing**

The incorrect belief that the medium moves with the wave is often reinforced by garbled secondhand knowledge of surfing. Anyone who has actually surfed knows that the front of the board pushes the water to the sides, creating a wake — the surfer can even drag his hand through the water, as in figure f. If the water was moving along with the wave and the surfer, this wouldn’t happen. The surfer is carried forward because forward is downhill, not because of any forward flow of the water. If the water was flowing forward, then a person floating in the water up to her neck would be carried along just as quickly as someone on a surfboard. In fact, it is even possible to surf down the back side of a wave, although the ride wouldn’t last very long because the surfer and the wave would quickly part company.

3. **A wave’s velocity depends on the medium.**

A material object can move with any velocity, and can be sped up or slowed down by a force that increases or decreases its kinetic energy. Not so with waves. The magnitude of a wave’s velocity depends on the properties of the medium (and perhaps also on the shape of the wave, for certain types of waves). Sound waves travel at about 340 m/s in air, 1000 m/s in helium. If you kick up water waves in a pool, you will find that kicking harder makes waves that are taller (and therefore carry more energy), not faster. The sound waves from an exploding stick of dynamite carry a lot of energy, but are no faster than any other waves. Thus although both waves and physical objects carry energy as they move through space, the energy of the wave relates to its amplitude, not to its speed.

In the following section we will give an example of the physical relationship between the wave speed and the properties of the medium.

**Breaking waves**

The velocity of water waves increases with depth. The crest of a wave travels faster than the trough, and this can cause the wave to break.

Once a wave is created, the only reason its speed will change is if it enters a different medium or if the properties of the medium change. It is not so surprising that a change in medium can slow down a wave, but the reverse can also happen. A sound wave traveling through a helium balloon will slow down when it emerges into the air, but if it enters another balloon it will speed back up again! Similarly, water waves travel more quickly over deeper water, so a wave will slow down as it passes over an underwater ridge, but speed
Hull speed

The speeds of most boats, and of some surface-swimming animals, are limited by the fact that they make a wave due to their motion through the water. The boat in figure h is going at the same speed as its own waves, and can’t go any faster. No matter how hard the boat pushes against the water, it can’t make the wave move ahead faster and get out of the way. The wave’s speed depends only on the medium. Adding energy to the wave doesn’t speed it up, it just increases its amplitude.

A water wave, unlike many other types of wave, has a speed that depends on its shape: a broader wave moves faster. The shape of the wave made by a boat tends to mold itself to the shape of the boat’s hull, so a boat with a longer hull makes a broader wave that moves faster. The maximum speed of a boat whose speed is limited by this effect is therefore closely related to the length of its hull, and the maximum speed is called the hull speed. Sailboats designed for racing are not just long and skinny to make them more streamlined — they are also long so that their hull speeds will be high.

Wave patterns

If the magnitude of a wave’s velocity vector is preordained, what about its direction? Waves spread out in all directions from every point on the disturbance that created them. If the disturbance is small, we may consider it as a single point, and in the case of water waves the resulting wave pattern is the familiar circular ripple, i/1. If, on the other hand, we lay a pole on the surface of the water and wiggle it up and down, we create a linear wave pattern, i/2. For a three-dimensional wave such as a sound wave, the analogous patterns would be spherical waves and plane waves, j.

Infinitely many patterns are possible, but linear or plane waves are often the simplest to analyze, because the velocity vector is in the same direction no matter what part of the wave we look at. Since all the velocity vectors are parallel to one another, the problem is effectively one-dimensional. Throughout this chapter and the next, we will restrict ourselves mainly to wave motion in one dimension, while not hesitating to broaden our horizons when it can be done without too much complication.
**Discussion questions**

A [see above]

B Sketch two positive wave pulses on a string that are overlapping but not right on top of each other, and draw their superposition. Do the same for a positive pulse running into a negative pulse.

C A traveling wave pulse is moving to the right on a string. Sketch the velocity vectors of the various parts of the string. Now do the same for a pulse moving to the left.

D In a spherical sound wave spreading out from a point, how would the energy of the wave fall off with distance?

---

**19.2 Waves on a string**

So far you have learned some counterintuitive things about the behavior of waves, but intuition can be trained. The first half of this section aims to build your intuition by investigating a simple, one-dimensional type of wave: a wave on a string. If you have ever stretched a string between the bottoms of two open-mouthed cans to talk to a friend, you were putting this type of wave to work. Stringed instruments are another good example. Although we usually think of a piano wire simply as vibrating, the hammer actually strikes it quickly and makes a dent in it, which then ripples out in both directions. Since this chapter is about free waves, not bounded ones, we pretend that our string is infinitely long.

After the qualitative discussion, we will use simple approximations to investigate the speed of a wave pulse on a string. This quick and dirty treatment is then followed by a rigorous attack using the methods of calculus, which may be skipped by the student who has not studied calculus. How far you penetrate in this section is up to you, and depends on your mathematical self-confidence. If you skip some of the math, you should nevertheless absorb the significance of the result, discussed on p. 504.

**Intuitive ideas**

Consider a string that has been struck, l/1, resulting in the creation of two wave pulses, 2, one traveling to the left and one to the right. This is analogous to the way ripples spread out in all directions from a splash in water, but on a one-dimensional string, “all directions” becomes “both directions.”

We can gain insight by modeling the string as a series of masses connected by springs. (In the actual string the mass and the springiness are both contributed by the molecules themselves.) If we look at various microscopic portions of the string, there will be some areas that are flat, m/1, some that are sloping but not curved, 2, and some that are curved, 3 and 4. In example 1 it is clear that both the forces on the central mass cancel out, so it will not accelerate. The