Discussion question A.

$r = b$. Referring to the magnified portion of the figure, $v_\perp$ can be related to $h$, because the two right triangles are similar:

\[
\frac{h}{\text{distance traveled}} = \frac{v_\perp}{|v|}
\]

The area can thus be rewritten as

\[
\text{area} = \frac{1}{2} r v_\perp (\text{distance traveled}) \frac{|v|}{|v|}.
\]

The distance traveled equals $|v|\Delta t$, so this simplifies to

\[
\text{area} = \frac{1}{2} r v_\perp \Delta t.
\]

We have found the following relationship between angular momentum and the rate at which area is swept out:

\[
L = 2 m \frac{\text{area}}{\Delta t}.
\]

The factor of 2 in front is simply a matter of convention, since any conserved quantity would be an equally valid conserved quantity if you multiplied it by a constant. The factor of $m$ was not relevant to Kepler, who did not know the planets’ masses, and who was only describing the motion of one planet at a time.

We thus find that Kepler’s equal-area law is equivalent to a statement that the planet’s angular momentum remains constant. But wait, why should it remain constant? — the planet is not a closed system, since it is being acted on by the sun’s gravitational force. There are two valid answers. The first is that it is actually the total angular momentum of the sun plus the planet that is conserved. The sun, however, is millions of times more massive than the typical planet, so it accelerates very little in response to the planet’s gravitational force. It is thus a good approximation to say that the sun doesn’t move at all, so that no angular momentum is transferred between it and the planet.

The second answer is that to change the planet’s angular momentum requires not just a force but a force applied in a certain way. In section 15.4 we discuss the transfer of angular momentum by a force, but the basic idea here is that a force directly in toward the axis does not change the angular momentum.

**Discussion questions**

A Suppose an object is simply traveling in a straight line at constant speed. If we pick some point not on the line and call it the axis of rotation, is area swept out by the object at a constant rate? Would it matter if we chose a different axis?

B The figure is a strobe photo of a pendulum bob, taken from underneath the pendulum looking straight up. The black string can’t be seen.
in the photograph. The bob was given a slight sideways push when it was released, so it did not swing in a plane. The bright spot marks the center, i.e., the position the bob would have if it hung straight down at us. Does the bob’s angular momentum appear to remain constant if we consider the center to be the axis of rotation? What if we choose a different axis?

Discussion question B.

15.3 Two theorems about angular momentum

With plain old momentum, $p$, we had the freedom to work in any inertial frame of reference we liked. The same object could have different values of momentum in two different frames, if the frames were not at rest with respect to each other. Conservation of momentum, however, would be true in either frame. As long as we employed a single frame consistently throughout a calculation, everything would work.

The same is true for angular momentum, and in addition there is an ambiguity that arises from the definition of an axis of rotation. For a wheel, the natural choice of an axis of rotation is obviously the axle, but what about an egg rotating on its side? The egg has an asymmetric shape, and thus no clearly defined geometric center. A similar issue arises for a cyclone, which does not even have a sharply defined shape, or for a complicated machine with many gears. The following theorem, the first of two presented in this section without proof, explains how to deal with this issue. Although I have put descriptive titles above both theorems, they have no generally accepted names.

**the choice of axis theorem**

It is entirely arbitrary what point one defines as the axis for purposes of calculating angular momentum. If a closed system’s angular momentum is conserved when calculated with one choice of axis, then it will also be conserved for any other choice. Likewise, any inertial frame of reference may be used.
Colliding asteroids described with different axes  example 4

Observers on planets A and B both see the two asteroids colliding. The asteroids are of equal mass and their impact speeds are the same. Astronomers on each planet decide to define their own planet as the axis of rotation. Planet A is twice as far from the collision as planet B. The asteroids collide and stick. For simplicity, assume planets A and B are both at rest.

With planet A as the axis, the two asteroids have the same amount of angular momentum, but one has positive angular momentum and the other has negative. Before the collision, the total angular momentum is therefore zero. After the collision, the two asteroids will have stopped moving, and again the total angular momentum is zero. The total angular momentum both before and after the collision is zero, so angular momentum is conserved if you choose planet A as the axis.

The only difference with planet B as axis is that $r$ is smaller by a factor of two, so all the angular momenta are halved. Even though the angular momenta are different than the ones calculated by planet A, angular momentum is still conserved.

The earth spins on its own axis once a day, but simultaneously travels in its circular one-year orbit around the sun, so any given part of it traces out a complicated loopy path. It would seem difficult to calculate the earth’s angular momentum, but it turns out that there is an intuitively appealing shortcut: we can simply add up the angular momentum due to its spin plus that arising from its center of mass’s circular motion around the sun. This is a special case of the following general theorem:

**the spin theorem**

An object’s angular momentum with respect to some outside axis A can be found by adding up two parts:

1. The first part is the object’s angular momentum found by using its own center of mass as the axis, i.e., the angular momentum the object has because it is spinning.
2. The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

A system with its center of mass at rest  example 5

In the special case of an object whose center of mass is at rest, the spin theorem implies that the object’s angular momentum is the same regardless of what axis we choose. (This is an even stronger statement than the choice of axis theorem, which only guarantees that angular momentum is conserved for any given choice of axis, without specifying that it is the same for all such choices.)
Angular momentum of a rigid object example 6

A motorcycle wheel has almost all its mass concentrated at the outside. If the wheel has mass $m$ and radius $r$, and the time required for one revolution is $T$, what is the spin part of its angular momentum?

This is an example of the commonly encountered special case of rigid motion, as opposed to the rotation of a system like a hurricane in which the different parts take different amounts of time to go around. We don’t really have to go through a laborious process of adding up contributions from all the many parts of a wheel, because they are all at about the same distance from the axis, and are all moving around the axis at about the same speed. The velocity is all perpendicular to the spokes,

$$v_{\perp} = v = \frac{\text{(circumference)}}{T} = \frac{2\pi r}{T},$$

and the angular momentum of the wheel about its center is

$$L = mv_{\perp}r = m(2\pi r/T)r = 2\pi mr^2/T.$$

Note that although the factors of $2\pi$ in this expression is peculiar to a wheel with its mass concentrated on the rim, the proportionality to $m/T$ would have been the same for any other rigidly rotating object. Although an object with a noncircular shape does not have a radius, it is also true in general that angular momentum is proportional to the square of the object’s size for fixed values of $m$ and $T$. For instance doubling an object’s size doubles both the $v_{\perp}$ and $r$ factors in the contribution of each of its parts to the total angular momentum, resulting in an overall factor of four increase.

The figure shows some examples of angular momenta of various shapes rotating about their centers of mass. The equations for their angular momenta were derived using calculus, as described in my calculus-based book Simple Nature. Do not memorize these equations!

The hammer throw example 7

In the men’s Olympic hammer throw, a steel ball of radius 6.1 cm is swung on the end of a wire of length 1.22 m. What fraction of the ball’s angular momentum comes from its rotation, as opposed to its motion through space?

It’s always important to solve problems symbolically first, and plug in numbers only at the end, so let the radius of the ball be $b$,
Example 8.

and the length of the wire \( \ell \). If the time the ball takes to go once around the circle is \( T \), then this is also the time it takes to revolve once around its own axis. Its speed is \( v = \frac{2\pi \ell}{T} \), so its angular momentum due to its motion through space is \( mv\ell = \frac{2\pi m\ell^2}{T} \). Its angular momentum due to its rotation around its own center is \( \left(\frac{4\pi}{5}\right)mb^2/T \). The ratio of these two angular momenta is \( \left(\frac{2}{5}\right)\left(b/\ell\right)^2 = 1.0 \times 10^{-3} \). The angular momentum due to the ball’s spin is extremely small.

### Toppling a rod 

**A rod of length \( b \) and mass \( m \) stands upright. We want to strike the rod at the bottom, causing it to fall and land flat. Find the momentum, \( p \), that should be delivered, in terms of \( m \), \( b \), and \( g \). Can this really be done without having the rod scrape on the floor?**

**This is a nice example of a question that can very nearly be answered based only on units. Since the three variables, \( m \), \( b \), and \( g \), all have different units, they can’t be added or subtracted. The only way to combine them mathematically is by multiplication or division. Multiplying one of them by itself is exponentiation, so in general we expect that the answer must be of the form**

\[
p = Am^j b^k g^l,
\]

where \( A \), \( j \), \( k \), and \( l \) are unitless constants. The result has to have units of kg\cdot m/s. To get kilograms to the first power, we need

\[ j = 1, \]
meters to the first power requires

\[ k + l = 1, \]

and seconds to the power \(-1\) implies

\[ l = 1/2. \]

We find \( j = 1, \ k = 1/2, \) and \( l = 1/2, \) so the solution must be of the form

\[ p = Am\sqrt{bg}. \]

Note that no physics was required!

Consideration of units, however, won’t help us to find the unitless constant \( A. \) Let \( t \) be the time the rod takes to fall, so that \( (1/2)gt^2 = b/2. \) If the rod is going to land exactly on its side, then the number of revolutions it completes while in the air must be \( 1/4, \) or \( 3/4, \) or \( 5/4, \ldots, \) but all the possibilities greater than \( 1/4 \) would cause the head of the rod to collide with the floor prematurely. The rod must therefore rotate at a rate that would cause it to complete a full rotation in a time \( T = 4t, \) and it has angular momentum \( L = (\pi/6)mb^2/T. \)

The momentum lost by the object striking the rod is \( p, \) and by conservation of momentum, this is the amount of momentum, in the horizontal direction, that the rod acquires. In other words, the rod will fly forward a little. However, this has no effect on the solution to the problem. More importantly, the object striking the rod loses angular momentum \( bp/2, \) which is also transferred to the rod. Equating this to the expression above for \( L, \) we find

\[ p = (\pi/12)m\sqrt{bg}. \]

Finally, we need to know whether this can really be done without having the foot of the rod scrape on the floor. The figure shows that the answer is no for this rod of finite width, but it appears that the answer would be yes for a sufficiently thin rod. This is analyzed further in homework problem 28 on page 428.

Discussion question

A  In the example of the colliding asteroids, suppose planet A was moving toward the top of the page, at the same speed as the bottom asteroid. How would planet A’s astronomers describe the angular momenta of the asteroids? Would angular momentum still be conserved?

15.4 Torque: the rate of transfer of angular momentum

Force can be interpreted as the rate of transfer of momentum. The equivalent in the case of angular momentum is called torque (rhymes with “fork”). Where force tells us how hard we are pushing or pulling on something, torque indicates how hard we are twisting on
it. Torque is represented by the Greek letter tau, $\tau$, and the rate of change of an object’s angular momentum equals the total torque acting on it,

$$\tau_{\text{total}} = \frac{\Delta L}{\Delta t}.$$  

(If the angular momentum does not change at a constant rate, the total torque equals the slope of the tangent line on a graph of $L$ versus $t$.)

As with force and momentum, it often happens that angular momentum recedes into the background and we focus our interest on the torques. The torque-focused point of view is exemplified by the fact that many scientifically untrained but mechanically apt people know all about torque, but none of them have heard of angular momentum. Car enthusiasts eagerly compare engines’ torques, and there is a tool called a torque wrench which allows one to apply a desired amount of torque to a screw and avoid overtightening it.

**Torque distinguished from force**

Of course a force is necessary in order to create a torque — you can’t twist a screw without pushing on the wrench — but force and torque are two different things. One distinction between them is direction. We use positive and negative signs to represent forces in the two possible directions along a line. The direction of a torque, however, is clockwise or counterclockwise, not a linear direction.

The other difference between torque and force is a matter of leverage. A given force applied at a door’s knob will change the door’s angular momentum twice as rapidly as the same force applied halfway between the knob and the hinge. The same amount of force produces different amounts of torque in these two cases.

It is possible to have a zero total torque with a nonzero total force. An airplane with four jet engines, q, would be designed so that their forces are balanced on the left and right. Their forces are all in the same direction, but the clockwise torques of two of the engines are canceled by the counterclockwise torques of the other two, giving zero total torque.

Conversely we can have zero total force and nonzero total torque. A merry-go-round’s engine needs to supply a nonzero torque on it to bring it up to speed, but there is zero total force on it. If there was not zero total force on it, its center of mass would accelerate!
Relationship between force and torque

How do we calculate the amount of torque produced by a given force? Since it depends on leverage, we should expect it to depend on the distance between the axis and the point of application of the force. We'll derive an equation relating torque to force for a particular very simple situation, and state without proof that the equation actually applies to all situations.

To try to pin down this relationship more precisely, let’s imagine hitting a tetherball, figure r. The boy applies a force $F$ to the ball for a short time $\Delta t$, accelerating the ball from rest to a velocity $v$. Since force is the rate of transfer of momentum, we have

$$F = \frac{m\Delta v}{\Delta t}.$$ 

Since the initial velocity is zero, $\Delta v$ is the same as the final velocity $v$. Multiplying both sides by $r$ gives

$$Fr = \frac{mvr}{\Delta t}.$$ 

But $mvr$ is simply the amount of angular momentum he’s given the ball, so $mvr/\Delta t$ also equals the amount of torque he applied. The result of this example is

$$\tau = Fr.$$ 

Figure r was drawn so that the force $F$ was in the direction tangent to the circle, i.e., perpendicular to the radius $r$. If the boy had applied a force parallel to the radius line, either directly inward or outward, then the ball would not have picked up any clockwise or counterclockwise angular momentum.

If a force acts at an angle other than 0 or 90° with respect to the line joining the object and the axis, it would be only the component of the force perpendicular to the line that would produce a torque,

$$\tau = F_\perp r.$$ 

Although this result was proved under a simplified set of circumstances, it is more generally valid:

relationship between force and torque

The rate at which a force transfers angular momentum to an object, i.e., the torque produced by the force, is given by

$$|\tau| = r|F_\perp|,$$

where $r$ is the distance from the axis to the point of application of the force, and $F_\perp$ is the component of the force that is perpendicular to the line joining the axis to the point of application.
The quantity \( r \). The equation is stated with absolute value signs because the positive and negative signs of force and torque indicate different things, so there is no useful relationship between them. The sign of the torque must be found by physical inspection of the case at hand.

From the equation, we see that the units of torque can be written as newtons multiplied by meters. Metric torque wrenches are calibrated in N·m, but American ones use foot-pounds, which is also a unit of distance multiplied by a unit of force. We know from our study of mechanical work that newtons multiplied by meters equal joules, but torque is a completely different quantity from work, and nobody writes torques with units of joules, even though it would be technically correct.

**self-check A**

Compare the magnitudes and signs of the four torques shown in the figure. ▶ Answer, p. 567

How torque depends on the direction of the force example 9 ▶ How can the torque applied to the wrench in the figure be expressed in terms of \( r, |F|, \) and the angle \( \theta \) between these two vectors?

▶ The force vector and its \( F_\perp \) component form the hypotenuse and one leg of a right triangle,

and the interior angle opposite to \( F_\perp \) equals \( \theta \). The absolute value of \( F_\perp \) can thus be expressed as

\[
F_\perp = |F| \sin \theta,
\]

leading to

\[
|\tau| = r|F| \sin \theta.
\]

Section 15.4 Torque: the rate of transfer of angular momentum

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Sometimes torque can be more neatly visualized in terms of the quantity \( r_\perp \) shown in figure t, which gives us a third way of expressing the relationship between torque and force:

\[
|\tau| = r_\perp |F|.
\]

Of course you would not want to go and memorize all three equations for torque. Starting from any one of them you could easily derive the other two using trigonometry. Familiarizing yourself with them can however clue you in to easier avenues of attack on certain problems.

**The torque due to gravity**

Up until now we’ve been thinking in terms of a force that acts at a single point on an object, such as the force of your hand on the wrench. This is of course an approximation, and for an extremely realistic calculation of your hand’s torque on the wrench you might need to add up the torques exerted by each square millimeter where your skin touches the wrench. This is seldom necessary. But in the case of a gravitational force, there is never any single point at which the force is applied. Our planet is exerting a separate tug on every brick in the Leaning Tower of Pisa, and the total gravitational torque on the tower is the sum of the torques contributed by all the little forces. Luckily there is a trick that allows us to avoid such a massive calculation. It turns out that for purposes of computing the total gravitational torque on an object, you can get the right answer by just pretending that the whole gravitational force acts at the object’s center of mass.

**Gravitational torque on an outstretched arm example 10**

▷ Your arm has a mass of 3.0 kg, and its center of mass is 30 cm from your shoulder. What is the gravitational torque on your arm when it is stretched out horizontally to one side, taking the shoulder to be the axis?

▷ The total gravitational force acting on your arm is

\[
|F| = (3.0 \text{ kg})(9.8 \text{ m/s}^2) = 29 \text{ N}.
\]

For the purpose of calculating the gravitational torque, we can treat the force as if it acted at the arm’s center of mass. The force is straight down, which is perpendicular to the line connecting the shoulder to the center of mass, so

\[
F_\perp = |F| = 29 \text{ N}.
\]

Continuing to pretend that the force acts at the center of the arm, \( r \) equals 30 cm = 0.30 m, so the torque is

\[
\tau = rF_\perp = 9 \text{ N} \cdot \text{m}.
\]
In 2005, Dr. Margo Lillie and her graduate student Tracy Boeche-ler published a study claiming to debunk cow tipping. Their claim was based on an analysis of the torques that would be required to tip a cow, which showed that one person wouldn’t be able to make enough torque to do it. A lively discussion ensued on the popular web site slashdot.org (“news for nerds, stuff that matters”) concerning the validity of the study. Personally, I had always assumed that cow-tipping was a group sport anyway, but as a physicist, I also had some quibbles with their calculation. Here’s my own analysis.

There are three forces on the cow: the force of gravity $F_W$, the ground’s normal force $F_N$, and the tippers’ force $F_A$.

As soon as the cow’s left hooves (on the right from our point of view) break contact with the ground, the ground’s force is being applied only to hooves on the other side. We don’t know the ground’s force, and we don’t want to find it. Therefore we take the axis to be at its point of application, so that its torque is zero.

For the purpose of computing torques, we can pretend that gravity acts at the cow’s center of mass, which I’ve placed a little lower than the center of its torso, since its legs and head also have some mass, and the legs are more massive than the head and stick out farther, so they lower the c.m. more than the head raises it. The angle $\theta_W$ between the vertical gravitational force and the line $r_W$ is about 14°. (An estimate by Matt Semke at the University of Nebraska-Lincoln gives 20°, which is in the same ballpark.)

To generate the maximum possible torque with the least possible force, the tippers want to push at a point as far as possible from the axis, which will be the shoulder on the other side, and they want to push at a 90 degree angle with respect to the radius line $r_A$.

When the tippers are just barely applying enough force to raise the cow’s hooves on one side, the total torque has to be just slightly more than zero. (In reality, they want to push a lot harder than this — hard enough to impart a lot of angular momentum to the cow fair in a short time, before it gets mad and hurts them. We’re just trying to calculate the bare minimum force they can possibly use, which is the question that science can answer.) Setting the total torque equal to zero,

$$\tau_N + \tau_W + \tau_A = 0,$$

and letting counterclockwise torques be positive, we have

$$0 - mgr_W \sin \theta_W + F_A r_A \sin 90^\circ = 0$$
\[
F_A = \frac{r_W}{r_A} mg \sin \theta_W \\
\approx \frac{1}{1.5} (680 \text{ kg})(9.8 \text{ m/s}^2) \sin 14^\circ \\
= 1100 \text{ N}.
\]

The 680 kg figure for the typical mass of a cow is due to Lillie and Boechler, who are veterinarians, so I assume it’s fairly accurate. My estimate of 1100 N comes out significantly lower than their 1400 N figure, mainly because their incorrect placement of the center of mass gives \( \theta_W = 24^\circ \). I don’t think 1100 N is an impossible amount of force to require of one big, strong person (it’s equivalent to lifting about 110 kg, or 240 pounds), but given that the tippers need to impart a large angular momentum fairly quickly, it’s probably true that several people would be required.

The main practical issue with cow tipping is that cows generally sleep lying down. Falling on its side can also seriously injure a cow.

**Discussion questions**

**A** This series of discussion questions deals with past students’ incorrect reasoning about the following problem.

Suppose a comet is at the point in its orbit shown in the figure. The only force on the comet is the sun’s gravitational force.

Throughout the question, define all torques and angular momenta using the sun as the axis.

1. Is the sun producing a nonzero torque on the comet? Explain.
2. Is the comet’s angular momentum increasing, decreasing, or staying the same? Explain.

Explain what is wrong with the following answers. In some cases, the answer is correct, but the reasoning leading up to it is wrong.

- Incorrect answer to part (1): “Yes, because the sun is exerting a force on the comet, and the comet is a certain distance from the sun.”
- Incorrect answer to part (1): “No, because the torques cancel out.”
- Incorrect answer to part (2): “Increasing, because the comet is speeding up.”

**B** Which claw hammer would make it easier to get the nail out of the wood if the same force was applied in the same direction?

**C** You whirl a rock over your head on the end of a string, and gradually pull in the string, eventually cutting the radius in half. What happens to the rock’s angular momentum? What changes occur in its speed, the time required for one revolution, and its acceleration? Why might the string break?

**D** A helicopter has, in addition to the huge fan blades on top, a smaller propeller mounted on the tail that rotates in a vertical plane. Why?
The photo shows an amusement park ride whose two cars rotate in opposite directions. Why is this a good design?
15.5 Statics

Equilibrium

There are many cases where a system is not closed but maintains constant angular momentum. When a merry-go-round is running at constant angular momentum, the engine’s torque is being canceled by the torque due to friction.

When an object has constant momentum and constant angular momentum, we say that it is in equilibrium. This is a scientific redefinition of the common English word, since in ordinary speech nobody would describe a car spinning out on an icy road as being in equilibrium.

Very commonly, however, we are interested in cases where an object is not only in equilibrium but also at rest, and this corresponds more closely to the usual meaning of the word. Trees and bridges have been designed by evolution and engineers to stay at rest, and to do so they must have not just zero total force acting on them but zero total torque. It is not enough that they don’t fall down, they also must not tip over. Statics is the branch of physics concerned with problems such as these.

Solving statics problems is now simply a matter of applying and combining some things you already know:

- You know the behaviors of the various types of forces, for example that a frictional force is always parallel to the surface of contact.
- You know about vector addition of forces. It is the vector sum of the forces that must equal zero to produce equilibrium.
- You know about torque. The total torque acting on an object must be zero if it is to be in equilibrium.
- You know that the choice of axis is arbitrary, so you can make a choice of axis that makes the problem easy to solve.

In general, this type of problem could involve four equations in four unknowns: three equations that say the force components add up to zero, and one equation that says the total torque is zero. Most cases you’ll encounter will not be this complicated. In the following example, only the equation for zero total torque is required in order to get an answer.
Example 12.

The abstract sculpture shown in figure x contains a cube of mass $m$ and sides of length $b$. The cube rests on top of a cylinder, which is off-center by a distance $a$. Find the tension in the cable.

There are four forces on the cube: a gravitational force $mg$, the force $F_T$ from the cable, the upward normal force from the cylinder, $F_N$, and the horizontal static frictional force from the cylinder, $F_S$.

The total force on the cube in the vertical direction is zero:

$$F_N - mg = 0.$$  

As our axis for defining torques, let’s choose the center of the cube. The cable’s torque is counterclockwise, the torque due to $F_N$ clockwise. Letting counterclockwise torques be positive, and using the convenient equation $\tau = r \times F$, we find the equation for the total torque:

$$bF_T - aF_N = 0.$$  

We could also write down the equation saying that the total horizontal force is zero, but that would bring in the cylinder’s frictional force on the cube, which we don’t know and don’t need to find. We already have two equations in the two unknowns $F_T$ and $F_N$, so there’s no need to make it into three equations in three unknowns. Solving the first equation for $F_N = mg$, we then substitute into the second equation to eliminate $F_N$, and solve for $F_T = (a/b)mg$.

As a check, our result makes sense when $a = 0$; the cube is balanced on the cylinder, so the cable goes slack.

Example 13.

A 10-kg flagpole is being held up by a lightweight horizontal cable, and is propped against the foot of a wall as shown in the figure. If the cable is only capable of supporting a tension of 70 N, how great can the angle $\alpha$ be without breaking the cable?

All three objects in the figure are supposed to be in equilibrium: the pole, the cable, and the wall. Whichever of the three objects we pick to investigate, all the forces and torques on it have to cancel out. It is not particularly helpful to analyze the forces and torques on the wall, since it has forces on it from the ground that are not given and that we don’t want to find. We could study the forces and torques on the cable, but that doesn’t let us use the given information about the pole. The object we need to analyze is the pole.

The pole has three forces on it, each of which may also result in a torque: (1) the gravitational force, (2) the cable’s force, and (3) the wall’s force.

We are free to define an axis of rotation at any point we wish, and it is helpful to define it to lie at the bottom end of the pole, since
by that definition the wall’s force on the pole is applied at \( r = 0 \) and thus makes no torque on the pole. This is good, because we don’t know what the wall’s force on the pole is, and we are not trying to find it.

With this choice of axis, there are two nonzero torques on the pole, a counterclockwise torque from the cable and a clockwise torque from gravity. Choosing to represent counterclockwise torques as positive numbers, and using the equation \(|\tau| = r|F|\sin\theta\), we have

\[
|F_{\text{cable}}| \sin \theta_{\text{cable}} - |F_{\text{grav}}| \sin \theta_{\text{grav}} = 0.
\]

A little geometry gives \( \theta_{\text{cable}} = 90^\circ - \alpha \) and \( \theta_{\text{grav}} = \alpha \), so

\[
|F_{\text{cable}}| \sin(90^\circ - \alpha) - |F_{\text{grav}}| \sin \alpha = 0.
\]

The gravitational force can be considered as acting at the pole’s center of mass, i.e., at its geometrical center, so \( r_{\text{cable}} \) is twice \( r_{\text{grav}} \), and we can simplify the equation to read

\[
2|F_{\text{cable}}| \sin(90^\circ - \alpha) - |F_{\text{grav}}| \sin \alpha = 0.
\]

These are all quantities we were given, except for \( \alpha \), which is the angle we want to find. To solve for \( \alpha \) we need to use the trig identity \( \sin(90^\circ - x) = \cos x \),

\[
2|F_{\text{cable}}| \cos \alpha - |F_{\text{grav}}| \sin \alpha = 0,
\]

which allows us to find

\[
\tan \alpha = 2 \frac{|F_{\text{cable}}|}{|F_{\text{grav}}|}
\]

\[
\alpha = \tan^{-1} \left( 2 \frac{|F_{\text{cable}}|}{|F_{\text{grav}}|} \right)
\]

\[
= \tan^{-1} \left( 2 \times \frac{70 \text{ N}}{98 \text{ N}} \right)
\]

\[
= 55^\circ.
\]
Stable and unstable equilibria

A pencil balanced upright on its tip could theoretically be in equilibrium, but even if it was initially perfectly balanced, it would topple in response to the first air current or vibration from a passing truck. The pencil can be put in equilibrium, but not in stable equilibrium. The things around us that we really do see staying still are all in stable equilibrium.

Why is one equilibrium stable and another unstable? Try pushing your own nose to the left or the right. If you push it a millimeter to the left, your head responds with a gentle force to the right, which keeps your nose from flying off of your face. If you push your nose a centimeter to the left, your face’s force on your nose becomes much stronger. The defining characteristic of a stable equilibrium is that the farther the object is moved away from equilibrium, the stronger the force is that tries to bring it back.

The opposite is true for an unstable equilibrium. In the top figure, the ball resting on the round hill theoretically has zero total force on it when it is exactly at the top. But in reality the total force will not be exactly zero, and the ball will begin to move off to one side. Once it has moved, the net force on the ball is greater than it was, and it accelerates more rapidly. In an unstable equilibrium, the farther the object gets from equilibrium, the stronger the force that pushes it farther from equilibrium.

This idea can be rephrased in terms of energy. The difference between the stable and unstable equilibria shown in figure z is that in the stable equilibrium, the potential energy is at a minimum, and moving to either side of equilibrium will increase it, whereas the unstable equilibrium represents a maximum.

Note that we are using the term “stable” in a weaker sense than in ordinary speech. A domino standing upright is stable in the sense we are using, since it will not spontaneously fall over in response to a sneeze from across the room or the vibration from a passing truck. We would only call it unstable in the technical sense if it could be toppled by any force, no matter how small. In everyday usage, of course, it would be considered unstable, since the force required to topple it is so small.

An application of calculus

Nancy Neutron is living in a uranium nucleus that is undergoing fission. Nancy’s potential energy as a function of position can be approximated by $PE = x^4 - x^2$, where all the units and numerical constants have been suppressed for simplicity. Use calculus to locate the equilibrium points, and determine whether they are stable or unstable.

The equilibrium points occur where the PE is at a minimum or maximum, and minima and maxima occur where the derivative
(which equals minus the force on Nancy) is zero. This derivative is \( \frac{dPE}{dx} = 4x^3 - 2x \), and setting it equal to zero, we have \( x = 0, \pm 1/\sqrt{2} \). Minima occur where the second derivative is positive, and maxima where it is negative. The second derivative is \( 12x^2 - 2 \), which is negative at \( x = 0 \) (unstable) and positive at \( x = \pm 1/\sqrt{2} \) (stable). Interpretation: the graph of the PE is shaped like a rounded letter ‘W’, with the two troughs representing the two halves of the splitting nucleus. Nancy is going to have to decide which half she wants to go with.

15.6 Simple Machines: the lever

Although we have discussed some simple machines such as the pulley, without the concept of torque we were not yet ready to address the lever, which is the machine nature used in designing living things, almost to the exclusion of all others. (We can speculate what life on our planet might have been like if living things had evolved wheels, gears, pulleys, and screws.) The figures show two examples of levers within your arm. Different muscles are used to flex and extend the arm, because muscles work only by contraction.

Analyzing example ac physically, there are two forces that do work. When we lift a load with our biceps muscle, the muscle does positive work, because it brings the bone in the forearm in the direction it is moving. The load’s force on the arm does negative work, because the arm moves in the direction opposite to the load’s force. This makes sense, because we expect our arm to do positive work on the load, so the load must do an equal amount of negative work on the arm. (If the biceps was lowering a load, the signs of the works would be reversed. Any muscle is capable of doing either positive or negative work.)

There is also a third force on the forearm: the force of the upper arm’s bone exerted on the forearm at the elbow joint (not shown with an arrow in the figure). This force does no work, because the elbow joint is not moving.

Because the elbow joint is motionless, it is natural to define our torques using the joint as the axis. The situation now becomes quite simple, because the upper arm bone’s force exerted at the elbow neither does work nor creates a torque. We can ignore it completely. In any lever there is such a point, called the fulcrum.

If we restrict ourselves to the case in which the forearm rotates with constant angular momentum, then we know that the total torque on the forearm is zero,

\[ \tau_{\text{muscle}} + \tau_{\text{load}} = 0. \]

If we choose to represent counterclockwise torques as positive, then the muscle’s torque is positive, and the load’s is negative. In terms
of their absolute values,

\[ |\tau_{\text{muscle}}| = |\tau_{\text{load}}|. \]

Assuming for simplicity that both forces act at angles of 90° with respect to the lines connecting the axis to the points at which they act, the absolute values of the torques are

\[ r_{\text{muscle}}F_{\text{muscle}} = r_{\text{load}}F_{\text{arm}}, \]

where \( r_{\text{muscle}} \), the distance from the elbow joint to the biceps' point of insertion on the forearm, is only a few cm, while \( r_{\text{load}} \) might be 30 cm or so. The force exerted by the muscle must therefore be about ten times the force exerted by the load. We thus see that this lever is a force reducer. In general, a lever may be used either to increase or to reduce a force.

Why did our arms evolve so as to reduce force? In general, your body is built for compactness and maximum speed of motion rather than maximum force. This is the main anatomical difference between us and the Neanderthals (their brains covered the same range of sizes as those of modern humans), and it seems to have worked for us.

As with all machines, the lever is incapable of changing the amount of mechanical work we can do. A lever that increases force will always reduce motion, and vice versa, leaving the amount of work unchanged.

It is worth noting how simple and yet how powerful this analysis was. It was simple because we were well prepared with the concepts of torque and mechanical work. In anatomy textbooks, whose readers are assumed not to know physics, there is usually a long and complicated discussion of the different types of levers. For example, the biceps lever, ac, would be classified as a class III lever, since it has the fulcrum and load on the ends and the muscle's force acting in the middle. The triceps, ad, is called a class I lever, because the load and muscle's force are on the ends and the fulcrum is in the middle. How tiresome! With a firm grasp of the concept of torque, we realize that all such examples can be analyzed in much the same way. Physics is at its best when it lets us understand many apparently complicated phenomena in terms of a few simple yet powerful concepts.
15.7  *Proof of Kepler’s elliptical orbit law*

Kepler determined purely empirically that the planets’ orbits were ellipses, without understanding the underlying reason in terms of physical law. Newton’s proof of this fact based on his laws of motion and law of gravity was considered his crowning achievement both by him and by his contemporaries, because it showed that the same physical laws could be used to analyze both the heavens and the earth. Newton’s proof was very lengthy, but by applying the more recent concepts of conservation of energy and angular momentum we can carry out the proof quite simply and succinctly, and without calculus.

The basic idea of the proof is that we want to describe the shape of the planet’s orbit with an equation, and then show that this equation is exactly the one that represents an ellipse. Newton’s original proof had to be very complicated because it was based directly on his laws of motion, which include time as a variable. To make any statement about the shape of the orbit, he had to eliminate time from his equations, leaving only space variables. But conservation laws tell us that certain things don’t change over time, so they have already had time eliminated from them.

There are many ways of representing a curve by an equation, of which the most familiar is \( y = ax + b \) for a line in two dimensions. It would be perfectly possible to describe a planet’s orbit using an \( x - y \) equation like this, but remember that we are applying conservation of angular momentum, and the space variables that occur in the equation for angular momentum are the distance from the axis, \( r \), and the angle between the velocity vector and the \( r \) vector, which we will call \( \phi \). The planet will have \( \phi = 90^\circ \) when it is moving perpendicular to the \( r \) vector, i.e., at the moments when it is at its smallest or greatest distances from the sun. When \( \phi \) is less than \( 90^\circ \) the planet is approaching the sun, and when it is greater than \( 90^\circ \) it is receding from it. Describing a curve with an \( r - \phi \) equation is like telling a driver in a parking lot a certain rule for what direction to steer based on the distance from a certain streetlight in the middle of the lot.

The proof is broken into the three parts for easier digestion. The first part is a simple and intuitively reasonable geometrical fact about ellipses, whose proof we relegate to the caption of figure af; you will not be missing much if you merely absorb the result without reading the proof.

1. If we use one of the two foci of an ellipse as an axis for defining the variables \( r \) and \( \phi \), then the angle between the tangent line and the line drawn to the other focus is the same as \( \phi \), i.e., the two angles labeled \( \phi \) in figure af are in fact equal.

The other two parts form the meat of our proof. We state the
results first and then prove them.

(2) A planet, moving under the influence of the sun’s gravity with less than the energy required to escape, obeys an equation of the form

$$\sin \varphi = \frac{1}{\sqrt{-pr^2 + qr}}$$

where $p$ and $q$ are positive constants that depend on the planet’s energy and angular momentum.

(3) A curve is an ellipse if and only if its $r - \varphi$ equation is of the form

$$\sin \varphi = \frac{1}{\sqrt{-pr^2 + qr}}$$

where $p$ and $q$ are positive constants that depend on the size and shape of the ellipse.

**Proof of part (2)**

The component of the planet’s velocity vector that is perpendicular to the $r$ vector is $v_\perp = v \sin \varphi$, so conservation of angular momentum tells us that $L = mrv \sin \varphi$ is a constant. Since the planet’s mass is a constant, this is the same as the condition $rv \sin \varphi = \text{constant}$.

Conservation of energy gives

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}.$$  

We solve the first equation for $v$ and plug into the second equation to eliminate $v$. Straightforward algebra then leads to the equation claimed above, with the constant $p$ being positive because of our assumption that the planet’s energy is insufficient to escape from the sun, i.e., its total energy is negative.

**Proof of part (3)**

We define the quantities $\alpha$, $d$, and $s$ as shown in the figure. The law of cosines gives

$$d^2 = r^2 + s^2 - 2rs \cos \alpha.$$  

Using $\alpha = 180^\circ - 2\varphi$ and the trigonometric identities $\cos(180^\circ - x) = -\cos x$ and $\cos 2x = 1 - 2 \sin^2 x$, we can rewrite this as

$$d^2 = r^2 + s^2 - 2rs \left(2\sin^2 \varphi - 1\right).$$

Straightforward algebra transforms this into

$$\sin \varphi = \sqrt{\frac{(r + s)^2 - d^2}{4rs}}.$$  

Since $r + s$ is constant, the top of the fraction is constant, and the denominator can be rewritten as $4rs = 4r(\text{constant} - r)$, which is equivalent to the desired form.
Angular momentum is a measure of rotational motion which is conserved for a closed system. This book only discusses angular momentum for rotation of material objects in two dimensions. Not all rotation is rigid like that of a wheel or a spinning top. An example of nonrigid rotation is a cyclone, in which the inner parts take less time to complete a revolution than the outer parts. In order to define a measure of rotational motion general enough to include nonrigid rotation, we define the angular momentum of a system by dividing it up into small parts, and adding up all the angular momenta of the small parts, which we think of as tiny particles. We arbitrarily choose some point in space, the axis, and we say that anything that changes its direction relative to that point possesses angular momentum. The angular momentum of a single particle is

\[ L = mv_\perp r, \]

where \( v_\perp \) is the component of its velocity perpendicular to the line.
joining it to the axis, and \( r \) is its distance from the axis. Positive and negative signs of angular momentum are used to indicate clockwise and counterclockwise rotation.

The \textit{choice of axis theorem} states that any axis may be used for defining angular momentum. If a system’s angular momentum is constant for one choice of axis, then it is also constant for any other choice of axis.

The \textit{spin theorem} states that an object’s angular momentum with respect to some outside axis \( A \) can be found by adding up two parts:

(1) The first part is the object’s angular momentum found by using its own center of mass as the axis, i.e., the angular momentum the object has because it is spinning.

(2) The other part equals the angular momentum that the object would have with respect to the axis \( A \) if it had all its mass concentrated at and moving with its center of mass.

Torque is the rate of change of angular momentum. The torque a force can produce is a measure of its ability to twist on an object. The relationship between force and torque is

\[
|\tau| = r|F_\perp|,
\]

where \( r \) is the distance from the axis to the point where the force is applied, and \( F_\perp \) is the component of the force perpendicular to the line connecting the axis to the point of application. Statics problems can be solved by setting the total force and total torque on an object equal to zero and solving for the unknowns.
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 You are trying to loosen a stuck bolt on your RV using a big wrench that is 50 cm long. If you hang from the wrench, and your mass is 55 kg, what is the maximum torque you can exert on the bolt?

2 A physical therapist wants her patient to rehabilitate his injured elbow by laying his arm flat on a table, and then lifting a 2.1 kg mass by bending his elbow. In this situation, the weight is 33 cm from his elbow. He calls her back, complaining that it hurts him to grasp the weight. He asks if he can strap a bigger weight onto his arm, only 17 cm from his elbow. How much mass should she tell him to use so that he will be exerting the same torque? (He is raising his forearm itself, as well as the weight.)

3 An object thrown straight up in the air is momentarily at rest when it reaches the top of its motion. Does that mean that it is in equilibrium at that point? Explain.

4 An object is observed to have constant angular momentum. Can you conclude that no torques are acting on it? Explain. [Based on a problem by Serway and Faughn.]

5 A person of weight \( W \) stands on the ball of one foot. Find the tension in the calf muscle and the force exerted by the shinbones on the bones of the foot, in terms of \( W \), \( a \), and \( b \). For simplicity, assume that all the forces are at 90-degree angles to the foot, i.e., neglect the angle between the foot and the floor.

6 Two objects have the same momentum vector. Assume that they are not spinning; they only have angular momentum due to their motion through space. Can you conclude that their angular momenta are the same? Explain. [Based on a problem by Serway and Faughn.]
7 The sun turns on its axis once every 26.0 days. Its mass is $2.0 \times 10^{30}$ kg and its radius is $7.0 \times 10^{8}$ m. Assume it is a rigid sphere of uniform density.

(a) What is the sun's angular momentum? 

In a few billion years, astrophysicists predict that the sun will use up all its sources of nuclear energy, and will collapse into a ball of exotic, dense matter known as a white dwarf. Assume that its radius becomes $5.8 \times 10^6$ m (similar to the size of the Earth.) Assume it does not lose any mass between now and then. (Don't be fooled by the photo, which makes it look like nearly all of the star was thrown off by the explosion. The visually prominent gas cloud is actually thinner than the best laboratory vacuum ever produced on earth. Certainly a little bit of mass is actually lost, but it is not at all unreasonable to make an approximation of zero loss of mass as we are doing.)

(b) What will its angular momentum be? 

(c) How long will it take to turn once on its axis? 

8 A uniform ladder of mass $m$ and length $L$ leans against a smooth wall, making an angle $\theta$ with respect to the ground. The dirt exerts a normal force and a frictional force on the ladder, producing a force vector with magnitude $F_1$ at an angle $\phi$ with respect to the ground. Since the wall is smooth, it exerts only a normal force on the ladder; let its magnitude be $F_2$. 

(a) Explain why $\phi$ must be greater than $\theta$. No math is needed. 

(b) Choose any numerical values you like for $m$ and $L$, and show that the ladder can be in equilibrium (zero torque and zero total force vector) for $\theta = 45.00^\circ$ and $\phi = 63.43^\circ$. 

9 Continuing problem 8, find an equation for $\phi$ in terms of $\theta$, and show that $m$ and $L$ do not enter into the equation. Do not assume any numerical values for any of the variables. You will need the trig identity $\sin(a - b) = \sin a \cos b - \sin b \cos a$. (As a numerical check on your result, you may wish to check that the angles given in part b of the previous problem satisfy your equation.) 

10 (a) Find the minimum horizontal force which, applied at the axle, will pull a wheel over a step. Invent algebra symbols for whatever quantities you find to be relevant, and give your answer in symbolic form. [Hints: There are four forces on the wheel at first, but only three when it lifts off. Normal forces are always perpendicular to the surface of contact. Note that the corner of the step cannot be perfectly sharp, so the surface of contact for this force really coincides with the surface of the wheel.] 

(b) Under what circumstances does your result become infinite? Give a physical interpretation.
11. A yo-yo of total mass \( m \) consists of two solid cylinders of radius \( R \), connected by a small spindle of negligible mass and radius \( r \). The top of the string is held motionless while the string unrolls from the spindle. Show that the acceleration of the yo-yo is \( g/(1 + R^2/2r^2) \). [Hint: The acceleration and the tension in the string are unknown. Use \( \tau = \Delta L/\Delta t \) and \( F = ma \) to determine these two unknowns.]

12. A ball is connected by a string to a vertical post. The ball is set in horizontal motion so that it starts winding the string around the post. Assume that the motion is confined to a horizontal plane, i.e., ignore gravity. Michelle and Astrid are trying to predict the final velocity of the ball when it reaches the post. Michelle says that according to conservation of angular momentum, the ball has to speed up as it approaches the post. Astrid says that according to conservation of energy, the ball has to keep a constant speed. Who is right? [Hint: How is this different from the case where you whirl a rock in a circle on a string and gradually reel in the string?]

13. In the 1950’s, serious articles began appearing in magazines like Life predicting that world domination would be achieved by the nation that could put nuclear bombs in orbiting space stations, from which they could be dropped at will. In fact it can be quite difficult to get an orbiting object to come down. Let the object have energy \( E = KE + PE \) and angular momentum \( L \). Assume that the energy is negative, i.e., the object is moving at less than escape velocity. Show that it can never reach a radius less than

\[
r_{\text{min}} = \frac{GMm}{2E} \left( -1 + \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}} \right).
\]

[Note that both factors are negative, giving a positive result.]

14. [Problem 14 has been deleted.]

15. [Problem 15 has been deleted.]

16. Two bars of length \( L \) are connected with a hinge and placed on a frictionless cylinder of radius \( r \). (a) Show that the angle \( \theta \) shown in the figure is related to the unitless ratio \( r/L \) by the equation

\[
\frac{r}{L} = \frac{\cos^2 \theta}{2 \tan \theta}.
\]

(b) Discuss the physical behavior of this equation for very large and very small values of \( r/L \).

17. You wish to determine the mass of a ship in a bottle without taking it out. Show that this can be done with the setup shown in the figure, with a scale supporting the bottle at one end, provided that it is possible to take readings with the ship slid to several different locations. Note that you can’t determine the position of
the ship’s center of mass just by looking at it, and likewise for the bottle. In particular, you can’t just say, “position the ship right on top of the fulcrum” or “position it right on top of the balance.”

18 Two atoms will interact through electrical forces between their protons and electrons. One fairly good approximation to the potential energy is the Lennard-Jones formula,

\[ PE(r) = k \left[ \left( \frac{a}{r} \right)^{12} - 2 \left( \frac{a}{r} \right)^{6} \right], \]

where \( r \) is the center-to-center distance between the atoms and \( k \) is a positive constant. Show that (a) there is an equilibrium point at \( r = a \),
(b) the equilibrium is stable, and
(c) the energy required to bring the atoms from their equilibrium separation to infinity is \( k \).

19 Suppose that we lived in a universe in which Newton’s law of gravity gave forces proportional to \( r^{-7} \) rather than \( r^{-2} \). Which, if any, of Kepler’s laws would still be true? Which would be completely false? Which would be different, but in a way that could be calculated with straightforward algebra?

20 The figure shows scale drawing of a pair of pliers being used to crack a nut, with an appropriately reduced centimeter grid. Warning: do not attempt this at home; it is bad manners. If the force required to crack the nut is 300 N, estimate the force required of the person’s hand.

21 Show that a sphere of radius \( R \) that is rolling without slipping has angular momentum and momentum in the ratio \( L/p = (2/5)R \).

22 Suppose a bowling ball is initially thrown so that it has no angular momentum at all, i.e., it is initially just sliding down the lane. Eventually kinetic friction will get it spinning fast enough so that it is rolling without slipping. Show that the final velocity of the ball equals \( 5/7 \) of its initial velocity. [Hint: You’ll need the result of problem 21.]

23 The rod in the figure is supported by the finger and the string.
(a) Find the tension, \( T \), in the string, and the force, \( F \), from the finger, in terms of \( m, b, L \), and \( g \).
(b) Comment on the cases \( b = L \) and \( b = L/2 \).
(c) Are any values of \( b \) unphysical?

24 Two horizontal tree branches on the same tree have equal diameters, but one branch is twice as long as the other. Give a quantitative comparison of the torques where the branches join the trunk. [Thanks to Bong Kang.]
Problem 25. (a) Alice says Cathy’s body has zero momentum, but Bob says Cathy’s momentum is nonzero. Nobody is lying or making a mistake. How is this possible? Give a concrete example.
(b) Alice and Bob agree that Dong’s body has nonzero momentum, but disagree about Dong’s angular momentum, which Alice says is zero, and Bob says is nonzero. Explain.

Problem 26. Penguins are playful animals. Tux the Penguin invents a new game using a natural circular depression in the ice. He waddles at top speed toward the crater, aiming off to the side, and then hops into the air and lands on his belly just inside its lip. He then belly-surfs, moving in a circle around the rim. The ice is frictionless, so his speed is constant. Is Tux’s angular momentum zero, or nonzero? What about the total torque acting on him? Take the center of the crater to be the axis. Explain your answers.

Problem 27. Make a rough estimate of the mechanical advantage of the lever shown in the figure. In other words, for a given amount of force applied on the handle, how many times greater is the resulting force on the cork?

Problem 28. In example 8 on page 405, prove that if the rod is sufficiently thin, it can be toppled without scraping on the floor.

Problem 29. A massless rod of length ℓ has weights, each of mass m, attached to its ends. The rod is initially put in a horizontal position, and laid on an off-center fulcrum located at a distance b from the rod’s center. The rod will topple. (a) Calculate the total gravitational torque on the rod directly, by adding the two torques. (b) Verify that this gives the same result as would have been obtained by taking the entire gravitational force as acting at the center of mass.

Problem 30. A skilled motorcyclist can ride up a ramp, fly through the air, and land on another ramp. Why would it be useful for the rider to speed up or slow down the back wheel while in the air?

Problem 31. (a) The bar of mass m is attached at the wall with a hinge, and is supported on the right by a massless cable. Find the tension, T, in the cable in terms of the angle θ.√
(b) Interpreting your answer to part a, what would be the best angle to use if we wanted to minimize the strain on the cable?
(c) Again interpreting your answer to part a, for what angles does the result misbehave mathematically? Interpret this physically.
A disk starts from rest and rotates about a fixed axis, subject to a constant torque. The work done by the torque during the first revolution is $W$. What is the work done by the torque during the second revolution? $\sqrt{\text{[problem by B. Shotwell]}}$
Exercise 15: Torque

Equipment:

- rulers with holes in them
- spring scales (two per group)

While one person holds the pencil which forms the axle for the ruler, the other members of the group pull on the scale and take readings. In each case, calculate the total torque on the ruler, and find out whether it equals zero to roughly within the accuracy of the experiment. Finish the calculations for each part before moving on to the next one.
Chapter 16

Thermodynamics

This chapter is optional, and should probably be omitted from a two-semester survey course. It can be covered at any time after chapter 13.

In a developing country like China, a refrigerator is the mark of a family that has arrived in the middle class, and a car is the ultimate symbol of wealth. Both of these are heat engines: devices for converting between heat and other forms of energy. Unfortunately for the Chinese, neither is a very efficient device. Burning fossil fuels has made China’s big cities the most polluted on the planet, and the country’s total energy supply isn’t sufficient to support American levels of energy consumption by more than a small fraction of China’s population. Could we somehow manipulate energy in a more efficient way?

Conservation of energy is a statement that the total amount of energy is constant at all times, which encourages us to believe that any energy transformation can be undone — indeed, the laws of physics you’ve learned so far don’t even distinguish the past from the future. If you get in a car and drive around the block, the net effect is to consume some of the energy you paid for at the gas station, using it to heat the neighborhood. There would not seem to be any fundamental physical principle to prevent you from
recapturing all that heat and using it again the next time you want to go for a drive. More modestly, why don’t engineers design a car engine so that it recaptures the heat energy that would otherwise be wasted via the radiator and the exhaust?

Hard experience, however, has shown that designers of more and more efficient engines run into a brick wall at a certain point. The generators that the electric company uses to produce energy at an oil-fueled plant are indeed much more efficient than a car engine, but even if one is willing to accept a device that is very large, expensive, and complex, it turns out to be impossible to make a perfectly efficient heat engine — not just impossible with present-day technology, but impossible due to a set of fundamental physical principles known as the science of thermodynamics. And thermodynamics isn’t just a pesky set of constraints on heat engines. Without thermodynamics, there is no way to explain the direction of time’s arrow — why we can remember the past but not the future, and why it’s easier to break Humpty Dumpty than to put him back together again.

16.1 Pressure and temperature

When we heat an object, we speed up the mind-bogglingly complex random motion of its molecules. One method for taming complexity is the conservation laws, since they tell us that certain things must remain constant regardless of what process is going on. Indeed, the law of conservation of energy is also known as the first law of thermodynamics.

But as alluded to in the introduction to this chapter, conservation of energy by itself is not powerful enough to explain certain empirical facts about heat. A second way to sidestep the complexity of heat is to ignore heat’s atomic nature and concentrate on quantities like temperature and pressure that tell us about a system’s properties as a whole. This approach is called macroscopic in contrast to the microscopic method of attack. Pressure and temperature were fairly well understood in the age of Newton and Galileo, hundreds of years before there was any firm evidence that atoms and molecules even existed.

Unlike the conserved quantities such as mass, energy, momentum, and angular momentum, neither pressure nor temperature is additive. Two cups of coffee have twice the heat energy of a single cup, but they do not have twice the temperature. Likewise, the painful pressure on your eardrums at the bottom of a pool is not affected if you insert or remove a partition between the two halves of the pool.

Pressure

We restrict ourselves to a discussion of pressure in fluids at rest and in equilibrium. In physics, the term “fluid” is used to mean
either a gas or a liquid. The important feature of a fluid can be demonstrated by comparing with a cube of jello on a plate. The jello is a solid. If you shake the plate from side to side, the jello will respond by shearing, i.e., by slanting its sides, but it will tend to spring back into its original shape. A solid can sustain shear forces, but a fluid cannot. A fluid does not resist a change in shape unless it involves a change in volume.

If you’re at the bottom of a pool, you can’t relieve the pain in your ears by turning your head. The water’s force on your eardrum is always the same, and is always perpendicular to the surface where the eardrum contacts the water. If your ear is on the east side of your head, the water’s force is to the west. If you keep your head in the same spot while turning around so your ear is on the north, the force will still be the same in magnitude, and it will change its direction so that it is still perpendicular to the eardrum: south. This shows that pressure has no direction in space, i.e., it is a scalar. The direction of the force is determined by the orientation of the surface on which the pressure acts, not by the pressure itself. A fluid flowing over a surface can also exert frictional forces, which are parallel to the surface, but the present discussion is restricted to fluids at rest.

Experiments also show that a fluid’s force on a surface is proportional to the surface area. The vast force of the water behind a dam, for example, in proportion to the dam’s great surface area. (The bottom of the dam experiences a higher proportion of its force.)

Based on these experimental results, it appears that the useful way to define pressure is as follows. The pressure of a fluid at a given point is defined as \( \frac{F_{\perp}}{A} \), where \( A \) is the area of a small surface inserted in the fluid at that point, and \( F_{\perp} \) is the component of the fluid’s force on the surface which is perpendicular to the surface.

This is essentially how a pressure gauge works. The reason that the surface must be small is so that there will not be any significant difference in pressure between one part of it and another part. The SI units of pressure are evidently N/m\(^2\), and this combination can be abbreviated as the pascal, 1 Pa=1 N/m\(^2\). The pascal turns out to be an inconveniently small unit, so car tires, for example, have recommended pressures imprinted on them in units of kilopascals.

\[ \text{Pressure in U.S. units} \]

In U.S. units, the unit of force is the pound, and the unit of distance is the inch. The unit of pressure is therefore pounds per square inch, or p.s.i. (Note that the pound is not a unit of mass.)
Atmospheric pressure in U.S. and metric units example 2

A figure that many people in the U.S. remember is that atmospheric pressure is about 15 pounds per square inch. What is this in metric units?

\[
\begin{align*}
15 \text{ lb} & \quad \text{1 in}^2 \\
& \quad \frac{68 \text{ N}}{(0.0254 \text{ m})^2} \\
& \quad = 1.0 \times 10^5 \text{ N/m}^2 \\
& \quad = 100 \text{ kPa}
\end{align*}
\]

Only pressure differences are normally significant.

If you spend enough time on an airplane, the pain in your ears subsides. This is because your body has gradually been able to admit more air into the cavity behind the eardrum. Once the pressure inside is equalized with the pressure outside, the inward and outward forces on your eardrums cancel out, and there is no physical sensation to tell you that anything unusual is going on. For this reason, it is normally only pressure differences that have any physical significance. Thus deep-sea fish are perfectly healthy in their habitat because their bodies have enough internal pressure to cancel the pressure from the water in which they live; if they are caught in a net and brought to the surface rapidly, they explode because their internal pressure is so much greater than the low pressure outside.

Getting killed by a pool pump example 3

My house has a pool, which I maintain myself. A pool always needs to have its water circulated through a filter for several hours a day in order to keep it clean. The filter is a large barrel with a strong clamp that holds the top and bottom halves together. My filter has a prominent warning label that warns me not to try to open the clamps while the pump is on, and it shows a cartoon of a person being struck by the top half of the pump. The cross-sectional area of the filter barrel is 0.25 m\(^2\). Like most pressure gauges, the one on my pool pump actually reads the difference in pressure between the pressure inside the pump and atmospheric pressure. The gauge reads 90 kPa. What is the force that is trying to pop open the filter?

If the gauge told us the absolute pressure of the water inside, we’d have to find the force of the water pushing outward and the force of the air pushing inward, and subtract in order to find the total force. Since air surrounds us all the time, we would have to do such a subtraction every time we wanted to calculate anything useful based on the gauge’s reading. The manufacturers of the gauge decided to save us from all this work by making it read the difference in pressure between inside and outside, so all we have
to do is multiply the gauge reading by the cross-sectional area of the filter:

\[
F = PA
\]
\[
= (90 \times 10^3 \text{ N/m}^2)(0.25 \text{ m}^2)
\]
\[
= 22000 \text{ N}
\]

That's a lot of force!

The word "suction" and other related words contain a hidden misunderstanding related to this point about pressure differences. When you suck water up through a straw, there is nothing in your mouth that is attracting the water upward. The force that lifts the water is from the pressure of the water in the cup. By creating a partial vacuum in your mouth, you decreased the air's downward force on the water so that it no longer exactly canceled the upward force.

### Variation of pressure with depth

The pressure within a fluid in equilibrium can only depend on depth, due to gravity. If the pressure could vary from side to side, then a piece of the fluid in between, b, would be subject to unequal forces from the parts of the fluid on its two sides. But fluids do not exhibit shear forces, so there would be no other force that could keep this piece of fluid from accelerating. This contradicts the assumption that the fluid was in equilibrium.

**self-check A**

How does this proof fail for solids?  

To find the variation with depth, we consider the vertical forces acting on a tiny, imaginary cube of the fluid having height \( \Delta y \) and areas \( dA \) on the top and bottom. Using positive numbers for upward forces, we have

\[
P_{\text{bottom}} \Delta A - P_{\text{top}} \Delta A - F_g = 0.
\]

The weight of the fluid is \( F_g = mg = \rho V g = \rho \Delta A \Delta y g \), where \( \rho \) is the density of the fluid, so the difference in pressure is

\[
\Delta P = -\rho g \Delta y.
\]

[variation in pressure with depth for a fluid of density \( \rho \) in equilibrium; positive \( y \) is up.]

The factor of \( \rho \) explains why we notice the difference in pressure when diving 3 m down in a pool, but not when going down 3 m of stairs. Note also that the equation only tells us the difference in pressure, not the absolute pressure. The pressure at the surface of a swimming pool equals the atmospheric pressure, not zero, even though the depth is zero at the surface. The blood in your body does not even have an upper surface.
Pressure of lava underneath a volcano example 4

A volcano has just finished erupting, and a pool of molten lava is lying at rest in the crater. The lava has come up through an opening inside the volcano that connects to the earth's molten mantle. The density of the lava is 4.1 g/cm³. What is the pressure in the lava underneath the base of the volcano, 3000 m below the surface of the pool?

\[ \Delta P = \rho g \Delta y \]
\[ = (4.1 \text{ g/cm}^3)(9.8 \text{ m/s}^2)(3000 \text{ m}) \]
\[ = (4.1 \times 10^6 \text{ g/m}^3)(9.8 \text{ m/s}^2)(3000 \text{ m}) \]
\[ = (4.1 \times 10^3 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(3000 \text{ m}) \]
\[ = 1.2 \times 10^8 \text{ N/m}^2 \]
\[ = 1.2 \times 10^8 \text{ Pa} \]

This is the difference between the pressure we want to find and atmospheric pressure at the surface. The latter, however, is tiny compared to the \( \Delta P \) we just calculated, so what we’ve found is essentially the pressure, \( P \).

Atmospheric pressure example 5

This example uses calculus.

Gases, unlike liquids, are quite compressible, and at a given temperature, the density of a gas is approximately proportional to the pressure. The proportionality constant is discussed in section 16.2, but for now let’s just call it \( k \), \( \rho = kP \). Using this fact, we can find the variation of atmospheric pressure with altitude, assuming constant temperature:

\[ \frac{dP}{P} = -kg \, dy \]
\[ \ln P = -kgy + \text{constant} \]
\[ P = \text{(constant)} e^{-kgy} \]

Pressure falls off exponentially with height. There is no sharp cutoff to the atmosphere, but the exponential gets extremely small by the time you’re ten or a hundred miles up.

Temperature

Thermal equilibrium

We use the term temperature casually, but what is it exactly? Roughly speaking, temperature is a measure of how concentrated the heat energy is in an object. A large, massive object with very little heat energy in it has a low temperature.
But physics deals with operational definitions, i.e., definitions of how to measure the thing in question. How do we measure temperature? One common feature of all temperature-measuring devices is that they must be left for a while in contact with the thing whose temperature is being measured. When you take your temperature with a fever thermometer, you wait for the mercury inside to come up to the same temperature as your body. The thermometer actually tells you the temperature of its own working fluid (in this case the mercury). In general, the idea of temperature depends on the concept of thermal equilibrium. When you mix cold eggs from the refrigerator with flour that has been at room temperature, they rapidly reach a compromise temperature. What determines this compromise temperature is conservation of energy, and the amount of energy required to heat or cool each substance by one degree. But without even having constructed a temperature scale, we can see that the important point is the phenomenon of thermal equilibrium itself: two objects left in contact will approach the same temperature. We also assume that if object A is at the same temperature as object B, and B is at the same temperature as C, then A is at the same temperature as C. This statement is sometimes known as the zeroth law of thermodynamics, so called because after the first, second, and third laws had been developed, it was realized that there was another law that was even more fundamental.

**Thermal expansion**

The familiar mercury thermometer operates on the principle that the mercury, its working fluid, expands when heated and contracts when cooled. In general, all substances expand and contract with changes in temperature. The zeroth law of thermodynamics guarantees that we can construct a comparative scale of temperatures that is independent of what type of thermometer we use. If a thermometer gives a certain reading when it’s in thermal equilibrium with object A, and also gives the same reading for object B, then A and B must be the same temperature, regardless of the details of how the thermometers works.

What about constructing a temperature scale in which every degree represents an equal step in temperature? The Celsius scale has 0 as the freezing point of water and 100 as its boiling point. The hidden assumption behind all this is that since two points define a line, any two thermometers that agree at two points must agree at all other points. In reality if we calibrate a mercury thermometer and an alcohol thermometer in this way, we will find that a graph of one thermometer’s reading versus the other is not a perfectly straight $y = x$ line. The subtle inconsistency becomes a drastic one when we try to extend the temperature scale through the points where mercury and alcohol boil or freeze. Gases, however, are much more consistent among themselves in their thermal expansion than...
A simplified version of an ideal gas thermometer. The whole instrument is allowed to come into thermal equilibrium with the substance whose temperature is to be measured, and the mouth of the cylinder is left open to standard pressure. The volume of the noble gas gives an indication of temperature.

The volume of 1 kg of neon gas as a function of temperature (at standard pressure). Although neon would actually condense into a liquid at some point, extrapolating the graph to zero volume gives the same temperature as for any other gas: absolute zero.

Absolute zero and the kelvin scale

We find that if we extrapolate a graph of volume versus temperature, the volume becomes zero at nearly the same temperature for all gases: −273°C. Real gases will all condense into liquids at some temperature above this, but an ideal gas would achieve zero volume at this temperature, known as absolute zero. The most useful temperature scale in scientific work is one whose zero is defined by absolute zero, rather than by some arbitrary standard like the melting point of water. The ideal temperature scale for scientific work, called the Kelvin scale, is the same as the Celsius scale, but shifted by 273 degrees to make its zero coincide with absolute zero. Scientists use the Celsius scale only for comparisons or when a change in temperature is all that is required for a calculation. Only on the Kelvin scale does it make sense to discuss ratios of temperatures, e.g., to say that one temperature is twice as hot as another.

Which temperature scale to use example 6

You open an astronomy book and encounter the equation

\[(\text{light emitted}) = (\text{constant}) \times T^4\]

for the light emitted by a star as a function of its surface temperature. What temperature scale is implied?

The equation tells us that doubling the temperature results in the emission of 16 times as much light. Such a ratio only makes sense if the Kelvin scale is used.
16.2 Microscopic description of an ideal gas

Evidence for the kinetic theory

Why does matter have the thermal properties it does? The basic answer must come from the fact that matter is made of atoms. How, then, do the atoms give rise to the bulk properties we observe? Gases, whose thermal properties are so simple, offer the best chance for us to construct a simple connection between the microscopic and macroscopic worlds.

A crucial observation is that although solids and liquids are nearly incompressible, gases can be compressed, as when we increase the amount of air in a car’s tire while hardly increasing its volume at all. This makes us suspect that the atoms in a solid are packed shoulder to shoulder, while a gas is mostly vacuum, with large spaces between molecules. Most liquids and solids have densities about 1000 times greater than most gases, so evidently each molecule in a gas is separated from its nearest neighbors by a space something like 10 times the size of the molecules themselves.

If gas molecules have nothing but empty space between them, why don’t the molecules in the room around you just fall to the floor? The only possible answer is that they are in rapid motion, continually rebounding from the walls, floor and ceiling. In chapter 12, we have already seen some of the evidence for the kinetic theory of heat, which states that heat is the kinetic energy of randomly moving molecules. This theory was proposed by Daniel Bernoulli in 1738, and met with considerable opposition, because there was no precedent for this kind of perpetual motion. No rubber ball, however elastic, rebounds from a wall with exactly as much energy as it originally had, nor do we ever observe a collision between balls in which none of the kinetic energy at all is converted to heat and sound. The analogy is a false one, however. A rubber ball consists of atoms, and when it is heated in a collision, the heat is a form of motion of those atoms. An individual molecule, however, cannot possess heat. Likewise sound is a form of bulk motion of molecules, so colliding molecules in a gas cannot convert their kinetic energy to sound. Molecules can indeed induce vibrations such as sound waves when they strike the walls of a container, but the vibrations of the walls are just as likely to impart energy to a gas molecule as to take energy from it. Indeed, this kind of exchange of energy is the mechanism by which the temperatures of the gas and its container become equilibrated.

Pressure, volume, and temperature

A gas exerts pressure on the walls of its container, and in the kinetic theory we interpret this apparently constant pressure as the averaged-out result of vast numbers of collisions occurring every second between the gas molecules and the walls. The empirical
facts about gases can be summarized by the relation

\[ PV \propto nT, \quad \text{[ideal gas]} \]

which really only holds exactly for an ideal gas. Here \( n \) is the number of molecules in the sample of gas.

\textit{Volume related to temperature} \hspace{1cm} \textit{example 7}

The proportionality of volume to temperature at fixed pressure was the basis for our definition of temperature.

\textit{Pressure related to temperature} \hspace{1cm} \textit{example 8}

Pressure is proportional to temperature when volume is held constant. An example is the increase in pressure in a car’s tires when the car has been driven on the freeway for a while and the tires and air have become hot.

We now connect these empirical facts to the kinetic theory of a classical ideal gas. For simplicity, we assume that the gas is monoatomic (i.e., each molecule has only one atom), and that it is confined to a cubical box of volume \( V \), with \( L \) being the length of each edge and \( A \) the area of any wall. An atom whose velocity has an \( x \) component \( v_x \) will collide regularly with the left-hand wall, traveling a distance \( 2L \) parallel to the \( x \) axis between collisions with that wall. The time between collisions is \( \Delta t = 2L/v_x \), and in each collision the \( x \) component of the atom’s momentum is reversed from \(-mv_x\) to \( mv_x\). The total force on the wall is

\[ F = \frac{\Delta p_{x,1}}{\Delta t_1} + \frac{\Delta p_{x,2}}{\Delta t_2} + \ldots \quad \text{[monoatomic ideal gas]}, \]

where the indices 1, 2, \ldots refer to the individual atoms. Substituting \( \Delta p_{x,i} = 2mv_{x,i} \) and \( \Delta t_i = 2L/v_{x,i} \), we have

\[ F = \frac{mv_{x,1}^2}{L} + \frac{mv_{x,2}^2}{L} + \ldots \quad \text{[monoatomic ideal gas]}. \]

The quantity \( mv_{x,i}^2 \) is twice the contribution to the kinetic energy from the part of the atom’s center of mass motion that is parallel to the \( x \) axis. Since we’re assuming a monoatomic gas, center of mass motion is the only type of motion that gives rise to kinetic energy. (A more complex molecule could rotate and vibrate as well.) If the quantity inside the sum included the \( y \) and \( z \) components, it would be twice the total kinetic energy of all the molecules. By symmetry, it must therefore equal \( 2/3 \) of the total kinetic energy, so

\[ F = \frac{2KE_{\text{total}}}{3L} \quad \text{[monoatomic ideal gas]}. \]

Dividing by \( A \) and using \( AL = V \), we have

\[ P = \frac{2KE_{\text{total}}}{3V} \quad \text{[monoatomic ideal gas]}. \]
This can be connected to the empirical relation \( PV \propto nT \) if we multiply by \( V \) on both sides and rewrite \( KE_{\text{total}} \) as \( nKE_{\text{av}} \), where \( KE_{\text{av}} \) is the average kinetic energy per molecule:

\[
P V = \frac{2}{3} nKE_{\text{av}} \quad \text{[monoatomic ideal gas]}
\]

For the first time we have an interpretation for the temperature based on a microscopic description of matter: in a monoatomic ideal gas, the temperature is a measure of the average kinetic energy per molecule. The proportionality between the two is \( KE_{\text{av}} = (3/2)kT \), where the constant of proportionality \( k \), known as Boltzmann’s constant, has a numerical value of \( 1.38 \times 10^{-23} \text{ J/K} \). In terms of Boltzmann’s constant, the relationship among the bulk quantities for an ideal gas becomes

\[
P V = nkT, \quad \text{[ideal gas]}
\]

which is known as the ideal gas law. Although I won’t prove it here, this equation applies to all ideal gases, even though the derivation assumed a monoatomic ideal gas in a cubical box. (You may have seen it written elsewhere as \( PV = NRT \), where \( N = n/N_A \) is the number of moles of atoms, \( R = kN_A \), and \( N_A = 6.0 \times 10^{23} \), called Avogadro’s number, is essentially the number of hydrogen atoms in 1 g of hydrogen.)

\[\text{Pressure in a car tire} \quad \text{example 9}\]

After driving on the freeway for a while, the air in your car’s tires heats up from 10°C to 35°C. How much does the pressure increase?

The tires may expand a little, but we assume this effect is small, so the volume is nearly constant. From the ideal gas law, the ratio of the pressures is the same as the ratio of the absolute temperatures,

\[
P_2/P_1 = T_2/T_1
\]

\[
= (308 \text{ K})/(283 \text{ K})
\]

\[= 1.09,
\]

or a 9% increase.

\[\text{Earth’s senescence} \quad \text{example 10}\]

Microbes were the only life on Earth up until the relatively recent advent of multicellular life, and are arguably still the dominant form of life on our planet. Furthermore, the sun has been gradually heating up ever since it first formed, and this continuing process will soon (“soon” in the sense of geological time) eliminate multicellular life again. Heat-induced decreases in the atmosphere’s \( \text{CO}_2 \) content will kill off all complex plants within about
500 million years, and although some animals may be able to live by eating algae, it will only be another few hundred million years at most until the planet is completely heat-sterilized.

Why is the sun getting brighter? The only thing that keeps a star like our sun from collapsing due to its own gravity is the pressure of its gases. The sun’s energy comes from nuclear reactions at its core, and the net result of these reactions is to fuse hydrogen atoms into helium atoms. It takes four hydrogens to make one helium, so the number of atoms in the sun is continuously decreasing. Since $PV = nkT$, this causes a decrease in pressure, which makes the core contract. As the core contracts, collisions between hydrogen atoms become more frequent, and the rate of fusion reactions increases.

A piston, a refrigerator, and a space suit example 11

Both sides of the equation $PV = nkT$ have units of energy. Suppose the pressure in a cylinder of gas pushes a piston out, as in the power stroke of an automobile engine. Let the cross-sectional area of the piston and cylinder be $A$, and let the piston travel a small distance $\Delta x$. Then the gas’s force on the piston $F = PA$ does an amount of mechanical work $W = F\Delta x = P\Delta x = P\Delta V$, where $\Delta V$ is the change in volume. This energy has to come from somewhere; it comes from cooling the gas. In a car, what this means is that we’re harvesting the energy released by burning the gasoline.

In a refrigerator, we use the same process to cool the gas, which then cools the food.

In a space suit, the quantity $P\Delta V$ represents the work the astronaut has to do because bending her limbs changes the volume of the suit. The suit inflates under pressure like a balloon, and doesn’t want to bend. This makes it very tiring to work for any significant period of time.
16.3 Entropy

Efficiency and grades of energy

Some forms of energy are more convenient than others in certain situations. You can’t run a spring-powered mechanical clock on a battery, and you can’t run a battery-powered clock with mechanical energy. However, there is no fundamental physical principle that prevents you from converting 100% of the electrical energy in a battery into mechanical energy or vice-versa. More efficient motors and generators are being designed every year. In general, the laws of physics permit perfectly efficient conversion within a broad class of forms of energy.

Heat is different. Friction tends to convert other forms of energy into heat even in the best lubricated machines. When we slide a book on a table, friction brings it to a stop and converts all its kinetic energy into heat, but we never observe the opposite process, in which a book spontaneously converts heat energy into mechanical energy and starts moving! Roughly speaking, heat is different because it is disorganized. Scrambling an egg is easy. Unscrambling it is harder.

We summarize these observations by saying that heat is a lower grade of energy than other forms such as mechanical energy.

Of course it is possible to convert heat into other forms of energy such as mechanical energy, and that is what a car engine does with the heat created by exploding the air-gasoline mixture. But a car engine is a tremendously inefficient device, and a great deal of the heat is simply wasted through the radiator and the exhaust. Engineers have never succeeded in creating a perfectly efficient device for converting heat energy into mechanical energy, and we now know that this is because of a deeper physical principle that is far more basic than the design of an engine.

Heat engines

Heat may be more useful in some forms than in other, i.e., there are different grades of heat energy. In figure k, the difference in temperature can be used to extract mechanical work with a fan blade. This principle is used in power plants, where steam is heated by burning oil or by nuclear reactions, and then allowed to expand through a turbine which has cooler steam on the other side. On a smaller scale, there is a Christmas toy that consists of a small propeller spun by the hot air rising from a set of candles, very much like the setup shown in the figure.

In figure l, however, no mechanical work can be extracted because there is no difference in temperature. Although the air in l has the same total amount of energy as the air in k, the heat in l is a lower grade of energy, since none of it is accessible for doing...
In general, we define a heat engine as any device that takes heat from a reservoir of hot matter, extracts some of the heat energy to do mechanical work, and expels a lesser amount of heat into a reservoir of cold matter. The efficiency of a heat engine equals the amount of useful work extracted, $W$, divided by the amount of energy we had to pay for in order to heat the hot reservoir. This latter amount of heat is the same as the amount of heat the engine extracts from the high-temperature reservoir, $Q_H$. (The letter $Q$ is the standard notation for a transfer of heat.) By conservation of energy, we have $Q_H = W + Q_L$, where $Q_L$ is the amount of heat expelled into the low-temperature reservoir, so the efficiency of a heat engine, $W/Q_H$, can be rewritten as

$$\text{efficiency} = 1 - \frac{Q_L}{Q_H}. \quad \text{[efficiency of any heat engine]}$$

It turns out that there is a particular type of heat engine, the Carnot engine, which, although not 100% efficient, is more efficient than any other. The grade of heat energy in a system can thus be unambiguously defined in terms of the amount of heat energy in it that cannot be extracted, even by a Carnot engine.

How can we build the most efficient possible engine? Let’s start with an unnecessarily inefficient engine like a car engine and see how it could be improved. The radiator and exhaust expel hot gases, which is a waste of heat energy. These gases are cooler than the exploded air-gas mixture inside the cylinder, but hotter than the air that surrounds the car. We could thus improve the engine’s efficiency by adding an auxiliary heat engine to it, which would operate with the first engine’s exhaust as its hot reservoir and the air as its cold reservoir. In general, any heat engine that expels heat at an intermediate temperature can be made more efficient by changing it so that it expels heat only at the temperature of the cold reservoir.

Similarly, any heat engine that absorbs some energy at an intermediate temperature can be made more efficient by adding an auxiliary heat engine to it which will operate between the hot reservoir and this intermediate temperature.

Based on these arguments, we define a Carnot engine as a heat engine that absorbs heat only from the hot reservoir and expels it only into the cold reservoir. Figures m-p show a realization of a Carnot engine using a piston in a cylinder filled with a monoatomic ideal gas. This gas, known as the working fluid, is separate from, but exchanges energy with, the hot and cold reservoirs. It turns out that this particular Carnot engine has an efficiency given by

$$\text{efficiency} = 1 - \frac{T_L}{T_H}, \quad \text{[efficiency of a Carnot engine]}$$
Entropy can be understood using the metaphor of a water wheel. Letting the water levels equalize is like letting the entropy maximize. Taking water from the high side and putting it into the low side increases the entropy. Water levels in this metaphor correspond to temperatures in the actual definition of entropy.

where \( T_L \) is the temperature of the cold reservoir and \( T_H \) is the temperature of the hot reservoir. (A proof of this fact is given in my book *Simple Nature*, which you can download for free.)

Even if you do not wish to dig into the details of the proof, the basic reason for the temperature dependence is not so hard to understand. Useful mechanical work is done on strokes m and n, in which the gas expands. The motion of the piston is in the same direction as the gas’s force on the piston, so positive work is done on the piston. In strokes o and p, however, the gas does negative work on the piston. We would like to avoid this negative work, but we must design the engine to perform a complete cycle. Luckily the pressures during the compression strokes are lower than the ones during the expansion strokes, so the engine doesn’t undo all its work with every cycle. The ratios of the pressures are in proportion to the ratios of the temperatures, so if \( T_L \) is 20% of \( T_H \), the engine is 80% efficient.

We have already proved that any engine that is not a Carnot engine is less than optimally efficient, and it is also true that all Carnot engines operating between a given pair of temperatures \( T_H \) and \( T_L \) have the same efficiency. Thus a Carnot engine is the most efficient possible heat engine.

**Entropy**

We would like to have some numerical way of measuring the grade of energy in a system. We want this quantity, called entropy, to have the following two properties:

(1) Entropy is additive. When we combine two systems and consider them as one, the entropy of the combined system equals the sum of the entropies of the two original systems. (Quantities like mass and energy also have this property.)

(2) The entropy of a system is not changed by operating a Carnot engine within it.

It turns out to be simpler and more useful to define changes in entropy than absolute entropies. Suppose as an example that a system contains some hot matter and some cold matter. It has a relatively high grade of energy because a heat engine could be used to extract mechanical work from it. But if we allow the hot and cold parts to equilibrate at some lukewarm temperature, the grade of energy has gotten worse. Thus putting heat into a hotter area is more useful than putting it into a cold area. Motivated by these considerations, we define a change in entropy as follows:

\[
\Delta S = \frac{Q}{T} \quad \text{[change in entropy when adding heat } Q \text{ to matter at temperature } T; \quad \Delta S \text{ is negative if heat is taken out]}
\]
A system with a higher grade of energy has a lower entropy.

Entropy is additive. example 12
Since changes in entropy are defined by an additive quantity (heat) divided by a non-additive one (temperature), entropy is additive.

Entropy isn’t changed by a Carnot engine. example 13
The efficiency of a heat engine is defined by

\[
\text{efficiency} = 1 - \frac{Q_L}{Q_H},
\]

and the efficiency of a Carnot engine is

\[
\text{efficiency} = 1 - \frac{T_L}{T_H},
\]

so for a Carnot engine we have \( \frac{Q_L}{Q_H} = \frac{T_L}{T_H} \), which can be rewritten as \( \frac{Q_L}{T_L} = \frac{Q_H}{T_H} \). The entropy lost by the hot reservoir is therefore the same as the entropy gained by the cold one.

Entropy increases in heat conduction. example 14
When a hot object gives up energy to a cold one, conservation of energy tells us that the amount of heat lost by the hot object is the same as the amount of heat gained by the cold one. The change in entropy is \(-\frac{Q}{T_H} + \frac{Q}{T_L}\), which is positive because \( T_L < T_H \).

Entropy is increased by a non-Carnot engine. example 15
The efficiency of a non-Carnot engine is less than \( 1 - \frac{T_L}{T_H} \), so \( \frac{Q_L}{Q_H} > \frac{T_L}{T_H} \) and \( \frac{Q_L}{T_L} > \frac{Q_H}{T_H} \). This means that the entropy increase in the cold reservoir is greater than the entropy decrease in the hot reservoir.

A book sliding to a stop example 16
A book slides across a table and comes to a stop. Once it stops, all its kinetic energy has been transformed into heat. As the book and table heat up, their entropies both increase, so the total entropy increases as well.

Examples 14-16 involved closed systems, and in all of them the total entropy either increased or stayed the same. It never decreased. Here are two examples of schemes for decreasing the entropy of a closed system, with explanations of why they don’t work.

Using a refrigerator to decrease entropy? example 17
A refrigerator takes heat from a cold area and dumps it into a hot area. (1) Does this lead to a net decrease in the entropy of a closed system? (2) Could you make a Carnot engine more efficient by running a refrigerator to cool its low-temperature reservoir and eject heat into its high-temperature reservoir?

(1) No. The heat that comes off of the radiator coils on the back of your kitchen fridge is a great deal more than the heat the fridge removes from inside; the difference is what it costs to run your fridge. The heat radiated from the coils is so much more
than the heat removed from the inside that the increase in the entropy of the air in the room is greater than the decrease of the entropy inside the fridge. The most efficient refrigerator is actually a Carnot engine running in reverse, which leads to neither an increase nor a decrease in entropy.

(2) No. The most efficient refrigerator is a reversed Carnot engine. You will not achieve anything by running one Carnot engine in reverse and another forward. They will just cancel each other out.

Maxwell’s daemon example 18

Physicist James Clerk Maxwell imagined pair of neighboring rooms, their air being initially in thermal equilibrium, having a partition across the middle with a tiny door. A miniscule daemon is posted at the door with a little ping-pong paddle, and his duty is to try to build up faster-moving air molecules in room B and slower ones in room A. For instance, when a fast molecule is headed through the door, going from A to B, he lets it by, but when a slower than average molecule tries the same thing, he hits it back into room A. Would this decrease the total entropy of the pair of rooms?

No. The daemon needs to eat, and we can think of his body as a little heat engine. His metabolism is less efficient than a Carnot engine, so he ends up increasing the entropy rather than decreasing it.

Observation such as these lead to the following hypothesis, known as the second law of thermodynamics:

\[ \Delta S \geq 0. \]

At present my arguments to support this statement may seem less than convincing, since they have so much to do with obscure facts about heat engines. A more satisfying and fundamental explanation for the continual increase in entropy was achieved by Ludwig Boltzmann, and you may wish to learn more about Boltzmann’s ideas from my book Simple Nature, which you can download for free. Briefly, Boltzmann realized that entropy was a measure of randomness or disorder at the atomic level, and disorder doesn’t spontaneously change into order.

To emphasize the fundamental and universal nature of the second law, here are a few examples.

Entropy and evolution example 19

A favorite argument of many creationists who don’t believe in evolution is that evolution would violate the second law of thermodynamics: the death and decay of a living thing releases heat (as
when a compost heap gets hot) and lessens the amount of energy available for doing useful work, while the reverse process, the emergence of life from nonliving matter, would require a decrease in entropy. Their argument is faulty, since the second law only applies to closed systems, and the earth is not a closed system. The earth is continuously receiving energy from the sun.

**The heat death of the universe** example 20
Victorian philosophers realized that living things had low entropy, as discussed in example 19, and spent a lot of time worrying about the heat death of the universe: eventually the universe would have to become a high-entropy, lukewarm soup, with no life or organized motion of any kind. Fortunately (?), we now know a great many other things that will make the universe inhospitable to life long before its entropy is maximized. Life on earth, for instance, will end when the sun evolves into a giant star and vaporizes our planet.

**Hawking radiation** example 21
Any process that could destroy heat (or convert it into nothing but mechanical work) would lead to a reduction in entropy. Black holes are supermassive stars whose gravity is so strong that nothing, not even light, can escape from them once it gets within a boundary known as the event horizon. Black holes are commonly observed to suck hot gas into them. Does this lead to a reduction in the entropy of the universe? Of course one could argue that the entropy is still there inside the black hole, but being able to “hide” entropy there amounts to the same thing as being able to destroy entropy.

The physicist Steven Hawking was bothered by this question, and finally realized that although the actual stuff that enters a black hole is lost forever, the black hole will gradually lose energy in the form of light emitted from just outside the event horizon. This light ends up reintroducing the original entropy back into the universe at large.
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 (a) Show that under conditions of standard pressure and temperature, the volume of a sample of an ideal gas depends only on the number of molecules in it.
   (b) One mole is defined as $6.0 \times 10^{23}$ atoms. Find the volume of one mole of an ideal gas, in units of liters, at standard temperature and pressure ($0{\degree}C$ and 101 kPa).

2 A gas in a cylinder expands its volume by an amount $\Delta V$, pushing out a piston. Show that the work done by the gas on the piston is given by $\Delta W = P\Delta V$.

3 (a) A helium atom contains 2 protons, 2 electrons, and 2 neutrons. Find the mass of a helium atom.
   (b) Find the number of atoms in 1 kg of helium.
   (c) Helium gas is monoatomic. Find the amount of heat needed to raise the temperature of 1 kg of helium by 1 degree C. (This is known as helium’s heat capacity at constant volume.)

4 Refrigerators, air conditioners, and heat pumps are heat engines that work in reverse. You put in mechanical work, and the effect is to take heat out of a cooler reservoir and deposit heat in a warmer one: $Q_L + W = Q_H$. As with the heat engines discussed previously, the efficiency is defined as the energy transfer you want ($Q_L$ for a refrigerator or air conditioner, $Q_H$ for a heat pump) divided by the energy transfer you pay for ($W$).

   Efficiencies are supposed to be unitless, but the efficiency of an air conditioner is normally given in terms of an EER rating (or a more complex version called an SEER). The EER is defined as $Q_L/W$, but expressed in the barbaric units of Btu/watt-hour. A typical EER rating for a residential air conditioner is about 10 Btu/watt-hour, corresponding to an efficiency of about 3. The standard temperatures used for testing an air conditioner’s efficiency are $80{\degree}F$ ($27{\degree}C$) inside and $95{\degree}F$ ($35{\degree}C$) outside.

   (a) What would be the EER rating of a reversed Carnot engine used as an air conditioner?
   (b) If you ran a 3-kW residential air conditioner, with an efficiency of 3, for one hour, what would be the effect on the total entropy of the universe? Is your answer consistent with the second law of thermodynamics?
5 (a) Estimate the pressure at the center of the Earth, assuming it is of constant density throughout. Use the technique of example 5 on page 436. Note that $g$ is not constant with respect to depth — it equals $Gmr/b^3$ for $r$, the distance from the center, less than $b$, the earth’s radius.¹ State your result in terms of $G$, $m$, and $b$.

(b) Show that your answer from part a has the right units for pressure.

(c) Evaluate the result numerically.

(d) Given that the earth’s atmosphere is on the order of one thousandth the thickness of the earth’s radius, and that the density of the earth is several thousand times greater than the density of the lower atmosphere, check that your result is of a reasonable order of magnitude.

6 (a) Determine the ratio between the escape velocities from the surfaces of the earth and the moon.

(b) The temperature during the lunar daytime gets up to about 130°C. In the extremely thin (almost nonexistent) lunar atmosphere, estimate how the typical velocity of a molecule would compare with that of the same type of molecule in the earth’s atmosphere. Assume that the earth’s atmosphere has a temperature of 0°C.

(c) Suppose you were to go to the moon and release some fluorocarbon gas, with molecular formula $C_nF_{2(n+2)}$. Estimate what is the smallest fluorocarbon molecule (lowest $n$) whose typical velocity would be lower than that of an $N_2$ molecule on earth in proportion to the moon’s lower escape velocity. The moon would be able to retain an atmosphere made of these molecules.

7 Most of the atoms in the universe are in the form of gas that is not part of any star or galaxy: the intergalactic medium (IGM). The IGM consists of about $10^{-5}$ atoms per cubic centimeter, with a typical temperature of about $10^3$ K. These are, in some sense, the density and temperature of the universe (not counting light, or the exotic particles known as “dark matter”). Calculate the pressure of the universe (or, speaking more carefully, the typical pressure due to the IGM).

8 A sample of gas is enclosed in a sealed chamber. The gas consists of molecules, which are then split in half through some process such as exposure to ultraviolet light, or passing an electric spark through the gas. The gas returns to the same temperature as the surrounding room, but the molecules remain split apart, at least for some amount of time. (To achieve these conditions, we would need an extremely dilute gas. Otherwise the recombination of the molecules would be faster than the cooling down to the same temperature as the room.) How does its pressure now compare with

¹Derivation: The shell theorem tells us that the gravitational field at $r$ is the same as if all the mass existing at greater depths was concentrated at the earth’s center. Since volume scales like the third power of distance, this constitutes a fraction $(r/b)^3$ of the earth’s mass, so the field is $(Gm/r^2)(r/b)^3 = Gmr/b^3$. 

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