Summary

Selected vocabulary

- work: the amount of energy transferred into or out of a system, excluding energy transferred by heat conduction

Notation

- \( W \): work

Summary

Work is a measure of the transfer of mechanical energy, i.e., the transfer of energy by a force rather than by heat conduction. When the force is constant, work can usually be calculated as

\[
W = F||d|, \quad [\text{only if the force is constant}]
\]

where \( d \) is simply a less cumbersome notation for \( \Delta r \), the vector from the initial position to the final position. Thus,

- A force in the same direction as the motion does positive work, i.e., transfers energy into the object on which it acts.
- A force in the opposite direction compared to the motion does negative work, i.e., transfers energy out of the object on which it acts.
- When there is no motion, no mechanical work is done. The human body burns calories when it exerts a force without moving, but this is an internal energy transfer of energy within the body, and thus does not fall within the scientific definition of work.
- A force perpendicular to the motion does no work.

When the force is not constant, the above equation should be generalized as the area under the graph of \( F|| \) versus \( d \).

Machines such as pulleys, levers, and gears may increase or decrease a force, but they can never increase or decrease the amount of work done. That would violate conservation of energy unless the machine had some source of stored energy or some way to accept and store up energy.

There are some situations in which the equation \( W = F|| |d| \) is ambiguous or not true, and these issues are discussed rigorously in section 13.6. However, problems can usually be avoided by analyzing the types of energy being transferred before plunging into the math. In any case there is no substitute for a physical understanding of the processes involved.

The techniques developed for calculating work can also be applied to the calculation of potential energy. We fix some position
as a reference position, and calculate the potential energy for some other position, \( x \), as

\[ PEx = -W_{\text{ref} \rightarrow x}. \]

The following two equations for potential energy have broader significance than might be suspected based on the limited situations in which they were derived:

\[ PE = \frac{1}{2}k(x - xo)^2. \]

[potential energy of a spring having spring constant \( k \), when stretched or compressed from the equilibrium position \( xo \); analogous equations apply for the twisting, bending, compression, or stretching of any object.]

\[ PE = -\frac{GMm}{r} \]

[gravitational potential energy of objects of masses \( M \) and \( m \), separated by a distance \( r \); an analogous equation applies to the electrical potential energy of an electron in an atom.]
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 Two speedboats are identical, but one has more people aboard than the other. Although the total masses of the two boats are unequal, suppose that they happen to have the same kinetic energy. In a boat, as in a car, it’s important to be able to stop in time to avoid hitting things. (a) If the frictional force from the water is the same in both cases, how will the boats’ stopping distances compare? Explain. (b) Compare the times required for the boats to stop.

2 In each of the following situations, is the work being done positive, negative, or zero? (a) a bull paws the ground; (b) a fishing boat pulls a net through the water behind it; (c) the water resists the motion of the net through it; (d) you stand behind a pickup truck and lower a bale of hay from the truck’s bed to the ground. Explain. [Based on a problem by Serway and Faughn.]

3 In the earth’s atmosphere, the molecules are constantly moving around. Because temperature is a measure of kinetic energy per molecule, the average kinetic energy of each type of molecule is the same, e.g., the average KE of the O₂ molecules is the same as the average KE of the N₂ molecules. (a) If the mass of an O₂ molecule is eight times greater than that of a He atom, what is the ratio of their average speeds? Which way is the ratio, i.e., which is typically moving faster? (b) Use your result from part a to explain why any helium occurring naturally in the atmosphere has long since escaped into outer space, never to return. (Helium is obtained commercially by extracting it from rocks.) You may want to do problem 21 first, for insight.

4 Weiping lifts a rock with a weight of 1.0 N through a height of 1.0 m, and then lowers it back down to the starting point. Bubba pushes a table 1.0 m across the floor at constant speed, requiring a force of 1.0 N, and then pushes it back to where it started. (a) Compare the total work done by Weiping and Bubba. (b) Check that your answers to part a make sense, using the definition of work: work is the transfer of energy. In your answer, you’ll need to discuss what specific type of energy is involved in each case.

5 In one of his more flamboyant moments, Galileo wrote “Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon.” Find the speed
of an ant that falls to earth from the distance of the moon at the moment when it is about to enter the atmosphere. Assume it is released from a point that is not actually near the moon, so the moon’s gravity is negligible. You will need the result of example 7 on p. 346.

6 [Problem 6 has been deleted.]

7 (a) The crew of an 18th century warship is raising the anchor. The anchor has a mass of 5000 kg. The water is 30 m deep. The chain to which the anchor is attached has a mass per unit length of 150 kg/m. Before they start raising the anchor, what is the total weight of the anchor plus the portion of the chain hanging out of the ship? (Assume that the buoyancy of the anchor is negligible.)

(b) After they have raised the anchor by 1 m, what is the weight they are raising?

(c) Define \( y = 0 \) when the anchor is resting on the bottom, and \( y = +30 \) m when it has been raised up to the ship. Draw a graph of the force the crew has to exert to raise the anchor and chain, as a function of \( y \). (Assume that they are raising it slowly, so water resistance is negligible.) It will not be a constant! Now find the area under the graph, and determine the work done by the crew in raising the anchor, in joules.

(d) Convert your answer from (c) into units of kcal.

8 In the power stroke of a car’s gasoline engine, the fuel-air mixture is ignited by the spark plug, explodes, and pushes the piston out. The exploding mixture’s force on the piston head is greatest at the beginning of the explosion, and decreases as the mixture expands. It can be approximated by \( F = a/x \), where \( x \) is the distance from the cylinder to the piston head, and \( a \) is a constant with units of N·m. (Actually \( a/x^{1.4} \) would be more accurate, but the problem works out more nicely with \( a/x! \) The piston begins its stroke at \( x = x_1 \), and ends at \( x = x_2 \). The 1965 Rambler had six cylinders, each with \( a = 220 \) N·m, \( x_1 = 1.2 \) cm, and \( x_2 = 10.2 \) cm.

(a) Draw a neat, accurate graph of \( F \) vs \( x \), on graph paper.

(b) From the area under the curve, derive the amount of work done in one stroke by one cylinder.

(c) Assume the engine is running at 4800 r.p.m., so that during one minute, each of the six cylinders performs 2400 power strokes. (Power strokes only happen every other revolution.) Find the engine’s power, in units of horsepower (1 hp=746 W).

(d) The compression ratio of an engine is defined as \( x_2/x_1 \). Explain in words why the car’s power would be exactly the same if \( x_1 \) and \( x_2 \) were, say, halved or tripled, maintaining the same compression ratio of 8.5. Explain why this would not quite be true with the more realistic force equation \( F = a/x^{1.4} \).
9 The magnitude of the force between two magnets separated by a distance $r$ can be approximated as $kr^{-3}$ for large values of $r$. The constant $k$ depends on the strengths of the magnets and the relative orientations of their north and south poles. Two magnets are released on a slippery surface at an initial distance $r_i$, and begin sliding towards each other. What will be the total kinetic energy of the two magnets when they reach a final distance $r_f$? (Ignore friction.)

10 A car starts from rest at $t = 0$, and starts speeding up with constant acceleration. (a) Find the car’s kinetic energy in terms of its mass, $m$, acceleration, $a$, and the time, $t$. (b) Your answer in the previous part also equals the amount of work, $W$, done from $t = 0$ until time $t$. Take the derivative of the previous expression to find the power expended by the car at time $t$. (c) Suppose two cars with the same mass both start from rest at the same time, but one has twice as much acceleration as the other. At any moment, how many times more power is being dissipated by the more quickly accelerating car? (The answer is not 2.)

11 A space probe of mass $m$ is dropped into a previously unexplored spherical cloud of gas and dust, and accelerates toward the center of the cloud under the influence of the cloud’s gravity. Measurements of its velocity allow its potential energy, $PE$, to be determined as a function of the distance $r$ from the cloud’s center. The mass in the cloud is distributed in a spherically symmetric way, so its density, $\rho(r)$, depends only on $r$ and not on the angular coordinates. Show that by finding $PE$, one can infer $\rho(r)$ as follows:

$$\rho(r) = \frac{1}{4\pi Gmr^2} \frac{d}{dr} \left( r^2 \frac{dPE}{dr} \right).$$

12 A rail gun is a device like a train on a track, with the train propelled by a powerful electrical pulse. Very high speeds have been demonstrated in test models, and rail guns have been proposed as an alternative to rockets for sending into outer space any object that would be strong enough to survive the extreme accelerations. Suppose that the rail gun capsule is launched straight up, and that the force of air friction acting on it is given by $F = be^{-cx}$, where $x$ is the altitude, $b$ and $c$ are constants, and $e$ is the base of natural logarithms. The exponential decay occurs because the atmosphere gets thinner with increasing altitude. (In reality, the force would probably drop off even faster than an exponential, because the capsule would be slowing down somewhat.) Find the amount of kinetic energy lost by the capsule due to air friction between when it is launched and when it is completely beyond the atmosphere. (Gravity is negligible, since the air friction force is much greater than the gravitational force.)
A certain binary star system consists of two stars with masses $m_1$ and $m_2$, separated by a distance $b$. A comet, originally nearly at rest in deep space, drops into the system and at a certain point in time arrives at the midpoint between the two stars. For that moment in time, find its velocity, $v$, symbolically in terms of $b$, $m_1$, $m_2$, and fundamental constants.

An airplane flies in the positive direction along the $x$ axis, through crosswinds that exert a force $\mathbf{F} = (a + bx)\mathbf{\hat{x}} + (c + dx)\mathbf{\hat{y}}$. Find the work done by the wind on the plane, and by the plane on the wind, in traveling from the origin to position $x$.

In 1935, Yukawa proposed an early theory of the force that held the neutrons and protons together in the nucleus. His equation for the potential energy of two such particles, at a center-to-center distance $r$, was $PE(r) = gr^{-1}e^{-r/a}$, where $g$ parametrizes the strength of the interaction, $e$ is the base of natural logarithms, and $a$ is about $10^{-15}$ m. Find the force between two nucleons that would be consistent with this equation for the potential energy.

Prove that the dot product defined in section 13.7 is rotationally invariant in the sense of section 7.5.

Fill in the details of the proof of $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$ on page 350.

Does it make sense to say that work is conserved?

(a) Suppose work is done in one-dimensional motion. What happens to the work if you reverse the direction of the positive coordinate axis? Base your answer directly on the definition of work as a transfer of mechanical energy. (b) Now answer the question based on the $W = Fd$ rule.

A microwave oven works by twisting molecules one way and then the other, counterclockwise and then clockwise about their own centers, millions of times a second. If you put an ice cube or a stick of butter in a microwave, you'll observe that the solid doesn't heat very quickly, although eventually melting begins in one small spot. Once this spot forms, it grows rapidly, while the rest of the solid remains solid; it appears that a microwave oven heats a liquid much more rapidly than a solid. Explain why this should happen, based on the atomic-level description of heat, solids, and liquids. (See, e.g., figure b on page 317.)

Don't repeat the following common mistakes:

In a solid, the atoms are packed more tightly and have less space between them. Not true. Ice floats because it’s less dense than water.

In a liquid, the atoms are moving much faster. No, the difference in average speed between ice at $-1^\circ$C and water at $1^\circ$C is only 0.4%.
Starting at a distance $r$ from a planet of mass $M$, how fast must an object be moving in order to have a hyperbolic orbit, i.e., one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it?

The figure, redrawn from Gray's Anatomy, shows the tension of which a muscle is capable. The variable $x$ is defined as the contraction of the muscle from its maximum length $L$, so that at $x = 0$ the muscle has length $L$, and at $x = L$ the muscle would theoretically have zero length. In reality, the muscle can only contract to $x = cL$, where $c$ is less than 1. When the muscle is extended to its maximum length, at $x = 0$, it is capable of the greatest tension, $T_0$. As the muscle contracts, however, it becomes weaker. Gray suggests approximating this function as a linear decrease, which would theoretically extrapolate to zero at $x = L$. (a) Find the maximum work the muscle can do in one contraction, in terms of $c$, $L$, and $T_0$.

(b) Show that your answer to part a has the right units.

(c) Show that your answer to part a has the right behavior when $c = 0$ and when $c = 1$.

(d) Gray also states that the absolute maximum tension $T_0$ has been found to be approximately proportional to the muscle’s cross-sectional area $A$ (which is presumably measured at $x = 0$), with proportionality constant $k$. Approximating the muscle as a cylinder, show that your answer from part a can be reexpressed in terms of the volume, $V$, eliminating $L$ and $A$.

(e) Evaluate your result numerically for a biceps muscle with a volume of 200 cm$^3$, with $c = 0.8$ and $k = 100$ N/cm$^2$ as estimated by Gray.

A car accelerates from rest. At low speeds, its acceleration is limited by static friction, so that if we press too hard on the gas, we will “burn rubber” (or, for many newer cars, a computerized traction-control system will override the gas pedal). At higher speeds, the limit on acceleration comes from the power of the engine, which puts a limit on how fast kinetic energy can be developed.

(a) Show that if a force $F$ is applied to an object moving at speed $v$, the power required is given by $P = vF$.

(b) Find the speed $v$ at which we cross over from the first regime described above to the second. At speeds higher than this, the engine does not have enough power to burn rubber. Express your result in terms of the car’s power $P$, its mass $m$, the coefficient of static friction $\mu_s$, and $g$.

(c) Show that your answer to part b has units that make sense.

(d) Show that the dependence of your answer on each of the four
variables makes sense physically.

(e) The 2010 Maserati Gran Turismo Convertible has a maximum power of $3.23 \times 10^5$ W (433 horsepower) and a mass (including a 50-kg driver) of $2.03 \times 10^3$ kg. (This power is the maximum the engine can supply at its optimum frequency of 7600 r.p.m. Presumably the automatic transmission is designed so a gear is available in which the engine will be running at very nearly this frequency when the car is moving at $v$.) Rubber on asphalt has $\mu_s \approx 0.9$. Find $v$ for this car. Answer: 18 m/s, or about 40 miles per hour.

(f) Our analysis has neglected air friction, which can probably be approximated as a force proportional to $v^2$. The existence of this force is the reason that the car has a maximum speed, which is 176 miles per hour. To get a feeling for how good an approximation it is to ignore air friction, find what fraction of the engine’s maximum power is being used to overcome air resistance when the car is moving at the speed $v$ found in part e. Answer: 1%

24 Most modern bow hunters in the U.S. use a fancy mechanical bow called a compound bow, which looks nothing like what most people imagine when they think of a bow and arrow. It has a system of pulleys designed to produce the force curve shown in the figure, where $F$ is the force required to pull the string back, and $x$ is the distance between the string and the center of the bow’s body. It is not a linear Hooke’s-law graph, as it would be for an old-fashioned bow. The big advantage of the design is that relatively little force is required to hold the bow stretched to point B on the graph. This is the force required from the hunter in order to hold the bow ready while waiting for a shot. Since it may be necessary to wait a long time, this force can’t be too big. An old-fashioned bow, designed to require the same amount of force when fully drawn, would shoot arrows at much lower speeds, since its graph would be a straight line from A to B. For the graph shown in the figure (taken from realistic data), find the speed at which a 26 g arrow is released, assuming that 70% of the mechanical work done by the hand is actually transmitted to the arrow. (The other 30% is lost to frictional heating inside the bow and kinetic energy of the recoiling and vibrating bow.) √
A mass moving in one dimension is attached to a horizontal spring. It slides on the surface below it, with equal coefficients of static and kinetic friction, $\mu_k = \mu_s$. The equilibrium position is $x = 0$. If the mass is pulled to some initial position and released from rest, it will complete some number of oscillations before friction brings it to a stop. When released from $x = a$ ($a > 0$), it completes exactly $1/4$ of an oscillation, i.e., it stops precisely at $x = 0$. Similarly, define $b > 0$ as the greatest $x$ from which it could be released and complete $1/2$ of an oscillation, stopping on the far side and not coming back toward equilibrium. Find $b/a$. Hint: To keep the algebra simple, set every fixed parameter of the system equal to 1.

"Big wall" climbing is a specialized type of rock climbing that involves going up tall cliffs such as the ones in Yosemite, usually with the climbers spending at least one night sleeping on a natural ledge or an artificial “portaledge.” In this style of climbing, each pitch of the climb involves strenuously hauling up several heavy bags of gear — a fact that has caused these climbs to be referred to as “vertical ditch digging.” (a) If an 80 kg haul bag has to be pulled up the full length of a 60 m rope, how much work is done? (b) Since it can be difficult to lift 80 kg, a 2:1 pulley is often used. The hauler then lifts the equivalent of 40 kg, but has to pull in 120 m of rope. How much work is done in this case?

A soccer ball of mass $m$ is moving at speed $v$ when you kick it in the same direction it is moving. You kick it with constant force $F$, and you want to triple the ball’s speed. Over what distance must your foot be in contact with the ball? [problem by B. Shotwell]
Chapter 14
Conservation of Momentum

In many subfields of physics these days, it is possible to read an entire issue of a journal without ever encountering an equation involving force or a reference to Newton’s laws of motion. In the last hundred and fifty years, an entirely different framework has been developed for physics, based on conservation laws.

The new approach is not just preferred because it is in fashion. It applies inside an atom or near a black hole, where Newton’s laws do not. Even in everyday situations the new approach can be superior. We have already seen how perpetual motion machines could be designed that were too complex to be easily debunked by Newton’s laws. The beauty of conservation laws is that they tell us something must remain the same, regardless of the complexity of the process.

So far we have discussed only two conservation laws, the laws of conservation of mass and energy. Is there any reason to believe that further conservation laws are needed in order to replace Newton’s laws as a complete description of nature? Yes. Conservation of mass and energy do not relate in any way to the three dimensions of space, because both are scalars. Conservation of energy, for instance, does not prevent the planet earth from abruptly making a 90-degree turn and heading straight into the sun, because kinetic energy does not depend on direction. In this chapter, we develop a new conserved quantity, called momentum, which is a vector.
14.1 Momentum

A conserved quantity of motion

Your first encounter with conservation of momentum may have come as a small child unjustly confined to a shopping cart. You spot something interesting to play with, like the display case of imported wine down at the end of the aisle, and decide to push the cart over there. But being imprisoned by Dad in the cart was not the only injustice that day. There was a far greater conspiracy to thwart your young id, one that originated in the laws of nature. Pushing forward did nudge the cart forward, but it pushed you backward. If the wheels of the cart were well lubricated, it wouldn’t matter how you jerked, yanked, or kicked off from the back of the cart. You could not cause any overall forward motion of the entire system consisting of the cart with you inside.

In the Newtonian framework, we describe this as arising from Newton’s third law. The cart made a force on you that was equal and opposite to your force on it. In the framework of conservation laws, we cannot attribute your frustration to conservation of energy. It would have been perfectly possible for you to transform some of the internal chemical energy stored in your body to kinetic energy of the cart and your body.

The following characteristics of the situation suggest that there may be a new conservation law involved:

A closed system is involved. All conservation laws deal with closed systems. You and the cart are a closed system, since the well-oiled wheels prevent the floor from making any forward force on you.

Something remains unchanged. The overall velocity of the system started out being zero, and you cannot change it. This vague reference to “overall velocity” can be made more precise: it is the velocity of the system’s center of mass that cannot be changed.

Something can be transferred back and forth without changing the total amount. If we define forward as positive and backward as negative, then one part of the system can gain positive motion if another part acquires negative motion. If we don’t want to worry about positive and negative signs, we can imagine that the whole cart was initially gliding forward on its well-oiled wheels. By kicking off from the back of the cart, you could increase your own velocity, but this inevitably causes the cart to slow down.
It thus appears that there is some numerical measure of an object’s quantity of motion that is conserved when you add up all the objects within a system.

**Momentum**

Although velocity has been referred to, it is not the total velocity of a closed system that remains constant. If it was, then firing a gun would cause the gun to recoil at the same velocity as the bullet! The gun does recoil, but at a much lower velocity than the bullet. Newton’s third law tells us

\[ F_{\text{gun on bullet}} = -F_{\text{bullet on gun}}, \]

and assuming a constant force for simplicity, Newton’s second law allows us to change this to

\[ m_{\text{bullet}} \frac{\Delta v_{\text{bullet}}}{\Delta t} = -m_{\text{gun}} \frac{\Delta v_{\text{gun}}}{\Delta t}. \]

Thus if the gun has 100 times more mass than the bullet, it will recoil at a velocity that is 100 times smaller and in the opposite direction, represented by the opposite sign. The quantity \( mv \) is therefore apparently a useful measure of motion, and we give it a name, momentum, and a symbol, \( p \). (As far as I know, the letter “p” was just chosen at random, since “m” was already being used for mass.) The situations discussed so far have been one-dimensional, but in three-dimensional situations it is treated as a vector.

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**definition of momentum for material objects**

The momentum of a material object, i.e., a piece of matter, is defined as

\[ p = mv, \]

the product of the object’s mass and its velocity vector.

The units of momentum are kg·m/s, and there is unfortunately no abbreviation for this clumsy combination of units.

The reasoning leading up to the definition of momentum was all based on the search for a conservation law, and the only reason why we bother to define such a quantity is that experiments show it is conserved:

**the law of conservation of momentum**

In any closed system, the vector sum of all the momenta remains constant,

\[ p_{1i} + p_{2i} + \ldots = p_{1f} + p_{2f} + \ldots, \]

where \( i \) labels the initial and \( f \) the final momenta. (A closed system is one on which no external forces act.)
This chapter first addresses the one-dimensional case, in which the direction of the momentum can be taken into account by using plus and minus signs. We then pass to three dimensions, necessitating the use of vector addition.

A subtle point about conservation laws is that they all refer to “closed systems,” but “closed” means different things in different cases. When discussing conservation of mass, “closed” means a system that doesn’t have matter moving in or out of it. With energy, we mean that there is no work or heat transfer occurring across the boundary of the system. For momentum conservation, “closed” means there are no external forces reaching into the system.

**A cannon example 1**

A cannon of mass 1000 kg fires a 10-kg shell at a velocity of 200 m/s. At what speed does the cannon recoil?

The law of conservation of momentum tells us that

\[
p_{\text{cannon},i} + p_{\text{shell},i} = p_{\text{cannon},f} + p_{\text{shell},f}.
\]

Choosing a coordinate system in which the cannon points in the positive direction, the given information is

\[
p_{\text{cannon},i} = 0 \\
p_{\text{shell},i} = 0 \\
p_{\text{shell},f} = 2000 \text{ kg} \cdot \text{m/s}.
\]

We must have \(p_{\text{cannon},f} = -2000 \text{ kg} \cdot \text{m/s}\), so the recoil velocity of the cannon is \(-2 \text{ m/s}\).

**Ion drive for propelling spacecraft example 2**

The experimental solar-powered ion drive of the Deep Space 1 space probe expels its xenon gas exhaust at a speed of 30,000 m/s, ten times faster than the exhaust velocity for a typical chemical-fuel rocket engine. Roughly how many times greater is the maximum speed this spacecraft can reach, compared with a chemical-fueled probe with the same mass of fuel (“reaction mass”) available for pushing out the back as exhaust?

Momentum equals mass multiplied by velocity. Both spacecraft are assumed to have the same amount of reaction mass, and the ion drive’s exhaust has a velocity ten times greater, so the momentum of its exhaust is ten times greater. Before the engine starts firing, neither the probe nor the exhaust has any momentum, so the total momentum of the system is zero. By conservation of momentum, the total momentum must also be zero after
The ion drive engine of the NASA Deep Space 1 probe, shown under construction (left) and being tested in a vacuum chamber (right) prior to its October 1998 launch. Intended mainly as a test vehicle for new technologies, the craft nevertheless carried out a successful scientific program that included a flyby of a comet.

All the exhaust has been expelled. If we define the positive direction as the direction the spacecraft is going, then the negative momentum of the exhaust is canceled by the positive momentum of the spacecraft. The ion drive allows a final speed that is ten times greater. (This simplified analysis ignores the fact that the reaction mass expelled later in the burn is not moving backward as fast, because of the forward speed of the already-moving spacecraft.)

Generalization of the momentum concept

As with all the conservation laws, the law of conservation of momentum has evolved over time. In the 1800’s it was found that a beam of light striking an object would give it some momentum, even though light has no mass, and would therefore have no momentum.
according to the above definition. Rather than discarding the principle of conservation of momentum, the physicists of the time decided to see if the definition of momentum could be extended to include momentum carried by light. The process is analogous to the process outlined on page 301 for identifying new forms of energy. The first step was the discovery that light could impart momentum to matter, and the second step was to show that the momentum possessed by light could be related in a definite way to observable properties of the light. They found that conservation of momentum could be successfully generalized by attributing to a beam of light a momentum vector in the direction of the light’s motion and having a magnitude proportional to the amount of energy the light possessed. The momentum of light is negligible under ordinary circumstances, e.g., a flashlight left on for an hour would only absorb about \(10^{-5}\) kg·m/s of momentum as it recoiled.

### Example 3

**The tail of a comet**

Momentum is not always equal to \(mv\). Like many comets, Halley’s comet has a very elongated elliptical orbit. About once per century, its orbit brings it close to the sun. The comet’s head, or nucleus, is composed of dirty ice, so the energy deposited by the intense sunlight boils off steam and dust, \(b\). The sunlight does not just carry energy, however — it also carries momentum. The momentum of the sunlight impacting on the smaller dust particles pushes them away from the sun, forming a tail, \(c\). By analogy with matter, for which momentum equals \(mv\), you would expect that massless light would have zero momentum, but the equation \(p = mv\) is not the correct one for light, and light does have momentum. (The gases typically form a second, distinct tail whose motion is controlled by the sun’s magnetic field.)

The reason for bringing this up is not so that you can plug numbers into formulas in these exotic situations. The point is that the conservation laws have proven so sturdy exactly because they can easily be amended to fit new circumstances. Newton’s laws are no longer at the center of the stage of physics because they did not have the same adaptability. More generally, the moral of this story is the provisional nature of scientific truth.

It should also be noted that conservation of momentum is not a consequence of Newton’s laws, as is often asserted in textbooks. Newton’s laws do not apply to light, and therefore could not possibly be used to prove anything about a concept as general as the conservation of momentum in its modern form.

### Momentum compared to kinetic energy

Momentum and kinetic energy are both measures of the quantity of motion, and a sideshow in the Newton-Leibnitz controversy over who invented calculus was an argument over whether \(mv\) (i.e., momentum) or \(mv^2\) (i.e., kinetic energy without the \(1/2\) in front)
was the “true” measure of motion. The modern student can certainly be excused for wondering why we need both quantities, when their complementary nature was not evident to the greatest minds of the 1700’s. The following table highlights their differences.

<table>
<thead>
<tr>
<th>kinetic energy . . .</th>
<th>momentum . . .</th>
</tr>
</thead>
<tbody>
<tr>
<td>is a scalar.</td>
<td>is a vector.</td>
</tr>
<tr>
<td>is not changed by a force perpendicular to the motion, which changes only the direction of the velocity vector.</td>
<td>is changed by any force, since a change in either the magnitude or the direction of the velocity vector will result in a change in the momentum vector.</td>
</tr>
<tr>
<td>is always positive, and cannot cancel out.</td>
<td>cancels with momentum in the opposite direction.</td>
</tr>
<tr>
<td>can be traded for other forms of energy that do not involve motion. KE is not a conserved quantity by itself.</td>
<td>is always conserved in a closed system.</td>
</tr>
<tr>
<td>is quadrupled if the velocity is doubled.</td>
<td>is doubled if the velocity is doubled.</td>
</tr>
</tbody>
</table>

A spinning top example 4
A spinning top has zero total momentum, because for every moving point, there is another point on the opposite side that cancels its momentum. It does, however, have kinetic energy.

Why a tuning fork has two prongs example 5
A tuning fork is made with two prongs so that they can vibrate in opposite directions, canceling their momenta. In a hypothetical version with only one prong, the momentum would have to oscillate, and this momentum would have to come from somewhere, such as the hand holding the fork. The result would be that vibrations would be transmitted to the hand and rapidly die out. In a two-prong fork, the two momenta cancel, but the energies don’t.

Momentum and kinetic energy in firing a rifle example 6
The rifle and bullet have zero momentum and zero kinetic energy to start with. When the trigger is pulled, the bullet gains some momentum in the forward direction, but this is canceled by the rifle’s backward momentum, so the total momentum is still zero. The kinetic energies of the gun and bullet are both positive scalars, however, and do not cancel. The total kinetic energy is allowed to increase, because kinetic energy is being traded for other forms of energy. Initially there is chemical energy in the gunpowder. This chemical energy is converted into heat, sound, and kinetic energy. The gun’s “backward” kinetic energy does not refrigerate the shooter’s shoulder!
The wobbly earth  
As the moon completes half a circle around the earth, its motion reverses direction. This does not involve any change in kinetic energy, and the earth’s gravitational force does not do any work on the moon. The reversed velocity vector does, however, imply a reversed momentum vector, so conservation of momentum in the closed earth-moon system tells us that the earth must also change its momentum. In fact, the earth wobbles in a little “orbit” about a point below its surface on the line connecting it and the moon. The two bodies’ momentum vectors always point in opposite directions and cancel each other out.

The earth and moon get a divorce  
Why can’t the moon suddenly decide to fly off one way and the earth the other way? It is not forbidden by conservation of momentum, because the moon’s newly acquired momentum in one direction could be canceled out by the change in the momentum of the earth, supposing the earth headed the opposite direction at the appropriate, slower speed. The catastrophe is forbidden by conservation of energy, because both their energies would have to increase greatly.

Momentum and kinetic energy of a glacier  
A cubic-kilometer glacier would have a mass of about $10^{12}$ kg. If it moves at a speed of $10^{-5}$ m/s, then its momentum is $10^7$ kg·m/s. This is the kind of heroic-scale result we expect, perhaps the equivalent of the space shuttle taking off, or all the cars in LA driving in the same direction at freeway speed. Its kinetic energy, however, is only 50 J, the equivalent of the calories contained in a poppy seed or the energy in a drop of gasoline too small to be seen without a microscope. The surprisingly small kinetic energy is because kinetic energy is proportional to the square of the velocity, and the square of a small number is an even smaller number.

Discussion questions
A If all the air molecules in the room settled down in a thin film on the floor, would that violate conservation of momentum? Conservation of energy?
B A refrigerator has coils in the back that get hot, and heat is molecular motion. These moving molecules have both energy and momentum. Why doesn’t the refrigerator need to be tied to the wall to keep it from recoiling from the momentum it loses out the back?
14.2 Collisions in one dimension

Physicists employ the term “collision” in a broader sense than ordinary usage, applying it to any situation where objects interact for a certain period of time. A bat hitting a baseball, a radioactively emitted particle damaging DNA, and a gun and a bullet going their separate ways are all examples of collisions in this sense. Physical contact is not even required. A comet swinging past the sun on a hyperbolic orbit is considered to undergo a collision, even though it never touches the sun. All that matters is that the comet and the sun exerted gravitational forces on each other.

The reason for broadening the term “collision” in this way is that all of these situations can be attacked mathematically using the same conservation laws in similar ways. In the first example, conservation of momentum is all that is required.

Getting rear-ended example 10

Ms. Chang is rear-ended at a stop light by Mr. Nelson, and sues to make him pay her medical bills. He testifies that he was only going 35 miles per hour when he hit Ms. Chang. She thinks he was going much faster than that. The cars skidded together after the impact, and measurements of the length of the skid marks and the coefficient of friction show that their joint velocity immediately after the impact was 19 miles per hour. Mr. Nelson’s Nissan weighs 3100 pounds, and Ms. Chang’s Cadillac weighs 5200 pounds. Is Mr. Nelson telling the truth?

Since the cars skidded together, we can write down the equation for conservation of momentum using only two velocities, \( v \) for Mr. Nelson’s velocity before the crash, and \( v' \) for their joint velocity afterward:

\[
m_N v = m_N v' + m_C v'.
\]

Solving for the unknown, \( v \), we find

\[
v = \left( 1 + \frac{m_C}{m_N} \right) v'.
\]

Although we are given the weights in pounds, a unit of force, the ratio of the masses is the same as the ratio of the weights, and we find \( v = 51 \) miles per hour. He is lying.

The above example was simple because both cars had the same velocity afterward. In many one-dimensional collisions, however, the two objects do not stick. If we wish to predict the result of such a collision, conservation of momentum does not suffice, because both velocities after the collision are unknown, so we have one equation in two unknowns.

Conservation of energy can provide a second equation, but its application is not as straightforward, because kinetic energy is only the particular form of energy that has to do with motion. In many
Gory details of the proof in example 11

The equation $A + B = C + D$ says that the change in one ball’s velocity is equal and opposite to the change in the other’s. We invent a symbol $x = C - A$ for the change in ball 1’s velocity. The second equation can then be rewritten as $A^2 + B^2 = (A + x)^2 + (B - x)^2$. Squaring out the quantities in parentheses and then simplifying, we get $0 = Ax - Bx + x^2$. The equation has the trivial solution $x = 0$, i.e., neither ball’s velocity is changed, but this is physically impossible because the balls can’t travel through each other like ghosts. Assuming $x \neq 0$, we can divide by $x$ and solve for $x = B - A$. This means that ball 1 has gained an amount of velocity exactly right to match ball 2’s initial velocity, and vice-versa. The balls must have swapped velocities.

A little experimentation with numbers shows that given values of $A$ and $B$, it is impossible to find $C$ and $D$ that satisfy these equations unless $C$ and $D$ equal $A$ and $B$, or $C$ and $D$ are the same as $A$ and $B$ but swapped around. A formal proof of this fact is given in the sidebar. In the special case where ball 2 is initially at rest, this tells us that ball 1 is stopped dead by the collision, and ball 2 heads off at the velocity originally possessed by ball 1. This behavior will be familiar to players of pool.

Often, as in the example above, the details of the algebra are the least interesting part of the problem, and considerable physical insight can be gained simply by counting the number of unknowns and comparing to the number of equations. Suppose a beginner at

Collisions of the superball type, in which almost no kinetic energy is converted to other forms of energy, can thus be analyzed more thoroughly, because they have $KE_f = KE_i$, as opposed to the less useful inequality $KE_f < KE_i$ for a case like a tennis ball bouncing on grass.

Pool balls colliding head-on example 11

Two pool balls collide head-on, so that the collision is restricted to one dimension. Pool balls are constructed so as to lose as little kinetic energy as possible in a collision, so under the assumption that no kinetic energy is converted to any other form of energy, what can we predict about the results of such a collision?

Pool balls have identical masses, so we use the same symbol $m$ for both. Conservation of momentum and no loss of kinetic energy give us the two equations

\[
\begin{align*}
 mv_{1i} + mv_{2i} &= mv_{1f} + mv_{2f} \\
 \frac{1}{2} mv_{1i}^2 + \frac{1}{2} mv_{2i}^2 &= \frac{1}{2} mv_{1f}^2 + \frac{1}{2} mv_{2f}^2
\end{align*}
\]

The masses and the factors of 1/2 can be divided out, and we eliminate the cumbersome subscripts by replacing the symbols $v_{1i,2i}$ with the symbols $A, B, C, \text{ and } D$:

\[
\begin{align*}
 A + B &= C + D \\
 A^2 + B^2 &= C^2 + D^2
\end{align*}
\]

This behavior will be familiar to players of pool.

Often, as in the example above, the details of the algebra are the least interesting part of the problem, and considerable physical insight can be gained simply by counting the number of unknowns and comparing to the number of equations. Suppose a beginner at
pool notices a case where her cue ball hits an initially stationary ball and stops dead. “Wow, what a good trick,” she thinks. “I bet I could never do that again in a million years.” But she tries again, and finds that she can’t help doing it even if she doesn’t want to. Luckily she has just learned about collisions in her physics course. Once she has written down the equations for conservation of energy and no loss of kinetic energy, she really doesn’t have to complete the algebra. She knows that she has two equations in two unknowns, so there must be a well-defined solution. Once she has seen the result of one such collision, she knows that the same thing must happen every time. The same thing would happen with colliding marbles or croquet balls. It doesn’t matter if the masses or velocities are different, because that just multiplies both equations by some constant factor.

The discovery of the neutron

This was the type of reasoning employed by James Chadwick in his 1932 discovery of the neutron. At the time, the atom was imagined to be made out of two types of fundamental particles, protons and electrons. The protons were far more massive, and clustered together in the atom’s core, or nucleus. Attractive electrical forces caused the electrons to orbit the nucleus in circles, in much the same way that gravitational forces kept the planets from cruising out of the solar system. Experiments showed that the helium nucleus, for instance, exerted exactly twice as much electrical force on an electron as a nucleus of hydrogen, the smallest atom, and this was explained by saying that helium had two protons to hydrogen’s one. The trouble was that according to this model, helium would have two electrons and two protons, giving it precisely twice the mass of a hydrogen atom with one of each. In fact, helium has about four times the mass of hydrogen.

Chadwick suspected that the helium nucleus possessed two additional particles of a new type, which did not participate in electrical forces at all, i.e., were electrically neutral. If these particles had very nearly the same mass as protons, then the four-to-one mass ratio of helium and hydrogen could be explained. In 1930, a new type of radiation was discovered that seemed to fit this description. It was electrically neutral, and seemed to be coming from the nuclei of light elements that had been exposed to other types of radiation. At this time, however, reports of new types of particles were a dime a dozen, and most of them turned out to be either clusters made of previously known particles or else previously known particles with higher energies. Many physicists believed that the “new” particle that had attracted Chadwick’s interest was really a previously known particle called a gamma ray, which was electrically neutral. Since gamma rays have no mass, Chadwick decided to try to determine the new particle’s mass and see if it was nonzero and approximately equal
to the mass of a proton.

Unfortunately a subatomic particle is not something you can just put on a scale and weigh. Chadwick came up with an ingenious solution. The masses of the nuclei of the various chemical elements were already known, and techniques had already been developed for measuring the speed of a rapidly moving nucleus. He therefore set out to bombard samples of selected elements with the mysterious new particles. When a direct, head-on collision occurred between a mystery particle and the nucleus of one of the target atoms, the nucleus would be knocked out of the atom, and he would measure its velocity.

Chadwick's subatomic pool table. A disk of the naturally occurring metal polonium provides a source of radiation capable of kicking neutrons out of the beryllium nuclei. The type of radiation emitted by the polonium is easily absorbed by a few mm of air, so the air has to be pumped out of the left-hand chamber. The neutrons, Chadwick's mystery particles, penetrate matter far more readily, and fly out through the wall and into the chamber on the right, which is filled with nitrogen or hydrogen gas. When a neutron collides with a nitrogen or hydrogen nucleus, it kicks it out of its atom at high speed, and this recoiling nucleus then rips apart thousands of other atoms of the gas. The result is an electrical pulse that can be detected in the wire on the right. Physicists had already calibrated this type of apparatus so that they could translate the strength of the electrical pulse into the velocity of the recoiling nucleus. The whole apparatus shown in the figure would fit in the palm of your hand, in dramatic contrast to today's giant particle accelerators.

Suppose, for instance, that we bombard a sample of hydrogen atoms with the mystery particles. Since the participants in the collision are fundamental particles, there is no way for kinetic energy to be converted into heat or any other form of energy, and Chadwick thus had two equations in three unknowns:

- equation #1: conservation of momentum
- equation #2: no loss of kinetic energy
unknown #1: mass of the mystery particle
unknown #2: initial velocity of the mystery particle
unknown #3: final velocity of the mystery particle

The number of unknowns is greater than the number of equations, so there is no unique solution. But by creating collisions with nuclei of another element, nitrogen, he gained two more equations at the expense of only one more unknown:

 equation #3: conservation of momentum in the new collision
 equation #4: no loss of kinetic energy in the new collision

unknown #4: final velocity of the mystery particle in the new collision

He was thus able to solve for all the unknowns, including the mass of the mystery particle, which was indeed within 1% of the mass of a proton. He named the new particle the neutron, since it is electrically neutral.

Discussion question
A Good pool players learn to make the cue ball spin, which can cause it not to stop dead in a head-on collision with a stationary ball. If this does not violate the laws of physics, what hidden assumption was there in the example above?

14.3 * Relationship of momentum to the center of mass

We have already discussed the idea of the center of mass on p. 67, but using the concept of momentum we can now find a mathematical method for defining the center of mass, explain why the motion of an object’s center of mass usually exhibits simpler motion than any other point, and gain a very simple and powerful way of understanding collisions.

The first step is to realize that the center of mass concept can be applied to systems containing more than one object. Even something like a wrench, which we think of as one object, is really made of many atoms. The center of mass is particularly easy to visualize in the case shown on the left, where two identical hockey pucks col-
Two hockey pucks collide. Their mutual center of mass traces the straight path shown by the dashed line.

It is clear on grounds of symmetry that their center of mass must be at the midpoint between them. After all, we previously defined the center of mass as the balance point, and if the two hockey pucks were joined with a very lightweight rod whose own mass was negligible, they would obviously balance at the midpoint. It doesn’t matter that the hockey pucks are two separate objects. It is still true that the motion of their center of mass is exceptionally simple, just like that of the wrench’s center of mass.

The $x$ coordinate of the hockey pucks’ center of mass is thus given by $x_{cm} = (x_1 + x_2)/2$, i.e., the arithmetic average of their $x$ coordinates. Why is its motion so simple? It has to do with conservation of momentum. Since the hockey pucks are not being acted on by any net external force, they constitute a closed system, and their total momentum is conserved. Their total momentum is

$$mv_1 + mv_2 = m(v_1 + v_2)$$

$$= m\left(\frac{\Delta x_1}{\Delta t} + \frac{\Delta x_2}{\Delta t}\right)$$

$$= \frac{m}{\Delta t} \Delta (x_1 + x_2)$$

$$= m\frac{2\Delta x_{cm}}{\Delta t}$$

$$= m_{total} v_{cm}$$

In other words, the total momentum of the system is the same as if all its mass was concentrated at the center of mass point. Since the total momentum is conserved, the $x$ component of the center of mass’s velocity vector cannot change. The same is also true for the other components, so the center of mass must move along a straight line at constant speed.

The above relationship between the total momentum and the motion of the center of mass applies to any system, even if it is not closed.

**total momentum related to center of mass motion**

The total momentum of any system is related to its total mass and the velocity of its center of mass by the equation

$$\mathbf{p}_{total} = m_{total} \mathbf{v}_{cm}.$$ 

What about a system containing objects with unequal masses, or containing more than two objects? The reasoning above can be generalized to a weighted average

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots},$$

with similar equations for the $y$ and $z$ coordinates.
Momentum in different frames of reference

Absolute motion is supposed to be undetectable, i.e., the laws of physics are supposed to be equally valid in all inertial frames of reference. If we first calculate some momenta in one frame of reference and find that momentum is conserved, and then rework the whole problem in some other frame of reference that is moving with respect to the first, the numerical values of the momenta will all be different. Even so, momentum will still be conserved. All that matters is that we work a single problem in one consistent frame of reference.

One way of proving this is to apply the equation $p_{\text{total}} = m_{\text{total}}v_{\text{cm}}$. If the velocity of frame B relative to frame A is $v_{BA}$, then the only effect of changing frames of reference is to change $v_{cm}$ from its original value to $v_{cm} + v_{BA}$. This adds a constant onto the momentum vector, which has no effect on conservation of momentum.

The center of mass frame of reference

A particularly useful frame of reference in many cases is the frame that moves along with the center of mass, called the center of mass (c.m.) frame. In this frame, the total momentum is zero. The following examples show how the center of mass frame can be a powerful tool for simplifying our understanding of collisions.

A collision of pool balls viewed in the c.m. frame example 12

If you move your head so that your eye is always above the point halfway in between the two pool balls, you are viewing things in the center of mass frame. In this frame, the balls come toward the center of mass at equal speeds. By symmetry, they must therefore recoil at equal speeds along the lines on which they entered. Since the balls have essentially swapped paths in the center of mass frame, the same must also be true in any other frame. This is the same result that required laborious algebra to prove previously without the concept of the center of mass frame.

The slingshot effect example 13

It is a counterintuitive fact that a spacecraft can pick up speed by swinging around a planet, if it arrives in the opposite direction compared to the planet’s motion. Although there is no physical contact, we treat the encounter as a one-dimensional collision, and analyze it in the center of mass frame. Figure j shows such a “collision,” with a space probe whipping around Jupiter. In the sun’s frame of reference, Jupiter is moving.

What about the center of mass frame? Since Jupiter is so much more massive than the spacecraft, the center of mass is essentially fixed at Jupiter’s center, and Jupiter has zero velocity in the center of mass frame, as shown in figure k. The c.m. frame is moving to the left compared to the sun-fixed frame used in j, so
the spacecraft’s initial velocity is greater in this frame.

Things are simpler in the center of mass frame, because it is more symmetric. In the complicated sun-fixed frame, the incoming leg of the encounter is rapid, because the two bodies are rushing toward each other, while their separation on the outbound leg is more gradual, because Jupiter is trying to catch up. In the c.m. frame, Jupiter is sitting still, and there is perfect symmetry between the incoming and outgoing legs, so by symmetry we have \( v_{1f} = -v_{1i} \). Going back to the sun-fixed frame, the spacecraft’s final velocity is increased by the frames’ motion relative to each other. In the sun-fixed frame, the spacecraft’s velocity has increased greatly.

The result can also be understood in terms of work and energy. In Jupiter’s frame, Jupiter is not doing any work on the spacecraft as it rounds the back of the planet, because the motion is perpendicular to the force. But in the sun’s frame, the spacecraft’s velocity vector at the same moment has a large component to the left, so Jupiter is doing work on it.

Discussion questions

A. Make up a numerical example of two unequal masses moving in one dimension at constant velocity, and verify the equation \( p_{total} = m_{total}v_{cm} \) over a time interval of one second.

B. A more massive tennis racquet or baseball bat makes the ball fly off faster. Explain why this is true, using the center of mass frame. For simplicity, assume that the racquet or bat is simply sitting still before the collision, and that the hitter’s hands do not make any force large enough to have a significant effect over the short duration of the impact.

14.4 Momentum transfer

The rate of change of momentum

As with conservation of energy, we need a way to measure and calculate the transfer of momentum into or out of a system when the system is not closed. In the case of energy, the answer was rather complicated, and entirely different techniques had to be used for measuring the transfer of mechanical energy (work) and the transfer of heat by conduction. For momentum, the situation is far simpler.

In the simplest case, the system consists of a single object acted on by a constant external force. Since it is only the object’s velocity that can change, not its mass, the momentum transferred is

\[
\Delta p = m \Delta v,
\]

which with the help of \( \mathbf{a} = \mathbf{F}/m \) and the constant-acceleration equation \( \mathbf{a} = \Delta \mathbf{v}/\Delta t \) becomes

\[
\Delta p = ma\Delta t = F\Delta t.
\]
Thus the rate of transfer of momentum, i.e., the number of kg·m/s absorbed per second, is simply the external force,

\[ F = \frac{\Delta p}{\Delta t}. \]

[relationship between the force on an object and the rate of change of its momentum; valid only if the force is constant]

This is just a restatement of Newton’s second law, and in fact Newton originally stated it this way. As shown in figure 1, the relationship between force and momentum is directly analogous to that between power and energy.

The situation is not materially altered for a system composed of many objects. There may be forces between the objects, but the internal forces cannot change the system’s momentum. (If they did, then removing the external forces would result in a closed system that could change its own momentum, like the mythical man who could pull himself up by his own bootstraps. That would violate conservation of momentum.) The equation above becomes

\[ F_{\text{total}} = \frac{\Delta p_{\text{total}}}{\Delta t}. \]

[relationship between the total external force on a system and the rate of change of its total momentum; valid only if the force is constant]

\[ \text{Walking into a lamppost} \]

\[ \text{example 14} \]

\[ \text{Starting from rest, you begin walking, bringing your momentum up to 100 kg·m/s. You walk straight into a lamppost. Why is the momentum change of } -100 \text{ kg·m/s caused by the lamppost so much more painful than the change of } +100 \text{ kg·m/s when you started walking?} \]

\[ \text{The situation is one-dimensional, so we can dispense with the vector notation. It probably takes you about 1 s to speed up initially, so the ground’s force on you is } F = \frac{\Delta p}{\Delta t} \approx 100 \text{ N. Your impact with the lamppost, however, is over in the blink of an eye, say } 1/10 \text{ s or less. Dividing by this much smaller } \Delta t \text{ gives a much larger force, perhaps thousands of newtons. (The negative sign simply indicates that the force is in the opposite direction.)} \]

This is also the principle of airbags in cars. The time required for the airbag to decelerate your head is fairly long, the time required for your face to travel 20 or 30 cm. Without an airbag, your face would hit the dashboard, and the time interval would be the much shorter time taken by your skull to move a couple of centimeters while your face compressed. Note that either way, the same amount of mechanical work has to be done on your head: enough to eliminate all its kinetic energy.
The $F - t$ graph for a tennis racquet hitting a ball might look like this. The amount of momentum transferred equals the area under the curve.

If you place a box on a frictionless surface, it will fall over with a very complicated motion that is hard to predict in detail. We know, however, that its center of mass moves in the same direction as its momentum vector points. There are two forces, a normal force and a gravitational force, both of which are vertical. (The gravitational force is actually many gravitational forces acting on all the atoms in the box.) The total force must be vertical, so the momentum vector must be purely vertical too, and the center of mass travels vertically. This is true even if the box bounces and tumbles. [Based on an example by Kleppner and Kolenkow.]

### The area under the force-time graph

Few real collisions involve a constant force. For example, when a tennis ball hits a racquet, the strings stretch and the ball flattens dramatically. They are both acting like springs that obey Hooke’s law, which says that the force is proportional to the amount of stretching or flattening. The force is therefore small at first, ramps up to a maximum when the ball is about to reverse directions, and ramps back down again as the ball is on its way back out. The equation $F = \Delta p/\Delta t$, derived under the assumption of constant acceleration, does not apply here, and the force does not even have a single well-defined numerical value that could be plugged in to the equation.

As with similar-looking equations such as $v = \Delta p/\Delta t$, the equation $F = \Delta p/\Delta t$ is correctly generalized by saying that the force is the slope of the $p - t$ graph.
Conversely, if we wish to find $\Delta p$ from a graph such as the one in figure o, one approach would be to divide the force by the mass of the ball, rescaling the $F$ axis to create a graph of acceleration versus time. The area under the acceleration-versus-time graph gives the change in velocity, which can then be multiplied by the mass to find the change in momentum. An unnecessary complication was introduced, however, because we began by dividing by the mass and ended by multiplying by it. It would have made just as much sense to find the area under the original $F - t$ graph, which would have given us the momentum change directly.

**Discussion question**

A Many collisions, like the collision of a bat with a baseball, appear to be instantaneous. Most people also would not imagine the bat and ball as bending or being compressed during the collision. Consider the following possibilities:

1. The collision is instantaneous.
2. The collision takes a finite amount of time, during which the ball and bat retain their shapes and remain in contact.
3. The collision takes a finite amount of time, during which the ball and bat are bending or being compressed.

How can two of these be ruled out based on energy or momentum considerations?

### 14.5 Momentum in three dimensions

In this section we discuss how the concepts applied previously to one-dimensional situations can be used as well in three dimensions. Often vector addition is all that is needed to solve a problem:

**An explosion**

Astronomers observe the planet Mars as the Martians fight a nuclear war. The Martian bombs are so powerful that they rip the planet into three separate pieces of liquified rock, all having the same mass. If one fragment flies off with velocity components

\[
\begin{align*}
    v_{1x} &= 0 \\
    v_{1y} &= 1.0 \times 10^4 \text{ km/hr},
\end{align*}
\]

and the second with

\[
\begin{align*}
    v_{2x} &= 1.0 \times 10^4 \text{ km/hr} \\
    v_{2y} &= 0,
\end{align*}
\]

(all in the center of mass frame) what is the magnitude of the third one’s velocity?
In the center of mass frame, the planet initially had zero momentum. After the explosion, the vector sum of the momenta must still be zero. Vector addition can be done by adding components, so

\[ \begin{align*}
mv_{1x} + mv_{2x} + mv_{3x} &= 0, \\
mv_{1y} + mv_{2y} + mv_{3y} &= 0,
\end{align*} \]

where we have used the same symbol \( m \) for all the terms, because the fragments all have the same mass. The masses can be eliminated by dividing each equation by \( m \), and we find

\[ \begin{align*}
v_{3x} &= -1.0 \times 10^4 \text{ km/hr} \\
v_{3y} &= -1.0 \times 10^4 \text{ km/hr}
\] which gives a magnitude of

\[ |v_3| = \sqrt{v_{3x}^2 + v_{3y}^2} \]
\[ = 1.4 \times 10^4 \text{ km/hr} \]

The center of mass

In three dimensions, we have the vector equations

\[ \mathbf{F}_{\text{total}} = \frac{\Delta \mathbf{p}_{\text{total}}}{\Delta t} \]

and

\[ \mathbf{p}_{\text{total}} = m_{\text{total}} \mathbf{v}_{\text{cm}}. \]

The following is an example of their use.

**The bola example 18**

The bola, similar to the North American lasso, is used by South American gauchos to catch small animals by tangling up their legs in the three leather thongs. The motion of the whirling bola through the air is extremely complicated, and would be a challenge to analyze mathematically. The motion of its center of mass, however, is much simpler. The only forces on it are gravitational, so

\[ \mathbf{F}_{\text{total}} = m_{\text{total}} \mathbf{g}. \]

Using the equation \( \mathbf{F}_{\text{total}} = \Delta \mathbf{p}_{\text{total}} / \Delta t \), we find

\[ \Delta \mathbf{p}_{\text{total}} / \Delta t = m_{\text{total}} \mathbf{g}. \]

and since the mass is constant, the equation \( \mathbf{p}_{\text{total}} = m_{\text{total}} \mathbf{v}_{\text{cm}} \) allows us to change this to

\[ m_{\text{total}} \Delta \mathbf{v}_{\text{cm}} / \Delta t = m_{\text{total}} \mathbf{g}. \]
The mass cancels, and $\Delta \mathbf{v}_{cm}/\Delta t$ is simply the acceleration of the center of mass, so

$$a_{cm} = g.$$  

In other words, the motion of the system is the same as if all its mass was concentrated at and moving with the center of mass. The bola has a constant downward acceleration equal to $g$, and flies along the same parabola as any other projectile thrown with the same initial center of mass velocity. Throwing a bola with the correct rotation is presumably a difficult skill, but making it hit its target is no harder than it is with a ball or a single rock.

[Based on an example by Kleppner and Kolenkow.]

**Counting equations and unknowns**

Counting equations and unknowns is just as useful as in one dimension, but every object’s momentum vector has three components, so an unknown momentum vector counts as three unknowns. Conservation of momentum is a single vector equation, but it says that all three components of the total momentum vector stay constant, so we count it as three equations. Of course if the motion happens to be confined to two dimensions, then we need only count vectors as having two components.

**A two-car crash with sticking**  
Suppose two cars collide, stick together, and skid off together. If we know the cars’ initial momentum vectors, we can count equations and unknowns as follows:

unknown #1: $x$ component of cars’ final, total momentum

unknown #2: $y$ component of cars’ final, total momentum

equation #1: conservation of the total $p_x$

equation #2: conservation of the total $p_y$

Since the number of equations equals the number of unknowns, there must be one unique solution for their total momentum vector after the crash. In other words, the speed and direction at which their common center of mass moves off together is unaffected by factors such as whether the cars collide center-to-center or catch each other a little off-center.

**Shooting pool**  
Two pool balls collide, and as before we assume there is no decrease in the total kinetic energy, i.e., no energy converted from KE into other forms. As in the previous example, we assume we are given the initial velocities and want to find the final velocities. The equations and unknowns are:

unknown #1: $x$ component of ball #1’s final momentum

unknown #2: $y$ component of ball #1’s final momentum
unknown #3: $x$ component of ball #2's final momentum
unknown #4: $y$ component of ball #2's final momentum
equation #1: conservation of the total $p_x$
equation #2: conservation of the total $p_y$
equation #3: no decrease in total KE

Note that we do not count the balls' final kinetic energies as unknowns, because knowing the momentum vector, one can always find the velocity and thus the kinetic energy. The number of equations is less than the number of unknowns, so no unique result is guaranteed. This is what makes pool an interesting game. By aiming the cue ball to one side of the target ball you can have some control over the balls' speeds and directions of motion after the collision.

It is not possible, however, to choose any combination of final speeds and directions. For instance, a certain shot may give the correct direction of motion for the target ball, making it go into a pocket, but may also have the undesired side-effect of making the cue ball go in a pocket.

Calculations with the momentum vector

The following example illustrates how a force is required in order to change the direction of the momentum vector, just as one would be required to change its magnitude.

**A turbine**

Example 21

In a hydroelectric plant, water flowing over a dam drives a turbine, which runs a generator to make electric power. The figure shows a simplified physical model of the water hitting the turbine, in which it is assumed that the stream of water comes in at a 45° angle with respect to the turbine blade, and bounces off at a 90° angle at nearly the same speed. The water flows at a rate $R$, in units of kg/s, and the speed of the water is $v$. What are the magnitude and direction of the water's force on the turbine?

In a time interval $\Delta t$, the mass of water that strikes the blade is $R \Delta t$, and the magnitude of its initial momentum is $mv = vR \Delta t$. The water's final momentum vector is of the same magnitude, but in the perpendicular direction. By Newton's third law, the water's force on the blade is equal and opposite to the blade's force on the water. Since the force is constant, we can use the equation

$$F_{\text{blade on water}} = \frac{\Delta p_{\text{water}}}{\Delta t}.$$ 

Choosing the $x$ axis to be to the right and the $y$ axis to be up, this...
can be broken down into components as

\[ F_{\text{blade on water}, x} = \frac{\Delta p_{\text{water}, x}}{\Delta t} \]
\[ = -v R \Delta t - 0 \]
\[ = -v R \]

and

\[ F_{\text{blade on water}, y} = \frac{\Delta p_{\text{water}, y}}{\Delta t} \]
\[ = 0 - (-v R \Delta t) \]
\[ = v R. \]

The water’s force on the blade thus has components

\[ F_{\text{water on blade}, x} = v R \]
\[ F_{\text{water on blade}, y} = -v R. \]

In situations like this, it is always a good idea to check that the result makes sense physically. The x component of the water’s force on the blade is positive, which is correct since we know the blade will be pushed to the right. The y component is negative, which also makes sense because the water must push the blade down. The magnitude of the water’s force on the blade is

\[ |F_{\text{water on blade}}| = \sqrt{2} v R \]

and its direction is at a 45-degree angle down and to the right.

**Discussion questions**

A. The figures show a jet of water striking two different objects. How does the total downward force compare in the two cases? How could this fact be used to create a better waterwheel? (Such a waterwheel is known as a Pelton wheel.)
14.6 Applications of calculus

By now you will have learned to recognize the circumlocutions I use in the sections without calculus in order to introduce calculus-like concepts without using the notation, terminology, or techniques of calculus. It will therefore come as no surprise to you that the rate of change of momentum can be represented with a derivative,

\[ F_{\text{total}} = \frac{dp_{\text{total}}}{dt}. \]

And of course the business about the area under the \( F - t \) curve is really an integral, \( \Delta p_{\text{total}} = \int F_{\text{total}} \, dt \), which can be made into an integral of a vector in the more general three-dimensional case:

\[ \Delta p_{\text{total}} = \int \mathbf{F}_{\text{total}} \, dt. \]

In the case of a material object that is neither losing nor picking up mass, these are just trivially rearranged versions of familiar equations, e.g., \( F = m \frac{dv}{dt} \) rewritten as \( F = \frac{d(mv)}{dt} \). The following is a less trivial example, where \( F = ma \) alone would not have been very easy to work with.

Rain falling into a moving cart example 22

If 1 kg/s of rain falls vertically into a 10-kg cart that is rolling without friction at an initial speed of 1.0 m/s, what is the effect on the speed of the cart when the rain first starts falling?

The rain and the cart make horizontal forces on each other, but there is no external horizontal force on the rain-plus-cart system, so the horizontal motion obeys

\[ F = \frac{d(mv)}{dt} = 0 \]

We use the product rule to find

\[ 0 = \frac{dm}{dt} v + m \frac{dv}{dt}. \]

We are trying to find how \( v \) changes, so we solve for \( dv/dt \),

\[ \frac{dv}{dt} = -\frac{v \frac{dm}{dt}}{m \frac{dt}{dt}} \]

\[ = -\left( \frac{1 \text{ m/s}}{10 \text{ kg}} \right) \left( \frac{1 \text{ kg/s}}{10 \text{ kg}} \right) \]

\[ = -0.1 \text{ m/s}^2. \]

(This is only at the moment when the rain starts to fall.)

Finally we note that there are cases where \( F = ma \) is not just less convenient than \( F = dp/dt \) but in fact \( F = ma \) is wrong and \( F = dp/dt \) is right. A good example is the formation of a comet’s
tail by sunlight. We cannot use $F = ma$ to describe this process, since we are dealing with a collision of light with matter, whereas Newton’s laws only apply to matter. The equation $F = \frac{dp}{dt}$, on the other hand, allows us to find the force experienced by an atom of gas in the comet’s tail if we know the rate at which the momentum vectors of light rays are being turned around by reflection from the atom.
Summary

Selected vocabulary

momentum . . . a measure of motion, equal to $mv$ for material objects

collision . . . . . an interaction between moving objects that lasts for a certain time

center of mass . . the balance point or average position of the mass in a system

Notation

$p$ . . . . . . . . . . the momentum vector

$cm$ . . . . . . . . . . center of mass, as in $x_{cm}, a_{cm}$, etc.

Other terminology and notation

impulse, $I, J$ . . the amount of momentum transferred, $\Delta p$

elastic collision . . one in which no KE is converted into other forms of energy

inelastic collision . . one in which some KE is converted to other forms of energy

Summary

If two objects interact via a force, Newton’s third law guarantees that any change in one’s velocity vector will be accompanied by a change in the other’s which is in the opposite direction. Intuitively, this means that if the two objects are not acted on by any external force, they cannot cooperate to change their overall state of motion. This can be made quantitative by saying that the quantity $m_1v_1 + m_2v_2$ must remain constant as long as the only forces are the internal ones between the two objects. This is a conservation law, called the conservation of momentum, and like the conservation of energy, it has evolved over time to include more and more phenomena unknown at the time the concept was invented. The momentum of a material object is

$$p = mv,$$

but this is more like a standard for comparison of momenta rather than a definition. For instance, light has momentum, but has no mass, and the above equation is not the right equation for light. The law of conservation of momentum says that the total momentum of any closed system, i.e., the vector sum of the momentum vectors of all the things in the system, is a constant.

An important application of the momentum concept is to collisions, i.e., interactions between moving objects that last for a certain amount of time while the objects are in contact or near each other. Conservation of momentum tells us that certain outcomes of a collision are impossible, and in some cases may even be sufficient to predict the motion after the collision. In other cases, conservation of momentum does not provide enough equations to find all the unknowns. In some collisions, such as the collision of a superball with

386 Chapter 14 Conservation of Momentum
the floor, very little kinetic energy is converted into other forms of energy, and this provides one more equation, which may suffice to predict the outcome.

The total momentum of a system can be related to its total mass and the velocity of its center of mass by the equation

\[ \mathbf{p}_{\text{total}} = m_{\text{total}} \mathbf{v}_{\text{cm}}. \]

The center of mass, introduced on an intuitive basis in book 1 as the “balance point” of an object, can be generalized to any system containing any number of objects, and is defined mathematically as the weighted average of the positions of all the parts of all the objects,

\[ x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots}, \]

with similar equations for the \( y \) and \( z \) coordinates.

The frame of reference moving with the center of mass of a closed system is always a valid inertial frame, and many problems can be greatly simplified by working them in the inertial frame. For example, any collision between two objects appears in the c.m. frame as a head-on one-dimensional collision.

When a system is not closed, the rate at which momentum is transferred in or out is simply the total force being exerted externally on the system. If the force is constant,

\[ F_{\text{total}} = \frac{\Delta \mathbf{p}_{\text{total}}}{\Delta t}. \]

When the force is not constant, the force equals the slope of the tangent line on a graph of \( p \) versus \( t \), and the change in momentum equals the area under the \( F - t \) graph.
Problems

Key

✓ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 Derive a formula expressing the kinetic energy of an object in terms of its momentum and mass.

2 Two people in a rowboat wish to move around without causing the boat to move. What should be true about their total momentum? Explain.

3 A learjet traveling due east at 300 mi/hr collides with a jumbo jet which was heading southwest at 150 mi/hr. The jumbo jet’s mass is five times greater than that of the learjet. When they collide, the learjet sticks into the fuselage of the jumbo jet, and they fall to earth together. Their engines stop functioning immediately after the collision. On a map, what will be the direction from the location of the collision to the place where the wreckage hits the ground? (Give an angle.)

4 A bullet leaves the barrel of a gun with a kinetic energy of 90 J. The gun barrel is 50 cm long. The gun has a mass of 4 kg, the bullet 10 g.
   (a) Find the bullet’s final velocity.
   (b) Find the bullet’s final momentum.
   (c) Find the momentum of the recoiling gun.
   (d) Find the kinetic energy of the recoiling gun, and explain why the recoiling gun does not kill the shooter.

5 The graph shows the force, in meganewtons, exerted by a rocket engine on the rocket as a function of time. If the rocket’s mass is 4000 kg, at what speed is the rocket moving when the engine
Problem 8

stops firing? Assume it goes straight up, and neglect the force of gravity, which is much less than a meganewton.

6 Cosmic rays are particles from outer space, mostly protons and atomic nuclei, that are continually bombarding the earth. Most of them, although they are moving extremely fast, have no discernible effect even if they hit your body, because their masses are so small. Their energies vary, however, and a very small minority of them have extremely large energies. In some cases the energy is as much as several Joules, which is comparable to the KE of a well thrown rock! If you are in a plane at a high altitude and are so incredibly unlucky as to be hit by one of these rare ultra-high-energy cosmic rays, what would you notice, the momentum imparted to your body, the energy dissipated in your body as heat, or both? Base your conclusions on numerical estimates, not just random speculation. (At these high speeds, one should really take into account the deviations from Newtonian physics described by Einstein’s special theory of relativity. Don’t worry about that, though.)

7 Show that for a body made up of many equal masses, the equation for the center of mass becomes a simple average of all the positions of the masses.

8 The figure shows a view from above of a collision about to happen between two air hockey pucks sliding without friction. They have the same speed, \( v_i \), before the collision, but the big puck is 2.3 times more massive than the small one. Their sides have sticky stuff on them, so when they collide, they will stick together. At what angle will they emerge from the collision? In addition to giving a numerical answer, please indicate by drawing on the figure how your angle is defined.

Solution, p. 558

9 A flexible rope of mass \( m \) and length \( L \) slides without friction over the edge of a table. Let \( x \) be the length of the rope that is hanging over the edge at a given moment in time.

(a) Show that \( x \) satisfies the equation of motion \( \frac{d^2 x}{dt^2} = \frac{gx}{L} \).

[Hint: Use \( F = \frac{dp}{dt} \), which allows you to handle the two parts of the rope separately even though mass is moving out of one part and into the other.]

(b) Give a physical explanation for the fact that a larger value of \( x \) on the right-hand side of the equation leads to a greater value of the acceleration on the left side.

(c) When we take the second derivative of the function \( x(t) \) we are supposed to get essentially the same function back again, except for a constant out in front. The function \( e^x \) has the property that it is unchanged by differentiation, so it is reasonable to look for solutions to this problem that are of the form \( x = be^{ct} \), where \( b \) and \( c \) are constants. Show that this does indeed provide a solution for two specific values of \( c \) (and for any value of \( b \)).

(d) Show that the sum of any two solutions to the equation of motion
is also a solution.

(e) Find the solution for the case where the rope starts at rest at  
\[ t = 0 \]  with some nonzero value of \( x \).

10    A very massive object with velocity \( v \) collides head-on with  
an object at rest whose mass is very small. No kinetic energy is  
converted into other forms. Prove that the low-mass object recoils  
with velocity \( 2v \). [Hint: Use the center-of-mass frame of reference.]

11    When the contents of a refrigerator cool down, the changed  
molecular speeds imply changes in both momentum and energy.  
Why, then, does a fridge transfer power through its radiator coils,  
but not force?

12    A 10-kg bowling ball moving at 2.0 m/s hits a 1.0-kg bowling  
pin, which is initially at rest. The other pins are all gone already,  
and the collision is head-on, so that the motion is one-dimensional.  
Assume that negligible amounts of heat and sound are produced.  
Find the velocity of the pin immediately after the collision.

13    A rocket ejects exhaust with an exhaust velocity \( u \). The rate  
at which the exhaust mass is used (mass per unit time) is \( b \). We  
assume that the rocket accelerates in a straight line starting from  
rest, and that no external forces act on it. Let the rocket’s initial  
mass (fuel plus the body and payload) be \( m_i \), and \( m_f \) be its final  
mass, after all the fuel is used up. (a) Find the rocket’s final velocity,  
\( v \), in terms of \( u \), \( m_i \), and \( m_f \). Neglect the effects of special relativity.  
(b) A typical exhaust velocity for chemical rocket engines is 4000  
m/s. Estimate the initial mass of a rocket that could accelerate a  
one-ton payload to 10% of the speed of light, and show that this  
design won’t work. (For the sake of the estimate, ignore the mass of  
the fuel tanks. The speed is fairly small compared to \( c \), so it’s not  
an unreasonable approximation to ignore relativity.)

14    A firework shoots up into the air, and just before it explodes  
it has a certain momentum and kinetic energy. What can you say  
about the momenta and kinetic energies of the pieces immediately  
after the explosion? [Based on a problem from PSSC Physics.]

15    Suppose a system consisting of pointlike particles has a total  
kinetic energy \( K_{cm} \) measured in the center-of-mass frame of reference.  
Since they are pointlike, they cannot have any energy due to  
internal motion.

(a) Prove that in a different frame of reference, moving with velocity \( u \) relative to the center-of-mass frame, the total kinetic energy  
equals \( K_{cm} + M|u|^2/2 \), where \( M \) is the total mass. [Hint: You can  
save yourself a lot of writing if you express the total kinetic energy  
using the dot product.]

(b) Use this to prove that if energy is conserved in one frame of
16 The big difference between the equations for momentum and kinetic energy is that one is proportional to $v$ and one to $v^2$. Both, however, are proportional to $m$. Suppose someone tells you that there’s a third quantity, funkosity, defined as $f = m^2v$, and that funkosity is conserved. How do you know your leg is being pulled?

Solution, p. 559

17 A mass $m$ moving at velocity $v$ collides with a stationary target having the same mass $m$. Find the maximum amount of energy that can be released as heat and sound.

18 Two blobs of putty collide head-on and stick. The collision is completely symmetric: the blobs are of equal mass, and they collide at equal speeds. What becomes of the energy the blobs had before the collision? The momentum?

19 The force acting on an object is $F = At^2$. The object is at rest at time $t = 0$. What is its momentum at $t = T$?

20 A bullet of mass $m$ strikes a block of mass $M$ which is hanging by a string of length $L$ from the ceiling. It is observed that, after the sticky collision, the maximum angle that the string makes with the vertical is $\theta$. This setup is called a ballistic pendulum, and it can be used to measure the speed of the bullet.

(a) What vertical height does the block reach?
(b) What was the speed of the block just after the collision?
(c) What was the speed of the bullet just before it struck the block?

21 A car of mass $M$ and a truck of mass $2M$ collide head-on with equal speeds $v$, and the collision is perfectly inelastic, i.e., the maximum possible amount of kinetic energy is transformed into heat and sound, consistent with conservation of momentum.

(a) What is the magnitude of the change in momentum of the car?
(b) What is the magnitude of the change in momentum of the truck?
(c) What is the final speed of the two vehicles?
(d) What fraction of the initial kinetic energy was lost as a result of the collision?
“Sure, and maybe the sun won’t come up tomorrow.” Of course, the sun only appears to go up and down because the earth spins, so the cliche should really refer to the unlikelihood of the earth’s stopping its rotation abruptly during the night. Why can’t it stop? It wouldn’t violate conservation of momentum, because the earth’s rotation doesn’t add anything to its momentum. While California spins in one direction, some equally massive part of India goes the opposite way, canceling its momentum. A halt to Earth’s rotation would entail a drop in kinetic energy, but that energy could simply be converted into some other form, such as heat.

Other examples along these lines are not hard to find. A hydrogen atom spins at the same rate for billions of years. A high-diver who is rotating when he comes off the board does not need to make
any physical effort to continue rotating, and indeed would be unable to stop rotating before he hit the water.

These observations have the hallmarks of a conservation law:

**A closed system is involved.** Nothing is making an effort to twist the earth, the hydrogen atom, or the high-diver. They are isolated from rotation-changing influences, i.e., they are closed systems.

**Something remains unchanged.** There appears to be a numerical quantity for measuring rotational motion such that the total amount of that quantity remains constant in a closed system.

**Something can be transferred back and forth without changing the total amount.** In figure a, the jumper wants to get his feet out in front of him so he can keep from doing a “face plant” when he lands. Bringing his feet forward would involve a certain quantity of counterclockwise rotation, but he didn’t start out with any rotation when he left the ground. Suppose we consider counterclockwise as positive and clockwise as negative. The only way his legs can acquire some positive rotation is if some other part of his body picks up an equal amount of negative rotation. This is why he swings his arms up behind him, clockwise.

![An early photograph of an old-fashioned long-jump.](image)

What numerical measure of rotational motion is conserved? Car engines and old-fashioned LP records have speeds of rotation measured in rotations per minute (r.p.m.), but the number of rotations per minute (or per second) is not a conserved quantity. A twirling figure skater, for instance, can pull her arms in to increase her r.p.m.’s. The first section of this chapter deals with the numerical definition of the quantity of rotation that results in a valid conservation law.
15.1 Conservation of angular momentum

When most people think of rotation, they think of a solid object like a wheel rotating in a circle around a fixed point. Examples of this type of rotation, called rigid rotation or rigid-body rotation, include a spinning top, a seated child’s swinging leg, and a helicopter’s spinning propeller. Rotation, however, is a much more general phenomenon, and includes noncircular examples such as a comet in an elliptical orbit around the sun, or a cyclone, in which the core completes a circle more quickly than the outer parts.

If there is a numerical measure of rotational motion that is a conserved quantity, then it must include nonrigid cases like these, since nonrigid rotation can be traded back and forth with rigid rotation. For instance, there is a trick for finding out if an egg is raw or hardboiled. If you spin a hardboiled egg and then stop it briefly with your finger, it stops dead. But if you do the same with a raw egg, it springs back into rotation because the soft interior was still swirling around within the momentarily motionless shell. The pattern of flow of the liquid part is presumably very complex and nonuniform due to the asymmetric shape of the egg and the different consistencies of the yolk and the white, but there is apparently some way to describe the liquid’s total amount of rotation with a single number, of which some percentage is given back to the shell when you release it.

The best strategy is to devise a way of defining the amount of rotation of a single small part of a system. The amount of rotation of a system such as a cyclone will then be defined as the total of all the contributions from its many small parts.

The quest for a conserved quantity of rotation even requires us to broaden the rotation concept to include cases where the motion doesn’t repeat or even curve around. If you throw a piece of putty at a door, the door will recoil and start rotating. The putty was traveling straight, not in a circle, but if there is to be a general conservation law that can cover this situation, it appears that we must describe the putty as having had some “rotation,” which it then gave up to the door. The best way of thinking about it is to attribute rotation to any moving object or part of an object that changes its angle in relation to the axis of rotation. In the putty-and-door example, the hinge of the door is the natural point to think of as an axis, and the putty changes its angle as seen by someone standing at the hinge. For this reason, the conserved quantity we are investigating is called angular momentum. The symbol for angular momentum can’t be \( a \) or \( m \), since those are used for acceleration and mass, so the symbol \( L \) is arbitrarily chosen instead.

Imagine a 1-kg blob of putty, thrown at the door at a speed of 1 m/s, which hits the door at a distance of 1 m from the hinge. We define this blob to have 1 unit of angular momentum. When
it hits the door, the door will recoil and start rotating. We can use the speed at which the door recoils as a measure of the angular momentum the blob brought in.\footnote{We assume that the door is much more massive than the blob. Under this assumption, the speed at which the door recoils is much less than the original speed of the blob, so the blob has lost essentially all its angular momentum, and given it to the door.}

Experiments show, not surprisingly, that a 2-kg blob thrown in the same way makes the door rotate twice as fast, so the angular momentum of the putty blob must be proportional to mass,

\[ L \propto m. \]

Similarly, experiments show that doubling the velocity of the blob will have a doubling effect on the result, so its angular momentum must be proportional to its velocity as well,

\[ L \propto mv. \]

You have undoubtedly had the experience of approaching a closed door with one of those bar-shaped handles on it and pushing on the wrong side, the side close to the hinges. You feel like an idiot, because you have so little leverage that you can hardly budge the door. The same would be true with the putty blob. Experiments would show that the amount of rotation the blob can give to the door is proportional to the distance, \( r \), from the axis of rotation, so angular momentum must also be proportional to \( r \),

\[ L \propto mvr. \]

We are almost done, but there is one missing ingredient. We know on grounds of symmetry that a putty ball thrown directly inward toward the hinge will have no angular momentum to give to the door. After all, there would not even be any way to decide whether the ball’s rotation was clockwise or counterclockwise in this situation. It is therefore only the component of the blob’s velocity vector perpendicular to the door that should be counted in its angular momentum,

\[ L = m v_\perp r. \]

More generally, \( v_\perp \) should be thought of as the component of the object’s velocity vector that is perpendicular to the line joining the object to the axis of rotation.

We find that this equation agrees with the definition of the original putty blob as having one unit of angular momentum, and we can now see that the units of angular momentum are \((\text{kg} \cdot \text{m/s}) \cdot \text{m}\), i.e., \(\text{kg} \cdot \text{m}^2/\text{s}\). This gives us a way of calculating the angular momentum of any material object or any system consisting of material objects:
The angular momentum of a moving particle is

\[ L = mv_\perp r, \]

where \( m \) is its mass, \( v_\perp \) is the component of its velocity vector perpendicular to the line joining it to the axis of rotation, and \( r \) is its distance from the axis. Positive and negative signs are used to describe opposite directions of rotation.

The angular momentum of a finite-sized object or a system of many objects is found by dividing it up into many small parts, applying the equation to each part, and adding to find the total amount of angular momentum.

Note that \( r \) is not necessarily the radius of a circle. (As implied by the qualifiers, matter isn’t the only thing that can have angular momentum. Light can also have angular momentum, and the above equation would not apply to light.)

Conservation of angular momentum has been verified over and over again by experiment, and is now believed to be one of the three most fundamental principles of physics, along with conservation of energy and momentum.

\[ A \text{ figure skater pulls her arms in} \] example 1

When a figure skater is twirling, there is very little friction between her and the ice, so she is essentially a closed system, and her angular momentum is conserved. If she pulls her arms in, she is decreasing \( r \) for all the atoms in her arms. It would violate conservation of angular momentum if she then continued rotating at the same speed, i.e., taking the same amount of time for each revolution, because her arms’ contributions to her angular momentum would have decreased, and no other part of her would have increased its angular momentum. This is impossible because it would violate conservation of angular momentum. If her total angular momentum is to remain constant, the decrease in \( r \) for her arms must be compensated for by an overall increase in her rate of rotation. That is, by pulling her arms in, she substantially reduces the time for each rotation.
Example 3. A view of the earth-moon system from above the north pole. All distances have been highly distorted for legibility. The earth's rotation is counterclockwise from this point of view (arrow). The moon's gravity creates a bulge on the side near it, because its gravitational pull is stronger there, and an "anti-bulge" on the far side, since its gravity there is weaker. For simplicity, let's focus on the tidal bulge closer to the moon. Its frictional force is trying to slow down the earth's rotation, so its force on the earth's solid crust is toward the bottom of the figure. By Newton's third law, the crust must thus make a force on the bulge which is toward the top of the figure. This causes the bulge to be pulled forward at a slight angle, and the bulge's gravity therefore pulls the moon forward, accelerating its orbital motion about the earth and flinging it outward.

Example 2. Changing the axis

An object's angular momentum can be different depending on the axis about which it rotates. Figure g shows two double-exposure photographs a viola player tipping the bow in order to cross from one string to another. Much more angular momentum is required when playing near the bow's handle, called the frog, as in the panel on the right; not only are most of the atoms in the bow at greater distances, \( r \), from the axis of rotation, but the ones in the tip also have more momentum, \( p \). It is difficult for the player to quickly transfer a large angular momentum into the bow, and then transfer it back out just as quickly. (In the language of section 15.4, large torques are required.) This is one of the reasons that string players tend to stay near the middle of the bow as much as possible.

Earth's slowing rotation and the receding moon

As noted in chapter 1, the earth's rotation is actually slowing down very gradually, with the kinetic energy being dissipated as heat by friction between the land and the tidal bulges raised in the seas by the earth's gravity. Does this mean that angular momentum is not really perfectly conserved? No, it just means that the earth is not quite a closed system by itself. If we consider the earth and moon as a system, then the angular momentum lost by the earth must be gained by the moon somehow. In fact very precise measurements of the distance between the earth and the moon have been carried out by bouncing laser beams off of a mirror left there by astronauts, and these measurements show that the moon is receding from the earth at a rate of 4 centimeters per year! The moon's greater value of \( r \) means that it has a greater
angular momentum, and the increase turns out to be exactly the amount lost by the earth. In the days of the dinosaurs, the days were significantly shorter, and the moon was closer and appeared bigger in the sky.

But what force is causing the moon to speed up, drawing it out into a larger orbit? It is the gravitational forces of the earth’s tidal bulges. The effect is described qualitatively in the caption of the figure. The result would obviously be extremely difficult to calculate directly, and this is one of those situations where a conservation law allows us to make precise quantitative statements about the outcome of a process when the calculation of the process itself would be prohibitively complex.

**Restriction to rotation in a plane**

Is angular momentum a vector, or is it a scalar? On p. 206, we defined the distinction between a vector and a scalar in terms of the quantity’s behavior when rotated. If rotation doesn’t change it, it’s a scalar. If rotation affects it in the same way that it would affect an arrow, then it’s a vector. Using these definitions, figure i shows that angular momentum cannot be a scalar.

It turns out that there is a way of defining angular momentum as a vector, but in this book the examples will be confined to a single plane of rotation, i.e., effectively two-dimensional situations. In this special case, we can choose to visualize the plane of rotation from one side or the other, and to define clockwise and counterclockwise rotation as having opposite signs of angular momentum.

Figure j shows a can rolling down a board. Although the can is three-dimensional, we can view it from the side and project out the third dimension, reducing the motion to rotation in a plane. This means that the axis is a point, even though the word “axis” often connotes a line in students’ minds, as in an $x$ or $y$ axis.
We reduce the motion to rotation in a plane, and the axis is then a point.

**Discussion question**

A Conservation of plain old momentum, \( p \), can be thought of as the greatly expanded and modified descendant of Galileo’s original principle of inertia, that no force is required to keep an object in motion. The principle of inertia is counterintuitive, and there are many situations in which it appears superficially that a force *is* needed to maintain motion, as maintained by Aristotle. Think of a situation in which conservation of angular momentum, \( L \), also seems to be violated, making it seem incorrectly that something external must act on a closed system to keep its angular momentum from “running down.”

**15.2 Angular momentum in planetary motion**

We now discuss the application of conservation of angular momentum to planetary motion, both because of its intrinsic importance and because it is a good way to develop a visual intuition for angular momentum.

Kepler’s law of equal areas states that the area swept out by a planet in a certain length of time is always the same. Angular momentum had not been invented in Kepler’s time, and he did not even know the most basic physical facts about the forces at work. He thought of this law as an entirely empirical and unexpectedly simple way of summarizing his data, a rule that succeeded in describing and predicting how the planets sped up and slowed down in their elliptical paths. It is now fairly simple, however, to show that the equal area law amounts to a statement that the planet’s angular momentum stays constant.

There is no simple geometrical rule for the area of a pie wedge cut out of an ellipse, but if we consider a very short time interval, as shown in figure k, the shaded shape swept out by the planet is very nearly a triangle. We do know how to compute the area of a triangle. It is one half the product of the base and the height:

\[
\text{area} = \frac{1}{2}bh.
\]

We wish to relate this to angular momentum, which contains the variables \( r \) and \( v_\perp \). If we consider the sun to be the axis of rotation, then the variable \( r \) is identical to the base of the triangle,