would be like swallowing a bathtub’s worth of energy, the continual
loss of body heat to one’s environment would be like an energy-hose
left on all day, and lifting a bag of cement would be like flicking
it with a few tiny energy-drops. The human body is tremendously
inefficient. The calories we “burn” in heavy exercise are almost all
dissipated directly as body heat.

You take the high road and I’ll take the low road. example 6

Figure f shows two ramps which two balls will roll down. Compare
their final speeds, when they reach point B. Assume friction
is negligible.

Each ball loses some energy because of its decreasing height
above the earth, and conservation of energy says that it must gain
an equal amount of energy of motion (minus a little heat created
by friction). The balls lose the same amount of height, so their
final speeds must be equal.

It’s impressive to note the complete impossibility of solving this
problem using only Newton’s laws. Even if the shape of the track
had been given mathematically, it would have been a formidable
task to compute the balls’ final speed based on vector addition of
the normal force and gravitational force at each point along the way.

How new forms of energy are discovered

Textbooks often give the impression that a sophisticated physics
concept was created by one person who had an inspiration one day,
but in reality it is more in the nature of science to rough out an idea
and then gradually refine it over many years. The idea of energy
was tinkered with from the early 1800’s on, and new types of energy
kept getting added to the list.

To establish the existence of a new form of energy, a physicist
has to

(1) show that it could be converted to and from other forms of
energy; and

(2) show that it related to some definite measurable property of
the object, for example its temperature, motion, position relative to
another object, or being in a solid or liquid state.

For example, energy is released when a piece of iron is soaked in
water, so apparently there is some form of energy already stored in
the iron. The release of this energy can also be related to a definite
measurable property of the chunk of metal: it turns reddish-orange.
There has been a chemical change in its physical state, which we
call rusting.

Although the list of types of energy kept getting longer and
longer, it was clear that many of the types were just variations on
a theme. There is an obvious similarity between the energy needed
to melt ice and to melt butter, or between the rusting of iron and many other chemical reactions. The topic of the next chapter is how this process of simplification reduced all the types of energy to a very small number (four, according to the way I’ve chosen to count them).

It might seem that if the principle of conservation of energy ever appeared to be violated, we could fix it up simply by inventing some new type of energy to compensate for the discrepancy. This would be like balancing your checkbook by adding in an imaginary deposit or withdrawal to make your figures agree with the bank’s statements. Step (2) above guards against this kind of chicanery. In the 1920s there were experiments that suggested energy was not conserved in radioactive processes. Precise measurements of the energy released in the radioactive decay of a given type of atom showed inconsistent results. One atom might decay and release, say, $1.1 \times 10^{-10}$ J of energy, which had presumably been stored in some mysterious form in the nucleus. But in a later measurement, an atom of exactly the same type might release $1.2 \times 10^{-10}$ J. Atoms of the same type are supposed to be identical, so both atoms were thought to have started out with the same energy. If the amount released was random, then apparently the total amount of energy was not the same after the decay as before, i.e., energy was not conserved.

Only later was it found that a previously unknown particle, which is very hard to detect, was being spewed out in the decay. The particle, now called a neutrino, was carrying off some energy, and if this previously unsuspected form of energy was added in, energy was found to be conserved after all. The discovery of the energy discrepancies is seen with hindsight as being step (1) in the establishment of a new form of energy, and the discovery of the neutrino was step (2). But during the decade or so between step (1) and step (2) (the accumulation of evidence was gradual), physicists had the admirable honesty to admit that the cherished principle of conservation of energy might have to be discarded.

**self-check A**

How would you carry out the two steps given above in order to establish that some form of energy was stored in a stretched or compressed spring?

▷ Answer, p. 567

**Mass Into Energy**

Einstein showed that mass itself could be converted to and from energy, according to his celebrated equation $E = mc^2$, in which $c$ is the speed of light. We thus speak of mass as simply another form of energy, and it is valid to measure it in units of joules. The mass of a 15-gram pencil corresponds to about $1.3 \times 10^{15}$ J. The issue is largely academic in the case of the pencil, because very violent processes such as nuclear reactions are required in order to convert any significant fraction of an object’s mass into energy. Cosmic rays, however, are continually striking you and your surroundings and converting part of their energy of motion...
into the mass of newly created particles. A single high-energy cosmic ray can create a “shower” of millions of previously nonexistent particles when it strikes the atmosphere. Einstein’s theories are discussed later in this book.

Even today, when the energy concept is relatively mature and stable, a new form of energy has been proposed based on observations of distant galaxies whose light began its voyage to us billions of years ago. Astronomers have found that the universe’s continuing expansion, resulting from the Big Bang, has not been decelerating as rapidly in the last few billion years as would have been expected from gravitational forces. They suggest that a new form of energy may be at work.

**Discussion question**

A I’m not making this up. XS Energy Drink has ads that read like this: *All the “Energy” ... Without the Sugar! Only 8 Calories!* Comment on this.

### 11.4 Kinetic energy

The technical term for the energy associated with motion is kinetic energy, from the Greek word for motion. (The root is the same as the root of the word “cinema” for a motion picture, and in French the term for kinetic energy is “énergie cinétique.”) To find how much kinetic energy is possessed by a given moving object, we must convert all its kinetic energy into heat energy, which we have chosen as the standard reference type of energy. We could do this, for example, by firing projectiles into a tank of water and measuring the increase in temperature of the water as a function of the projectile’s mass and velocity. Consider the following data from a series of three such experiments:

<table>
<thead>
<tr>
<th>m (kg)</th>
<th>v (m/s)</th>
<th>energy (J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>1.00</td>
<td>0.50</td>
</tr>
<tr>
<td>1.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>2.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Comparing the first experiment with the second, we see that doubling the object’s velocity doesn’t just double its energy, it quadruples it. If we compare the first and third lines, however, we find that doubling the mass only doubles the energy. This suggests that kinetic energy is proportional to mass and to the square of velocity, \( KE \propto mv^2 \), and further experiments of this type would indeed establish such a general rule. The proportionality factor equals 0.5 because of the design of the metric system, so the kinetic energy of a moving object is given by

\[
KE = \frac{1}{2}mv^2.
\]

The metric system is based on the meter, kilogram, and second, with other units being derived from those. Comparing the units on
the left and right sides of the equation shows that the joule can be reexpressed in terms of the basic units as kg·m$^2$/s$^2$.

\textbf{Energy released by a comet impact}  
\textbf{example 7}

Comet Shoemaker-Levy, which struck the planet Jupiter in 1994, had a mass of roughly $4 \times 10^{13}$ kg, and was moving at a speed of 60 km/s. Compare the kinetic energy released in the impact to the total energy in the world's nuclear arsenals, which is $2 \times 10^{19}$ J. Assume for the sake of simplicity that Jupiter was at rest.

Since we assume Jupiter was at rest, we can imagine that the comet stopped completely on impact, and 100% of its kinetic energy was converted to heat and sound. We first convert the speed to mks units, $v = 6 \times 10^4$ m/s, and then plug in to the equation to find that the comet's kinetic energy was roughly $7 \times 10^{22}$ J, or about 3000 times the energy in the world's nuclear arsenals.

\textbf{Energy and relative motion}

Galileo’s Aristotelian enemies (and it is no exaggeration to call them enemies!) would probably have objected to conservation of energy. Galilean got in trouble by claiming that an object in motion would continue in motion indefinitely in the absence of a force. This is not so different from the idea that an object’s kinetic energy stays the same unless there is a mechanism like frictional heating for converting that energy into some other form.

More subtly, however, it’s not immediately obvious that what we’ve learned so far about energy is strictly mathematically consistent with Galileo’s principle that motion is relative. Suppose we verify that a certain process, say the collision of two pool balls, conserves energy as measured in a certain frame of reference: the sum of the balls’ kinetic energies before the collision is equal to their sum after the collision. But what if we were to measure everything in a frame of reference that was in a different state of motion? It’s not immediately obvious that the total energy before the collision will still equal the total energy after the collision. It \textit{does} still work out. Homework problem 13, p. 312, gives a simple numerical example, and the general proof is taken up in problem 15 on p. 390 (with the solution given in the back of the book).

\textbf{Why kinetic energy obeys the equation it does}

I’ve presented the magic expression for kinetic energy, $(1/2)mv^2$, as a purely empirical fact. Does it have any deeper reason that might be knowable to us mere mortals? Yes and no. It contains three factors, and we need to consider each separately.

The reason for the factor of $1/2$ is understandable, but only as an arbitrary historical choice. The metric system was designed so that some of the equations relating to energy would come out looking simple, at the expense of some others, which had to have
inconvenient conversion factors in front. If we were using the old British Engineering System of units in this course, then we’d have the British Thermal Unit (BTU) as our unit of energy. In that system, the equation you’d learn for kinetic energy would have an inconvenient proportionality constant, \( KE = \left(1.29 \times 10^{-3}\right) mv^2 \), with \( KE \) measured in units of BTUs, \( v \) measured in feet per second, and so on. At the expense of this inconvenient equation for kinetic energy, the designers of the British Engineering System got a simple rule for calculating the energy required to heat water: one BTU per degree Fahrenheit per pound. The inventor of kinetic energy, Thomas Young, actually defined it as \( KE = mv^2 \), which meant that all his other equations had to be different from ours by a factor of two. All these systems of units work just fine as long as they are not combined with one another in an inconsistent way.

The proportionality to \( m \) is inevitable because the energy concept is based on the idea that we add up energy contributions from all the objects within a system. Therefore it is logically necessary that a 2 kg object moving at 1 m/s have the same kinetic energy as two 1 kg objects moving side-by-side at the same speed.

What about the proportionality to \( v^2 \)? Consider:

1. It’s surprisingly hard to tamper with this factor without breaking things: see discussion questions A and B on p. 306.

2. The proportionality to \( v^2 \) is not even correct, except as a low-velocity approximation. Experiments show deviations from the \( v^2 \) rule at high speeds (figure g), an effect that is related to Einstein’s theory of relativity.

3. As described on p. 304, we want conservation of energy to keep working when we switch frames of reference. The fact that this does work for \( KE \propto v^2 \) is intimately connected with the assumption that when we change frames, velocities add as described in section 2.5. This assumption turns out to be an approximation, which only works well at low velocities.

4. Conservation laws are of more general validity than Newton’s laws, which apply to material objects moving at low speeds. Under the conditions where Newton’s laws are accurate, they follow logically from the conservation laws. Therefore we need kinetic energy to have low-velocity behavior that ends up correctly reproducing Newton’s laws.

So under a certain set of low-velocity approximations, \( KE \propto v^2 \) is what works. We verify in problem 15, p. 390, that it satisfies criterion 3, and we show in section 13.6, p. 348, that it is the only such relation that satisfies criterion 4.
Discussion questions

A Suppose that, like Young or Einstein, you were trying out different equations for kinetic energy to see if they agreed with the experimental data. Based on the meaning of positive and negative signs of velocity, why would you suspect that a proportionality to $mv$ would be less likely than $mv^2$?

B As in discussion question A, try to think of an argument showing that $m(v^2 + v^4)$ is not a possible formula for kinetic energy.

C The figure shows a pendulum that is released at A and caught by a peg as it passes through the vertical, B. To what height will the bob rise on the right?

11.5 Power

A car may have plenty of energy in its gas tank, but still may not be able to increase its kinetic energy rapidly. A Porsche doesn’t necessarily have more energy in its gas tank than a Hyundai, it is just able to transfer it more quickly. The rate of transferring energy from one form to another is called power. The definition can be written as an equation,

$$P = \frac{\Delta E}{\Delta t},$$

where the use of the delta notation in the symbol $\Delta E$ has the usual interpretation: the final amount of energy in a certain form minus the initial amount that was present in that form. Power has units of J/s, which are abbreviated as watts, W (rhymes with “lots”).

If the rate of energy transfer is not constant, the power at any instant can be defined as the slope of the tangent line on a graph of $E$ versus $t$. Likewise $\Delta E$ can be extracted from the area under the $P$-versus-$t$ curve.

\[\text{Converting kilowatt-hours to joules example 8}\]

- The electric company bills you for energy in units of kilowatt-hours (kilowatts multiplied by hours) rather than in SI units of joules. How many joules is a kilowatt-hour?

- $1 \text{ kilowatt-hour} = (1 \text{ kW})(1 \text{ hour}) = (1000 \text{ J/s})(3600 \text{ s}) = 3.6 \text{ MJ}.$

\[\text{Human wattage example 9}\]

- A typical person consumes 2000 kcal of food in a day, and converts nearly all of that directly to heat. Compare the person’s heat output to the rate of energy consumption of a 100-watt lightbulb.

- Looking up the conversion factor from calories to joules, we find

$$\Delta E = 2000 \text{ kcal} \times \frac{1000 \text{ cal}}{1 \text{ kcal}} \times \frac{4.18 \text{ J}}{1 \text{ cal}} = 8 \times 10^6 \text{ J}$$

for our daily energy consumption. Converting the time interval likewise into mks,

$$\Delta t = 1 \text{ day} \times \frac{24 \text{ hours}}{1 \text{ day}} \times \frac{60 \text{ min}}{1 \text{ hour}} \times \frac{60 \text{ s}}{1 \text{ min}} = 9 \times 10^4 \text{ s}.$$
Dividing, we find that our power dissipated as heat is $90 \text{ J/s} = 90 \text{ W}$, about the same as a lightbulb.

It is easy to confuse the concepts of force, energy, and power, especially since they are synonyms in ordinary speech. The table on the following page may help to clear this up:

<table>
<thead>
<tr>
<th></th>
<th>force</th>
<th>energy</th>
<th>power</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>conceptual</strong></td>
<td>A force is an interaction between two objects that causes a push or a pull. A force can be defined as anything that is capable of changing an object’s state of motion.</td>
<td>Heating an object, making it move faster, or increasing its distance from another object that is attracting it are all examples of things that would require fuel or physical effort. All these things can be quantified using a single scale of measurement, and we describe them all as forms of energy.</td>
<td>Power is the rate at which energy is transformed from one form to another or transferred from one object to another.</td>
</tr>
<tr>
<td><strong>definition</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>operational</strong></td>
<td>A spring scale can be used to measure force.</td>
<td>If we define a unit of energy as the amount required to heat a certain amount of water by a $1^\circ\text{C}$, then we can measure any other quantity of energy by transferring it into heat in water and measuring the temperature increase.</td>
<td>Measure the change in the amount of some form of energy possessed by an object, and divide by the amount of time required for the change to occur.</td>
</tr>
<tr>
<td><strong>definition</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>scalar or vector?</strong></td>
<td>vector — has a direction in space which is the direction in which it pulls or pushes</td>
<td>scalar — has no direction in space</td>
<td>scalar — has no direction in space</td>
</tr>
<tr>
<td><strong>unit</strong></td>
<td>newtons (N)</td>
<td>joules (J)</td>
<td>watts (W) = joules/s</td>
</tr>
<tr>
<td><strong>Can it run out? Does it cost money?</strong></td>
<td>No. I don’t have to pay a monthly bill for the meganewtons of force required to hold up my house.</td>
<td>Yes. We pay money for gasoline, electrical energy, batteries, etc., because they contain energy.</td>
<td>More power means you are paying money at a higher rate. A 100-W lightbulb costs a certain number of cents per hour.</td>
</tr>
<tr>
<td><strong>Can it be a property of an object?</strong></td>
<td>No. A force is a relationship between two interacting objects. A home-run baseball doesn’t “have” force.</td>
<td>Yes. What a home-run baseball has is kinetic energy, not force.</td>
<td>Not really. A 100-W lightbulb doesn’t “have” 100 W. 100 J/s is the rate at which it converts electrical energy into light.</td>
</tr>
</tbody>
</table>
Summary
Selected vocabulary
energy . . . . A numerical scale used to measure the heat, motion, or other properties that would require fuel or physical effort to put into an object; a scalar quantity with units of joules (J).

power . . . . The rate of transferring energy; a scalar quantity with units of watts (W).

kinetic energy . . The energy an object possesses because of its motion.

heat . . . . A form of energy that relates to temperature. Heat is different from temperature because an object with twice as much mass requires twice as much heat to increase its temperature by the same amount. Heat is measured in joules, temperature in degrees. (In standard terminology, there is another, finer distinction between heat and thermal energy, which is discussed below. In this book, I informally refer to both as heat.)

temperature . . What a thermometer measures. Objects left in contact with each other tend to reach the same temperature. Cf. heat. As discussed in more detail in chapter 2, temperature is essentially a measure of the average kinetic energy per molecule.

Notation
E . . . . . . . . energy
J . . . . . . . . joules, the SI unit of energy
KE . . . . . . . kinetic energy
P . . . . . . . . power
W . . . . . . . . watts, the SI unit of power; equivalent to J/s

Other terminology and notation
Q or ∆Q . . . . the amount of heat transferred into or out of an object
K or T . . . . alternative symbols for kinetic energy, used in the scientific literature and in most advanced textbooks

thermal energy . Careful writers make a distinction between heat and thermal energy, but the distinction is often ignored in casual speech, even among physicists. Properly, thermal energy is used to mean the total amount of energy possessed by an object, while heat indicates the amount of thermal energy transferred in or out. The term heat is used in this book to include both meanings.
Summary

Heating an object, making it move faster, or increasing its distance from another object that is attracting it are all examples of things that would require fuel or physical effort. All these things can be quantified using a single scale of measurement, and we describe them all as forms of energy. The SI unit of energy is the Joule. The reason why energy is a useful and important quantity is that it is always conserved. That is, it cannot be created or destroyed but only transferred between objects or changed from one form to another. Conservation of energy is the most important and broadly applicable of all the laws of physics, more fundamental and general even than Newton’s laws of motion.

Heating an object requires a certain amount of energy per degree of temperature and per unit mass, which depends on the substance of which the object consists. Heat and temperature are completely different things. Heat is a form of energy, and its SI unit is the joule (J). Temperature is not a measure of energy. Heating twice as much of something requires twice as much heat, but double the amount of a substance does not have double the temperature.

The energy that an object possesses because of its motion is called kinetic energy. Kinetic energy is related to the mass of the object and the magnitude of its velocity vector by the equation

\[ KE = \frac{1}{2}mv^2. \]

Power is the rate at which energy is transformed from one form to another or transferred from one object to another,

\[ P = \frac{\Delta E}{\Delta t}. \]  [only for constant power]

The SI unit of power is the watt (W).
Problems

Key
✓ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 This problem is now problem 14 in chapter 12, on page 329.

2 Can kinetic energy ever be less than zero? Explain. [Based on a problem by Serway and Faughn.]

3 Estimate the kinetic energy of an Olympic sprinter.

4 You are driving your car, and you hit a brick wall head on, at full speed. The car has a mass of 1500 kg. The kinetic energy released is a measure of how much destruction will be done to the car and to your body. Calculate the energy released if you are traveling at (a) 40 mi/hr, and again (b) if you’re going 80 mi/hr. What is counterintuitive about this, and what implication does this have for driving at high speeds?

5 A closed system can be a bad thing — for an astronaut sealed inside a space suit, getting rid of body heat can be difficult. Suppose a 60-kg astronaut is performing vigorous physical activity, expending 200 W of power. If none of the heat can escape from her space suit, how long will it take before her body temperature rises by 6°C (11°F), an amount sufficient to kill her? Assume that the amount of heat required to raise her body temperature by 1°C is the same as it would be for an equal mass of water. Express your answer in units of minutes.

6 All stars, including our sun, show variations in their light output to some degree. Some stars vary their brightness by a factor of two or even more, but our sun has remained relatively steady during the hundred years or so that accurate data have been collected. Nevertheless, it is possible that climate variations such as ice ages are related to long-term irregularities in the sun’s light output. If the sun was to increase its light output even slightly, it could melt enough Antarctic ice to flood all the world’s coastal cities. The total sunlight that falls on Antarctica amounts to about $1 \times 10^{16}$ watts. In the absence of natural or human-caused climate change, this heat input to the poles is balanced by the loss of heat via winds, ocean currents, and emission of infrared light, so that there is no net melting or freezing of ice at the poles from year to year. Suppose that the sun changes its light output by some small percentage, but there is no change in the rate of heat loss by the polar caps. Estimate the percentage by which the sun’s light output would have to increase in order to melt enough ice to raise the level of the oceans by 10 meters over a period of 10 years. (This would be enough to flood New York, London, and many other cities.) Melting 1 kg of ice requires $3 \times 10^3$ J.
7 A bullet flies through the air, passes through a paperback book, and then continues to fly through the air beyond the book. When is there a force? When is there energy? ▶ Solution, p. 558

8 Experiments show that the power consumed by a boat’s engine is approximately proportional to the third power of its speed. (We assume that it is moving at constant speed.) (a) When a boat is cruising at constant speed, what type of energy transformation do you think is being performed? (b) If you upgrade to a motor with double the power, by what factor is your boat’s cruising speed increased? [Based on a problem by Arnold Arons.]
▶ Solution, p. 559

9 Object A has a kinetic energy of 13.4 J. Object B has a mass that is greater by a factor of 3.77, but is moving more slowly by a factor of 2.34. What is object B’s kinetic energy? [Based on a problem by Arnold Arons.]
▶ Solution, p. 559

10 The moon doesn’t really just orbit the Earth. By Newton’s third law, the moon’s gravitational force on the earth is the same as the earth’s force on the moon, and the earth must respond to the moon’s force by accelerating. If we consider the earth and moon in isolation and ignore outside forces, then Newton’s first law says their common center of mass doesn’t accelerate, i.e., the earth wobbles around the center of mass of the earth-moon system once per month, and the moon also orbits around this point. The moon’s mass is 81 times smaller than the earth’s. Compare the kinetic energies of the earth and moon. (We know that the center of mass is a kind of balance point, so it must be closer to the earth than to the moon. In fact, the distance from the earth to the center of mass is 1/81 of the distance from the moon to the center of mass, which makes sense intuitively, and can be proved rigorously using the equation on page 374.)

11 My 1.25 kW microwave oven takes 126 seconds to bring 250 g of water from room temperature to a boil. What percentage of the power is being wasted? Where might the rest of the energy be going? ▶ Solution, p. 559
12 The multiflash photograph shows a collision between two pool balls. The ball that was initially at rest shows up as a dark image in its initial position, because its image was exposed several times before it was struck and began moving. By making measurements on the figure, determine numerically whether or not energy appears to have been conserved in the collision. What systematic effects would limit the accuracy of your test? [From an example in PSSC Physics.]

13 This problem is a numerical example of the imaginary experiment discussed on p. 304 regarding the relationship between energy and relative motion. Let’s say that the pool balls both have masses of 1.00 kg. Suppose that in the frame of reference of the pool table, the cue ball moves at a speed of 1.00 m/s toward the eight ball, which is initially at rest. The collision is head-on, and as you can verify for yourself the next time you’re playing pool, the result of such a collision is that the incoming ball stops dead and the ball that was struck takes off with the same speed originally possessed by the incoming ball. (This is actually a bit of an idealization. To keep things simple, we’re ignoring the spin of the balls, and we assume that no energy is liberated by the collision as heat or sound.) (a) Calculate the total initial kinetic energy and the total final kinetic energy, and verify that they are equal. (b) Now carry out the whole calculation again in the frame of reference that is moving in the same direction that the cue ball was initially moving, but at a speed of 0.50 m/s. In this frame of reference, both balls have nonzero initial and final velocities, which are different from what they were in the table’s frame. [See also problem 15 on p. 390.]
One theory about the destruction of the space shuttle Columbia in 2003 is that one of its wings had been damaged on liftoff by a chunk of foam insulation that fell off of one of its external fuel tanks. The New York Times reported on June 5, 2003, that NASA engineers had recreated the impact to see if it would damage a mock-up of the shuttle’s wing. “Before last week’s test, many engineers at NASA said they thought lightweight foam could not harm the seemingly tough composite panels, and privately predicted that the foam would bounce off harmlessly, like a Nerf ball.” In fact, the 0.80 kg piece of foam, moving at 240 m/s, did serious damage. A member of the board investigating the disaster said this demonstrated that “people’s intuitive sense of physics is sometimes way off.” (a) Compute the kinetic energy of the foam, and (b) compare with the energy of an 80 kg boulder moving at 2.4 m/s (the speed it would have if you dropped it from about knee-level).

(c) The boulder is a hundred times more massive, but its speed is a hundred times smaller, so what’s counterintuitive about your results?

The figure above is from a classic 1920 physics textbook by Millikan and Gale. It represents a method for raising the water from the pond up to the water tower, at a higher level, without using a pump. Water is allowed into the drive pipe, and once it is flowing fast enough, it forces the valve at the bottom closed. Explain how this works in terms of conservation of mass and energy. (Cf. example 1 on page 295.)
The following table gives the amount of energy required in order to heat, melt, or boil a gram of water.

<table>
<thead>
<tr>
<th>Process</th>
<th>Energy (J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heat 1 g of ice by 1°C</td>
<td>2.05</td>
</tr>
<tr>
<td>Melt 1 g of ice</td>
<td>333</td>
</tr>
<tr>
<td>Heat 1 g of water by 1°C</td>
<td>4.19</td>
</tr>
<tr>
<td>Boil 1 g of water</td>
<td>2500</td>
</tr>
<tr>
<td>Heat 1 g of steam by 1°C</td>
<td>2.01</td>
</tr>
</tbody>
</table>

(a) How much energy is required in order to convert 1.00 g of ice at -20 °C into steam at 137 °C?

(b) What is the minimum amount of hot water that could melt 1.00 g of ice?

17 Estimate the kinetic energy of a buzzing fly’s wing. (You may wish to review section 1.4 on order-of-magnitude estimates.)

18 A blade of grass moves upward as it grows. Estimate its kinetic energy. (You may wish to review section 1.4 on order-of-magnitude estimates.)
Do these forms of energy have anything in common?

Chapter 12
Simplifying the Energy Zoo

Variety is the spice of life, not of science. The figure shows a few examples from the bewildering array of forms of energy that surrounds us. The physicist’s psyche rebels against the prospect of a long laundry list of types of energy, each of which would require its own equations, concepts, notation, and terminology. The point at which we’ve arrived in the study of energy is analogous to the period in the 1960’s when a half a dozen new subatomic particles were being discovered every year in particle accelerators. It was an embarrassment. Physicists began to speak of the “particle zoo,” and it seemed that the subatomic world was distressingly complex. The particle zoo was simplified by the realization that most of the
new particles being whipped up were simply clusters of a previously unsuspected set of more fundamental particles (which were whimsically dubbed quarks, a made-up word from a line of poetry by James Joyce, “Three quarks for Master Mark.”) The energy zoo can also be simplified, and it is the purpose of this chapter to demonstrate the hidden similarities between forms of energy as seemingly different as heat and motion.

A vivid demonstration that heat is a form of motion. A small amount of boiling water is poured into the empty can, which rapidly fills up with hot steam. The can is then sealed tightly, and soon crumples. This can be explained as follows. The high temperature of the steam is interpreted as a high average speed of random motions of its molecules. Before the lid was put on the can, the rapidly moving steam molecules pushed their way out of the can, forcing the slower air molecules out of the way. As the steam inside the can thinned out, a stable situation was soon achieved, in which the force from the less dense steam molecules moving at high speed balanced against the force from the more dense but slower air molecules outside. The cap was put on, and after a while the steam inside the can reached the same temperature as the air outside. The force from the cool, thin steam no longer matched the force from the cool, dense air outside, and the imbalance of forces crushed the can.

12.1 Heat is kinetic energy

What is heat really? Is it an invisible fluid that your bare feet soak up from a hot sidewalk? Can one ever remove all the heat from an object? Is there a maximum to the temperature scale?

The theory of heat as a fluid seemed to explain why colder objects absorbed heat from hotter ones, but once it became clear that heat was a form of energy, it began to seem unlikely that a material substance could transform itself into and out of all those other forms of energy like motion or light. For instance, a compost pile gets hot, and we describe this as a case where, through the action of bacteria, chemical energy stored in the plant cuttings is transformed into heat energy. The heating occurs even if there is no nearby warmer object that could have been leaking “heat fluid” into the pile.

An alternative interpretation of heat was suggested by the theory that matter is made of atoms. Since gases are thousands of times less dense than solids or liquids, the atoms (or clusters of atoms called molecules) in a gas must be far apart. In that case, what is keeping all the air molecules from settling into a thin film on the floor of the room in which you are reading this book? The simplest explanation is that they are moving very rapidly, continually ricocheting off of
the floor, walls, and ceiling. Though bizarre, the cloud-of-bullets image of a gas did give a natural explanation for the surprising ability of something as tenuous as a gas to exert huge forces. Your car’s tires can hold it up because you have pumped extra molecules into them. The inside of the tire gets hit by molecules more often than the outside, forcing it to stretch and stiffen.

The outward forces of the air in your car’s tires increase even further when you drive on the freeway for a while, heating up the rubber and the air inside. This type of observation leads naturally to the conclusion that hotter matter differs from colder in that its atoms’ random motion is more rapid. In a liquid, the motion could be visualized as people in a milling crowd shoving past each other more quickly. In a solid, where the atoms are packed together, the motion is a random vibration of each atom as it knocks against its neighbors.

We thus achieve a great simplification in the theory of heat. Heat is simply a form of kinetic energy, the total kinetic energy of random motion of all the atoms in an object. With this new understanding, it becomes possible to answer at one stroke the questions posed at the beginning of the section. Yes, it is at least theoretically possible to remove all the heat from an object. The coldest possible temperature, known as absolute zero, is that at which all the atoms have zero velocity, so that their kinetic energies, \((1/2)mv^2\), are all zero. No, there is no maximum amount of heat that a certain quantity of matter can have, and no maximum to the temperature scale, since arbitrarily large values of \(v\) can create arbitrarily large amounts of kinetic energy per atom.

The kinetic theory of heat also provides a simple explanation of the true nature of temperature. Temperature is a measure of the amount of energy per molecule, whereas heat is the total amount of energy possessed by all the molecules in an object.

There is an entire branch of physics, called thermodynamics, that deals with heat and temperature and forms the basis for technologies such as refrigeration. Thermodynamics is discussed in more detail in optional chapter 16, and I have provided here only a brief overview of the thermodynamic concepts that relate directly to energy, glossing over at least one point that would be dealt with more carefully in a thermodynamics course: it is really only true for a gas that all the heat is in the form of kinetic energy. In solids and liquids, the atoms are close enough to each other to exert intense electrical forces on each other, and there is therefore another type of energy involved, the energy associated with the atoms’ distances from each other. Strictly speaking, heat energy is defined not as energy associated with random motion of molecules but as any form of energy that can be conducted between objects in contact, without any force.
12.2 Potential energy: energy of distance or closeness

We have already seen many examples of energy related to the distance between interacting objects. When two objects participate in an attractive noncontact force, energy is required to bring them farther apart. In both of the perpetual motion machines that started off the previous chapter, one of the types of energy involved was the energy associated with the distance between the balls and the earth, which attract each other gravitationally. In the perpetual motion machine with the magnet on the pedestal, there was also energy associated with the distance between the magnet and the iron ball, which were attracting each other.

The opposite happens with repulsive forces: two socks with the same type of static electric charge will repel each other, and cannot be pushed closer together without supplying energy.

In general, the term potential energy, with algebra symbol $PE$, is used for the energy associated with the distance between two objects that attract or repel each other via a force that depends on the distance between them. Forces that are not determined by distance do not have potential energy associated with them. For instance, the normal force acts only between objects that have zero distance between them, and depends on other factors besides the fact that the distance is zero. There is no potential energy associated with the normal force.

The following are some commonplace examples of potential energy:

**gravitational potential energy**: The skateboarder in the photo has risen from the bottom of the pool, converting kinetic energy into gravitational potential energy. After being at rest for an instant, he will go back down, converting PE back into KE.

**magnetic potential energy**: When a magnetic compass needle is allowed to rotate, the poles of the compass change their distances from the earth’s north and south magnetic poles, converting magnetic potential energy into kinetic energy. (Eventually the kinetic energy is all changed into heat by friction, and the needle settles down in the position that minimizes its potential energy.)

**electrical potential energy**: Socks coming out of the dryer cling together because of attractive electrical forces. Energy is required in order to separate them.

**potential energy of bending or stretching**: The force between the two ends of a spring depends on the distance between
them, i.e., on the length of the spring. If a car is pressed down on its shock absorbers and then released, the potential energy stored in the spring is transformed into kinetic and gravitational potential energy as the car bounces back up.

I have deliberately avoided introducing the term potential energy up until this point, because it tends to produce unfortunate connotations in the minds of students who have not yet been inoculated with a careful description of the construction of a numerical energy scale. Specifically, there is a tendency to generalize the term inappropriately to apply to any situation where there is the “potential” for something to happen: “I took a break from digging, but I had potential energy because I knew I’d be ready to work hard again in a few minutes.”

An equation for gravitational potential energy

All the vital points about potential energy can be made by focusing on the example of gravitational potential energy. For simplicity, we treat only vertical motion, and motion close to the surface of the earth, where the gravitational force is nearly constant. (The generalization to the three dimensions and varying forces is more easily accomplished using the concept of work, which is the subject of the next chapter.)

To find an equation for gravitational PE, we examine the case of free fall, in which energy is transformed between kinetic energy and gravitational PE. Whatever energy is lost in one form is gained in an equal amount in the other form, so using the notation $\Delta KE$ to stand for $KE_f - KE_i$ and a similar notation for PE, we have

\[ \Delta KE = -\Delta PE_{grav}. \]

It will be convenient to refer to the object as falling, so that PE is being changed into KE, but the math applies equally well to an object slowing down on its way up. We know an equation for kinetic energy,

\[ KE = \frac{1}{2}mv^2, \]

so if we can relate $v$ to height, $y$, we will be able to relate $\Delta PE$ to $y$, which would tell us what we want to know about potential energy. The $y$ component of the velocity can be connected to the height via the constant acceleration equation

\[ v_f^2 = v_i^2 + 2a\Delta y, \]

and Newton’s second law provides the acceleration,

\[ a = F/m, \]
in terms of the gravitational force.

The algebra is simple because both equation [2] and equation [3] have velocity to the second power. Equation [2] can be solved for $v^2$ to give $v^2 = 2KE/m$, and substituting this into equation [3], we find

$$\frac{2 KE_f}{m} = \frac{2 KE_i}{m} + 2a \Delta y.$$

Making use of equations [1] and [4] gives the simple result

$$\Delta PE_{grav} = -F \Delta y.$$  [change in gravitational PE resulting from a change in height $\Delta y$; $F$ is the gravitational force on the object, i.e., its weight; valid only near the surface of the earth, where $F$ is constant]

**Dropping a rock**

example 1

▷ If you drop a 1-kg rock from a height of 1 m, how many joules of KE does it have on impact with the ground? (Assume that any energy transformed into heat by air friction is negligible.)

▷ If we choose the $y$ axis to point up, then $F_y$ is negative, and equals $-(1 \text{ kg})(g) = -9.8 \text{ N}$. A decrease in $y$ is represented by a negative value of $\Delta y$, $\Delta y = -1 \text{ m}$, so the change in potential energy is $-(-9.8 \text{ N})(-1 \text{ m}) \approx -10 \text{ J}$. (The proof that newtons multiplied by meters give units of joules is left as a homework problem.) Conservation of energy says that the loss of this amount of PE must be accompanied by a corresponding increase in KE of 10 J.

It may be dismaying to note how many minus signs had to be handled correctly even in this relatively simple example: a total of four. Rather than depending on yourself to avoid any mistakes with signs, it is better to check whether the final result make sense physically. If it doesn’t, just reverse the sign.

Although the equation for gravitational potential energy was derived by imagining a situation where it was transformed into kinetic energy, the equation can be used in any context, because all the types of energy are freely convertible into each other.
Gravitational PE converted directly into heat example 2

A 50-kg firefighter slides down a 5-m pole at constant velocity. How much heat is produced?

Since she slides down at constant velocity, there is no change in KE. Heat and gravitational PE are the only forms of energy that change. Ignoring plus and minus signs, the gravitational force on her body equals $mg$, and the amount of energy transformed is

$$(mg)(5 \text{ m}) = 2500 \text{ J}.$$  

On physical grounds, we know that there must have been an increase (positive change) in the heat energy in her hands and in the flagpole.

Here are some questions and answers about the interpretation of the equation $\Delta PE_{grav} = -F\Delta y$ for gravitational potential energy.

**Question:** In a nutshell, why is there a minus sign in the equation?

**Answer:** It is because we increase the PE by moving the object in the opposite direction compared to the gravitational force.

**Question:** Why do we only get an equation for the change in potential energy? Don’t I really want an equation for the potential energy itself?

**Answer:** No, you really don’t. This relates to a basic fact about potential energy, which is that it is not a well defined quantity in the absolute sense. Only changes in potential energy are unambiguously defined. If you and I both observe a rock falling, and agree that it deposits 10 J of energy in the dirt when it hits, then we will be forced to agree that the 10 J of KE must have come from a loss of 10 joules of PE. But I might claim that it started with 37 J of PE and ended with 27, while you might swear just as truthfully that it had 109 J initially and 99 at the end. It is possible to pick some specific height as a reference level and say that the PE is zero there, but it’s easier and safer just to work with changes in PE and avoid absolute PE altogether.

**Question:** You referred to potential energy as the energy that two objects have because of their distance from each other. If a rock falls, the object is the rock. Where’s the other object?

**Answer:** Newton’s third law guarantees that there will always be two objects. The other object is the planet earth.

**Question:** If the other object is the earth, are we talking about the distance from the rock to the center of the earth or the distance from the rock to the surface of the earth?

**Answer:** It doesn’t matter. All that matters is the change in distance, $\Delta y$, not $y$. Measuring from the earth’s center or its surface are just two equally valid choices of a reference point for defining absolute PE.

**Question:** Which object contains the PE, the rock or the earth?
All these energy transformations turn out at the atomic level to be changes in potential energy resulting from changes in the distances between atoms.

**Answer:** We may refer casually to the PE of the rock, but technically the PE is a relationship between the earth and the rock, and we should refer to the earth and the rock together as possessing the PE.

**Question:** How would this be any different for a force other than gravity?

**Answer:** It wouldn’t. The result was derived under the assumption of constant force, but the result would be valid for any other situation where two objects interacted through a constant force. Gravity is unusual, however, in that the gravitational force on an object is so nearly constant under ordinary conditions. The magnetic force between a magnet and a refrigerator, on the other hand, changes drastically with distance. The math is a little more complex for a varying force, but the concepts are the same.

**Question:** Suppose a pencil is balanced on its tip and then falls over. The pencil is simultaneously changing its height and rotating, so the height change is different for different parts of the object. The bottom of the pencil doesn’t lose any height at all. What do you do in this situation?

**Answer:** The general philosophy of energy is that an object’s energy is found by adding up the energy of every little part of it. You could thus add up the changes in potential energy of all the little parts of the pencil to find the total change in potential energy. Luckily there’s an easier way! The derivation of the equation for gravitational potential energy used Newton’s second law, which deals with the acceleration of the object’s center of mass (i.e., its balance point). If you just define $\Delta y$ as the height change of the center of mass, everything works out. A huge Ferris wheel can be rotated without putting in or taking out any PE, because its center of mass is staying at the same height.

**Self-check A**

A ball thrown straight up will have the same speed on impact with the ground as a ball thrown straight down at the same speed. How can this be explained using potential energy?  

**Answer, p. 567**

**Discussion question**

You throw a steel ball up in the air. How can you prove based on conservation of energy that it has the same speed when it falls back into your hand? What if you throw a feather up — is energy not conserved in this case?
12.3 All energy is potential or kinetic

In the same way that we found that a change in temperature is really only a change in kinetic energy at the atomic level, we now find that every other form of energy turns out to be a form of potential energy. Boiling, for instance, means knocking some of the atoms (or molecules) out of the liquid and into the space above, where they constitute a gas. There is a net attractive force between essentially any two atoms that are next to each other, which is why matter always prefers to be packed tightly in the solid or liquid state unless we supply enough potential energy to pull it apart into a gas. This explains why water stops getting hotter when it reaches the boiling point: the power being pumped into the water by your stove begins going into potential energy rather than kinetic energy.

As shown in figure e, every stored form of energy that we encounter in everyday life turns out to be a form of potential energy at the atomic level. The forces between atoms are electrical and magnetic in nature, so these are actually electrical and magnetic potential energies.

Although light is a topic of the second half of this course, it is useful to have a preview of how it fits in here. Light is a wave composed of oscillating electric and magnetic fields, so we can include it under the category of electrical and magnetic potential energy.

Even if we wish to include nuclear reactions in the picture, there still turn out to be only four fundamental types of energy:

- kinetic energy (including heat)
- gravitational PE
- electrical and magnetic PE (including light)
- nuclear PE

Discussion question

A Referring back to the pictures at the beginning of the chapter, how do all these forms of energy fit into the shortened list of categories given above?
12.4 Applications

Heat transfer

Conduction

When you hold a hot potato in your hand, energy is transferred from the hot object to the cooler one. Our microscopic picture of this process (figure b, p. 317) tells us that the heat transfer can only occur at the surface of contact, where one layer of atoms in the potato skin make contact with one such layer in the hand. This type of heat transfer is called conduction, and its rate is proportional to both the surface area and the temperature difference.

Convection

In a gas or a liquid, a faster method of heat transfer can occur, because hotter or colder parts of the fluid can flow, physically transporting their heat energy from one place to another. This mechanism of heat transfer, convection, is at work in Los Angeles when hot Santa Ana winds blow in from the Mojave Desert. On a cold day, the reason you feel warmer when there is no wind is that your skin warms a thin layer of air near it by conduction. If a gust of wind comes along, convection robs you of this layer. A thermos bottle has inner and outer walls separated by a layer of vacuum, which prevents heat transport by conduction or convection, except for a tiny amount of conduction through the thin connection between the walls, near the neck, which has a small cross-sectional area.

Radiation

The glow of the sun or a candle flame is an example of heat transfer by radiation. In this context, “radiation” just means anything that radiates outward from a source, including, in these examples, ordinary visible light. The power is proportional to the surface area of the radiating object. It also depends very dramatically on the radiator’s absolute temperature, $P \propto T^4$.

We can easily understand the reason for radiation based on the picture of heat as random kinetic energy at the atomic scale. Atoms are made out of subatomic particles, such as electrons and nuclei, that carry electric charge. When a charged particle vibrates, it creates wave disturbances in the electric and magnetic fields, and the waves have a frequency (number of vibrations per second) that matches the frequency of the particle’s motion. If this frequency is in the right range, they constitute visible light (see section 24.5.3, p. 713). In figure g, the nuclear and electrical potential energy in the plutonium pellet cause the pellet to heat up, and an equilibrium is reached, in which the heat is radiated away just as quickly as it is produced. When an object is closer to room temperature, it glows in the invisible infrared part of the spectrum (figure h).
Earth’s energy equilibrium

Our planet receives a nearly constant amount of energy from the sun (about $1.8 \times 10^{17}$ W). If it hadn’t had any mechanism for getting rid of that energy, the result would have been some kind of catastrophic explosion soon after its formation. Even a 10% imbalance between energy input and output, if maintained steadily from the time of the Roman Empire until the present, would have been enough to raise the oceans to a boil. So evidently the earth does dump this energy somehow. How does it do it? Our planet is surrounded by the vacuum of outer space, like the ultimate thermos bottle. Therefore it can’t expel heat by conduction or convection, but it does radiate in the infrared, and this is the only available mechanism for cooling.

Global warming

It was realized starting around 1930 that this created a dangerous vulnerability in our biosphere. Our atmosphere is only about 0.04% carbon dioxide, but carbon dioxide is an extraordinarily efficient absorber of infrared light. It is, however, transparent to visible light. Therefore any increase in the concentration of carbon dioxide would decrease the efficiency of cooling by radiation, while allowing in just as much heat input from visible light. When we burn fossil fuels such as gasoline or coal, we release into the atmosphere carbon that had previously been locked away underground. This results in a shift to a new energy balance. The average temperature $T$ of the land and oceans increases until the $T^4$ dependence of radiation compensates for the additional absorption of infrared light.

By about 1980, a clear scientific consensus had emerged that this effect was real, that it was caused by human activity, and that it had resulted in an abrupt increase in the earth’s average temperature. We know, for example, from radioisotope studies that the effect has not been caused by the release of carbon dioxide in volcanic eruptions. The temperature increase has been verified by multiple independent methods, including studies of tree rings and coral reefs. Detailed computer models have correctly predicted a number of effects that were later verified empirically, including a rise in sea levels, and day-night and pole-equator variations. There is no longer any controversy among climate scientists about the existence or cause of the effect.

One solution to the problem is to replace fossil fuels with renewable sources of energy such as solar power and wind. However, these cannot be brought online fast enough to prevent severe warming in the next few decades, so nuclear power is also a critical piece of the puzzle (see section 26.4.9, p. 788).
Summary

Selected vocabulary
potential energy the energy having to do with the distance between two objects that interact via a noncontact force

Notation
PE . . . . . . . . . potential energy

Other terminology and notation
U or V . . . . . symbols used for potential energy in the scientific literature and in most advanced textbooks

Summary

Historically, the energy concept was only invented to include a few phenomena, but it was later generalized more and more to apply to new situations, for example nuclear reactions. This generalizing process resulted in an undesirably long list of types of energy, each of which apparently behaved according to its own rules.

The first step in simplifying the picture came with the realization that heat was a form of random motion on the atomic level, i.e., heat was nothing more than the kinetic energy of atoms.

A second and even greater simplification was achieved with the realization that all the other apparently mysterious forms of energy actually had to do with changing the distances between atoms (or similar processes in nuclei). This type of energy, which relates to the distance between objects that interact via a force, is therefore of great importance. We call it potential energy.

Most of the important ideas about potential energy can be understood by studying the example of gravitational potential energy. The change in an object’s gravitational potential energy is given by

$$\Delta PE_{grav} = -F_{grav}\Delta y,$$

[if $F_{grav}$ is constant, i.e., the motion is all near the Earth’s surface]

The most important thing to understand about potential energy is that there is no unambiguous way to define it in an absolute sense. The only thing that everyone can agree on is how much the potential energy has changed from one moment in time to some later moment in time.
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 Can gravitational potential energy ever be negative? Note that the question refers to $PE$, not $\Delta PE$, so that you must think about how the choice of a reference level comes into play. [Based on a problem by Serway and Faughn.]

2 A ball rolls up a ramp, turns around, and comes back down. When does it have the greatest gravitational potential energy? The greatest kinetic energy? [Based on a problem by Serway and Faughn.]

3 (a) You release a magnet on a tabletop near a big piece of iron, and the magnet leaps across the table to the iron. Does the magnetic potential energy increase, or decrease? Explain. (b) Suppose instead that you have two repelling magnets. You give them an initial push towards each other, so they decelerate while approaching each other. Does the magnetic potential energy increase, or decrease? Explain.

4 Let $E_b$ be the energy required to boil one kg of water. (a) Find an equation for the minimum height from which a bucket of water must be dropped if the energy released on impact is to vaporize it. Assume that all the heat goes into the water, not into the dirt it strikes, and ignore the relatively small amount of energy required to heat the water from room temperature to 100°C. [Numerical check, not for credit: Plugging in $E_b = 2.3 \text{ MJ/kg}$ should give a result of 230 km.] √
(b) Show that the units of your answer in part a come out right based on the units given for $E_b$.

5 A grasshopper with a mass of 110 mg falls from rest from a height of 310 cm. On the way down, it dissipates 1.1 mJ of heat due to air resistance. At what speed, in m/s, does it hit the ground? ▷ Solution, p. 559

6 A person on a bicycle is to coast down a ramp of height $h$ and then pass through a circular loop of radius $r$. What is the smallest value of $h$ for which the cyclist will complete the loop without falling? (Ignore the kinetic energy of the spinning wheels.) √
7. A skateboarder starts at rest nearly at the top of a giant cylinder, and begins rolling down its side. (If he started exactly at rest and exactly at the top, he would never get going!) Show that his board loses contact with the pipe after he has dropped by a height equal to one third the radius of the pipe.  
Solution, p. 559  

8. (a) A circular hoop of mass $m$ and radius $r$ spins like a wheel while its center remains at rest. Its period (time required for one revolution) is $T$. Show that its kinetic energy equals $2\pi^2mr^2/T^2$.  
(b) If such a hoop rolls with its center moving at velocity $v$, its kinetic energy equals $(1/2)mv^2$, plus the amount of kinetic energy found in the first part of this problem. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have.  

9. Students are often tempted to think of potential energy and kinetic energy as if they were always related to each other, like yin and yang. To show this is incorrect, give examples of physical situations in which (a) PE is converted to another form of PE, and (b) KE is converted to another form of KE.  
Solution, p. 560  

10. Lord Kelvin, a physicist, told the story of how he encountered James Joule when Joule was on his honeymoon. As he traveled, Joule would stop with his wife at various waterfalls, and measure the difference in temperature between the top of the waterfall and the still water at the bottom. (a) It would surprise most people to learn that the temperature increased. Why should there be any such effect, and why would Joule care? How would this relate to the energy concept, of which he was the principal inventor? (b) How much of a gain in temperature should there be between the top and bottom of a 50-meter waterfall? (c) What assumptions did you have to make in order to calculate your answer to part b? In reality, would the temperature change be more than or less than what you calculated? [Based on a problem by Arnold Arons.]  

11. Make an order-of-magnitude estimate of the power represented by the loss of gravitational energy of the water going over Niagara Falls. If the hydroelectric plant at the bottom of the falls could convert 100% of this to electrical power, roughly how many households could be powered?  
Solution, p. 560  

12. When you buy a helium-filled balloon, the seller has to inflate it from a large metal cylinder of the compressed gas. The helium inside the cylinder has energy, as can be demonstrated for example by releasing a little of it into the air: you hear a hissing sound, and that sound energy must have come from somewhere. The total amount of energy in the cylinder is very large, and if the valve is inadvertently damaged or broken off, the cylinder can behave like a bomb or a rocket. 

Suppose the company that puts the gas in the cylinders prepares
cylinder A with half the normal amount of pure helium, and cylinder B with the normal amount. Cylinder B has twice as much energy, and yet the temperatures of both cylinders are the same. Explain, at the atomic level, what form of energy is involved, and why cylinder B has twice as much.

13 At a given temperature, the average kinetic energy per molecule is a fixed value, so for instance in air, the more massive oxygen molecules are moving more slowly on the average than the nitrogen molecules. The ratio of the masses of oxygen and nitrogen molecules is 16.00 to 14.01. Now suppose a vessel containing some air is surrounded by a vacuum, and the vessel has a tiny hole in it, which allows the air to slowly leak out. The molecules are bouncing around randomly, so a given molecule will have to “try” many times before it gets lucky enough to head out through the hole. Find the rate at which oxygen leaks divided by the rate at which nitrogen leaks. (Define this rate according to the fraction of the gas that leaks out in a given time, not the mass or number of molecules leaked per unit time.)

14 Explain in terms of conservation of energy why sweating cools your body, even though the sweat is at the same temperature as your body. Describe the forms of energy involved in this energy transformation. Why don’t you get the same cooling effect if you wipe the sweat off with a towel? Hint: The sweat is evaporating.

15 Anya and Ivan lean over a balcony side by side. Anya throws a penny downward with an initial speed of 5 m/s. Ivan throws a penny upward with the same speed. Both pennies end up on the ground below. Compare their kinetic energies and velocities on impact.

16 Problem 16 has been deleted.
17 The figure shows two unequal masses, $M$ and $m$, connected by a string running over a pulley. This system was analyzed previously in problem 10 on p. 181, using Newton’s laws.

(a) Analyze the system using conservation of energy instead. Find the speed the weights gain after being released from rest and traveling a distance $h$. √

(b) Use your result from part a to find the acceleration, reproducing the result of the earlier problem. √

18 The rock climber in the figure has mass $m$ and is on a slope $\theta$ above the horizontal. At a distance $x$ down the slope below him is a ledge. He is tied in to a climbing rope and being belayed from above, so that if he slips he won’t simply plunge to his death. Climbing ropes are intentionally made out of stretchy material so that in a fall, the climber gets a gentle catch rather than a violent force that would hurt (see example 2, p. 334). However, the rope should not be more stretchy than necessary because of situations like this one: if the rope were to stretch by more than $x$, the climber would hit the ledge.

(a) Find the spring constant that the rope should have in order to limit the amount of rope stretch to $x$. √

(b) Show that your answer to part a has the right units.

(c) Analyze the mathematical dependence of the result on each of the variables, and verify that it makes sense physically.
Chapter 13
Work: The Transfer of Mechanical Energy

13.1 Work: the transfer of mechanical energy

The concept of work

The mass contained in a closed system is a conserved quantity, but if the system is not closed, we also have ways of measuring the amount of mass that goes in or out. The water company does this with a meter that records your water use.

Likewise, we often have a system that is not closed, and would like to know how much energy comes in or out. Energy, however, is not a physical substance like water, so energy transfer cannot be measured with the same kind of meter. How can we tell, for instance, how much useful energy a tractor can “put out” on one tank of gas?

The law of conservation of energy guarantees that all the chem-
ical energy in the gasoline will reappear in some form, but not necessarily in a form that is useful for doing farm work. Tractors, like cars, are extremely inefficient, and typically 90% of the energy they consume is converted directly into heat, which is carried away by the exhaust and the air flowing over the radiator. We wish to distinguish the energy that comes out directly as heat from the energy that serves to accelerate a trailer or to plow a field, so we define a technical meaning of the ordinary word “work” to express the distinction:

**definition of work**

Work is the amount of energy transferred into or out of a system, not counting energy transferred by heat conduction.

**self-check A**

Based on this definition, is work a vector, or a scalar? What are its units?

The conduction of heat is to be distinguished from heating by friction. When a hot potato heats up your hands by conduction, the energy transfer occurs without any force, but when friction heats your car’s brake shoes, there is a force involved. The transfer of energy with and without a force are measured by completely different methods, so we wish to include heat transfer by frictional heating under the definition of work, but not heat transfer by conduction. The definition of work could thus be restated as the amount of energy transferred by forces.

**Calculating work as force multiplied by distance**

The examples in figures b-d show that there are many different ways in which energy can be transferred. Even so, all these examples have two things in common:

1. A force is involved.
2. The tractor travels some distance as it does the work.

In b, the increase in the height of the weight, $\Delta y$, is the same as the distance the tractor travels, which we’ll call $d$. For simplicity, we discuss the case where the tractor raises the weight at constant speed, so that there is no change in the kinetic energy of the weight, and we assume that there is negligible friction in the pulley, so that the force the tractor applies to the rope is the same as the rope’s upward force on the weight. By Newton’s first law, these forces are also of the same magnitude as the earth’s gravitational force on the weight. The increase in the weight’s potential energy is given by $F\Delta y$, so the work done by the tractor on the weight equals $Fd$, the product of the force and the distance moved:

$$W = Fd.$$
In example c, the tractor’s force on the trailer accelerates it, increasing its kinetic energy. If frictional forces on the trailer are negligible, then the increase in the trailer’s kinetic energy can be found using the same algebra that was used on page 319 to find the potential energy due to gravity. Just as in example b, we have

\[ W = Fd. \]

Does this equation always give the right answer? Well, sort of. In example d, there are two quantities of work you might want to calculate: the work done by the tractor on the plow and the work done by the plow on the dirt. These two quantities can’t both equal \( Fd \). Most of the energy transmitted through the cable goes into frictional heating of the plow and the dirt. The work done by the plow on the dirt is less than the work done by the tractor on the plow, by an amount equal to the heat absorbed by the plow. It turns out that the equation \( W = Fd \) gives the work done by the tractor, not the work done by the plow. How are you supposed to know when the equation will work and when it won’t? The somewhat complex answer is postponed until section 13.6. Until then, we will restrict ourselves to examples in which \( W = Fd \) gives the right answer; essentially the reason the ambiguities come up is that when one surface is slipping past another, \( d \) may be hard to define, because the two surfaces move different distances.

We have also been using examples in which the force is in the same direction as the motion, and the force is constant. (If the force was not constant, we would have to represent it with a function, not a symbol that stands for a number.) To summarize, we have:

**rule for calculating work (simplest version)**
The work done by a force can be calculated as

\[ W = Fd, \]

if the force is constant and in the same direction as the motion. Some ambiguities are encountered in cases such as kinetic friction.

---

e / The baseball pitcher put kinetic energy into the ball, so he did work on it. To do the greatest possible amount of work, he applied the greatest possible force over the greatest possible distance.
Example 1.

Example 1. Surprisingly, the climber is in more danger at 1 than at 2. The distance $d$ is the amount by which the rope will stretch while work is done to transfer the kinetic energy of a fall out of her body.

Example 2. Counterintuitively, the rock climber may be in more danger in figure g/1 than later when she gets up to position g/2.

Along her route, the climber has placed removable rock anchors (not shown) and carabiners attached to the anchors. She clips the rope into each carabiner so that it can travel but can’t pop out. In both 1 and 2, she has ascended a certain distance above her last anchor, so that if she falls, she will drop through a height $h$ that is about twice this distance, and this fall height is about the same in both cases. In fact, $h$ is somewhat larger than twice her height above the last anchor, because the rope is intentionally designed to stretch under the big force of a falling climber who suddenly brings it taut.

To see why we want a stretchy rope, consider the equation $F = W/d$ in the case where $d$ is zero; $F$ would theoretically become infinite. In a fall, the climber loses a fixed amount of gravitational energy $mgh$. This is transformed into an equal amount of kinetic energy as she falls, and eventually this kinetic energy has to be transferred out of her body when the rope comes up taut. If the rope was not stretchy, then the distance traveled at the point where the rope attaches to her harness would be zero, and the force exerted would theoretically be infinite. Before the rope reached the theoretically infinite tension $F$ it would break (or her back would break, or her anchors would be pulled out of the rock). We want the rope to be stretchy enough to make $d$ fairly big, so that dividing $W$ by $d$ gives a small force.\(^1\)

In g/1 and g/2, the fall $h$ is about the same. What is different is the length $L$ of rope that has been paid out. A longer rope can stretch more, so the distance $d$ traveled after the “catch” is proportional to $L$. Combining $F = W/d$, $W \propto h$, and $d \propto L$, we have $F \propto h/L$. For these reasons, rock climbers define a fall factor $f = h/L$. The larger fall factor in g/1 is more dangerous.

---

\(^1\)Actually $F$ isn’t constant, because the tension in the rope increases steadily as it stretches, but this is irrelevant to the present analysis.
Machines can increase force, but not work.

Figure h shows a pulley arrangement for doubling the force supplied by the tractor (book 1, section 5.6). The tension in the left-hand rope is equal throughout, assuming negligible friction, so there are two forces pulling the pulley to the left, each equal to the original force exerted by the tractor on the rope. This doubled force is transmitted through the right-hand rope to the stump.

![Diagram of pulley arrangement](image)

It might seem as though this arrangement would also double the work done by the tractor, but look again. As the tractor moves forward 2 meters, 1 meter of rope comes around the pulley, and the pulley moves 1 m to the left. Although the pulley exerts double the force on the stump, the pulley and stump only move half as far, so the work done on the stump is no greater than it would have been without the pulley.

The same is true for any mechanical arrangement that increases or decreases force, such as the gears on a ten-speed bike. You can’t get out more work than you put in, because that would violate conservation of energy. If you shift gears so that your force on the pedals is amplified, the result is that you just have to spin the pedals more times.

No work is done without motion.

It strikes most students as nonsensical when they are told that if they stand still and hold a heavy bag of cement, they are doing no work on the bag. Even if it makes sense mathematically that $W = Fd$ gives zero when $d$ is zero, it seems to violate common sense. You would certainly become tired! The solution is simple. Physicists have taken over the common word “work” and given it a new technical meaning, which is the transfer of energy. The energy of the bag of cement is not changing, and that is what the physicist means by saying no work is done on the bag.

There is a transformation of energy, but it is taking place entirely within your own muscles, which are converting chemical energy into heat. Physiologically, a human muscle is not like a tree limb, which can support a weight indefinitely without the expenditure of energy. Each muscle cell’s contraction is generated by zillions of little molecular machines, which take turns supporting the tension. When a
Whenever energy is transferred out of the spring, the same amount has to be transferred into the ball, and vice versa. As the spring compresses, the ball is doing positive work on the spring (giving up its KE and transferring energy into the spring as PE), and as it decompresses the ball is doing negative work (extracting energy).

**Positive and negative work**

When object A transfers energy to object B, we say that A does positive work on B. B is said to do negative work on A. In other words, a machine like a tractor is defined as doing positive work. This use of the plus and minus signs relates in a logical and consistent way to their use in indicating the directions of force and motion in one dimension. In figure i, suppose we choose a coordinate system with the $x$ axis pointing to the right. Then the force the spring exerts on the ball is always a positive number. The ball’s motion, however, changes directions. The symbol $d$ is really just a shorter way of writing the familiar quantity $\Delta x$, whose positive and negative signs indicate direction.

While the ball is moving to the left, we use $d < 0$ to represent its direction of motion, and the work done by the spring, $Fd$, comes out negative. This indicates that the spring is taking kinetic energy out of the ball, and accepting it in the form of its own potential energy.

As the ball is reaccelerated to the right, it has $d > 0$, $Fd$ is positive, and the spring does positive work on the ball. Potential energy is transferred out of the spring and deposited in the ball as kinetic energy.

In summary:

**rule for calculating work (including cases of negative work)**

The work done by a force can be calculated as 

$$W = Fd,$$

if the force is constant and along the same line as the motion. The quantity $d$ is to be interpreted as a synonym for $\Delta x$, i.e., positive and negative signs are used to indicate the direction of motion. Some ambiguities are encountered in cases such as kinetic friction.

**self-check B**

In figure i, what about the work done by the ball on the spring? 

> **Answer**, p. 567

There are many examples where the transfer of energy out of an object cancels out the transfer of energy in. When the tractor pulls the plow with a rope, the rope does negative work on the tractor and positive work on the plow. The total work done by the rope is zero, which makes sense, since it is not changing its energy.

It may seem that when your arms do negative work by lowering particular molecule goes on or off duty, it moves, and since it moves while exerting a force, it is doing work. There is work, but it is work done by one molecule in a muscle cell on another.
a bag of cement, the cement is not really transferring energy into your body. If your body was storing potential energy like a compressed spring, you would be able to raise and lower a weight all day, recycling the same energy. The bag of cement does transfer energy into your body, but your body accepts it as heat, not as potential energy. The tension in the muscles that control the speed of the motion also results in the conversion of chemical energy to heat, for the same physiological reasons discussed previously in the case where you just hold the bag still.

One of the advantages of electric cars over gasoline-powered cars is that it is just as easy to put energy back in a battery as it is to take energy out. When you step on the brakes in a gas car, the brake shoes do negative work on the rest of the car. The kinetic energy of the car is transmitted through the brakes and accepted by the brake shoes in the form of heat. The energy cannot be recovered. Electric cars, however, are designed to use regenerative braking. The brakes don’t use friction at all. They are electrical, and when you step on the brake, the negative work done by the brakes means they accept the energy and put it in the battery for later use. This is one of the reasons why an electric car is far better for the environment than a gas car, even if the ultimate source of the electrical energy happens to be the burning of oil in the electric company’s plant. The electric car recycles the same energy over and over, and only dissipates heat due to air friction and rolling resistance, not braking. (The electric company’s power plant can also be fitted with expensive pollution-reduction equipment that would be prohibitively expensive or bulky for a passenger car.)
Discussion questions

A  Besides the presence of a force, what other things differentiate the processes of frictional heating and heat conduction?

B  Criticize the following incorrect statement: “A force doesn’t do any work unless it’s causing the object to move.”

C  To stop your car, you must first have time to react, and then it takes some time for the car to slow down. Both of these times contribute to the distance you will travel before you can stop. The figure shows how the average stopping distance increases with speed. Because the stopping distance increases more and more rapidly as you go faster, the rule of one car length per 10 m.p.h. of speed is not conservative enough at high speeds. In terms of work and kinetic energy, what is the reason for the more rapid increase at high speeds?

13.2 Work in three dimensions

A force perpendicular to the motion does no work.

Suppose work is being done to change an object’s kinetic energy. A force in the same direction as its motion will speed it up, and a force in the opposite direction will slow it down. As we have already seen, this is described as doing positive work or doing negative work on the object. All the examples discussed up until now have been of motion in one dimension, but in three dimensions the force can be at any angle \( \theta \) with respect to the direction of motion.

What if the force is perpendicular to the direction of motion? We have already seen that a force perpendicular to the motion results in circular motion at constant speed. The kinetic energy does not change, and we conclude that no work is done when the force is perpendicular to the motion.

So far we have been reasoning about the case of a single force acting on an object, and changing only its kinetic energy. The result is more generally true, however. For instance, imagine a hockey puck sliding across the ice. The ice makes an upward normal force, but does not transfer energy to or from the puck.
Forces at other angles

Suppose the force is at some other angle with respect to the motion, say $\theta = 45^\circ$. Such a force could be broken down into two components, one along the direction of the motion and the other perpendicular to it. The force vector equals the vector sum of its two components, and the principle of vector addition of forces thus tells us that the work done by the total force cannot be any different than the sum of the works that would be done by the two forces by themselves. Since the component perpendicular to the motion does no work, the work done by the force must be

$$W = F_\parallel |d|, \quad \text{[work done by a constant force]}$$

where the vector $d$ is simply a less cumbersome version of the notation $\Delta r$. This result can be rewritten via trigonometry as

$$W = |F||d| \cos \theta. \quad \text{[work done by a constant force]}$$

Even though this equation has vectors in it, it depends only on their magnitudes, and the magnitude of a vector is a scalar. Work is therefore still a scalar quantity, which only makes sense if it is defined as the transfer of energy. Ten gallons of gasoline have the ability to do a certain amount of mechanical work, and when you pull in to a full-service gas station you don’t have to say “Fill ’er up with 10 gallons of south-going gas.”

Students often wonder why this equation involves a cosine rather than a sine, or ask if it would ever be a sine. In vector addition, the treatment of sines and cosines seemed more equal and democratic, so why is the cosine so special now? The answer is that if we are going to describe, say, a velocity vector, we must give both the component parallel to the $x$ axis and the component perpendicular to the $x$ axis (i.e., the $y$ component). In calculating work, however, the force component perpendicular to the motion is irrelevant — it changes the direction of motion without increasing or decreasing the energy of the object on which it acts. In this context, it is only the parallel force component that matters, so only the cosine occurs.

**Self-check C**

(a) Work is the transfer of energy. According to this definition, is the horse in the picture doing work on the pack? (b) If you calculate work by the method described in this section, is the horse in figure o doing work on the pack?

If you exert a force of 21 N on a push broom, at an angle 35 degrees below horizontal, and walk for 5.0 m, how much work do you do? What is the physical significance of this quantity of work?

Using the second equation above, the work done equals

$$(21 \text{ N})(5.0 \text{ m})(\cos 35^\circ) = 86 \text{ J}.$$
The form of energy being transferred is heat in the floor and the broom’s bristles. This comes from the chemical energy stored in your body. (The majority of the calories you burn are dissipated directly as heat inside your body rather than doing any work on the broom. The 86 J is only the amount of energy transferred through the broom’s handle.)

As a violinist draws the bow across a string, the bow hairs exert both a normal force and a kinetic frictional force on the string. The normal force is perpendicular to the direction of motion, and does no work. However, the frictional force is in the same direction as the motion of the bow, so it does work: energy is transferred to the string, causing it to vibrate.

One way of playing a violin more loudly is to use longer strokes. Since \( W = Fd \), the greater distance results in more work.

A second way of getting a louder sound is to press the bow more firmly against the strings. This increases the normal force, and although the normal force itself does no work, an increase in the normal force has the side effect of increasing the frictional force, thereby increasing \( W = Fd \).

The violinist moves the bow back and forth, and sound is produced on both the “up-bow” (the stroke toward the player’s left) and the “down-bow” (to the right). One may, for example, play a series of notes in alternation between up-bows and down-bows. However, if the notes are of unequal length, the up and down motions tend to be unequal, and if the player is not careful, she can run out of bow in the middle of a note! To keep this from happening, one can move the bow more quickly on the shorter notes, but the resulting increase in \( d \) will make the shorter notes louder than they should be. A skilled player compensates by reducing the force.

13.3 *Varying force*

Up until now we have done no actual calculations of work in cases where the force was not constant. The question of how to treat such cases is mathematically analogous to the issue of how to generalize the equation (distance) = (velocity)(time) to cases where the velocity was not constant. There, we found that the correct generalization was to find the area under the graph of velocity versus time. The equivalent thing can be done with work:

**general rule for calculating work**

The work done by a force \( F \) equals the area under the curve on a graph of \( F_\parallel \) versus \( x \). (Some ambiguities are encountered in cases such as kinetic friction.)
The examples in this section are ones in which the force is varying, but is always along the same line as the motion, so \( F \) is the same as \( F_{\parallel} \).

**self-check D**

In which of the following examples would it be OK to calculate work using \( Fd \), and in which ones would you have to use the area under the \( F-x \) graph?

(a) A fishing boat cruises with a net dragging behind it.

(b) A magnet leaps onto a refrigerator from a distance.

(c) Earth’s gravity does work on an outward-bound space probe.

Answer, p. 567

An important and straightforward example is the calculation of the work done by a spring that obeys Hooke’s law,

\[
F \approx -k(x - x_o).
\]

The minus sign is because this is the force being exerted by the spring, not the force that would have to act on the spring to keep it at this position. That is, if the position of the cart in figure p is to the right of equilibrium, the spring pulls back to the left, and vice-versa.

We calculate the work done when the spring is initially at equilibrium and then decelerates the car as the car moves to the right. The work done by the spring on the cart equals the minus area of the shaded triangle, because the triangle hangs below the \( x \) axis. The area of a triangle is half its base multiplied by its height, so

\[
W = -\frac{1}{2}k(x - x_o)^2.
\]

This is the amount of kinetic energy lost by the cart as the spring decelerates it.

It was straightforward to calculate the work done by the spring in this case because the graph of \( F \) versus \( x \) was a straight line, giving a triangular area. But if the curve had not been so geometrically simple, it might not have been possible to find a simple equation for the work done, or an equation might have been derivable only using calculus. Optional section 13.4 gives an important example of such an application of calculus.

**Energy production in the sun**

The sun produces energy through nuclear reactions in which nuclei collide and stick together. The figure depicts one such reaction, in which a single proton (hydrogen nucleus) collides with a carbon nucleus, consisting of six protons and six neutrons. Neutrons and protons attract other neutrons and protons via the
Example 5. The strong nuclear force, so as the proton approaches the carbon nucleus it is accelerated. In the language of energy, we say that it loses nuclear potential energy and gains kinetic energy. Together, the seven protons and six neutrons make a nitrogen nucleus. Within the newly put-together nucleus, the neutrons and protons are continually colliding, and the new proton's extra kinetic energy is rapidly shared out among all the neutrons and protons. Soon afterward, the nucleus calms down by releasing some energy in the form of a gamma ray, which helps to heat the sun.

The graph shows the force between the carbon nucleus and the proton as the proton is on its way in, with the distance in units of femtometers (1 fm=10^{-15} m). Amusingly, the force turns out to be a few newtons: on the same order of magnitude as the forces we encounter ordinarily on the human scale. Keep in mind, however, that a force this big exerted on a single subatomic particle such as a proton will produce a truly fantastic acceleration (on the order of 10^{27} m/s^2).
Why does the force have a peak around $x = 3$ fm, and become smaller once the proton has actually merged with the nucleus? At $x = 3$ fm, the proton is at the edge of the crowd of protons and neutrons. It feels many attractive forces from the left, and none from the right. The forces add up to a large value. However if it later finds itself at the center of the nucleus, $x = 0$, there are forces pulling it from all directions, and these force vectors cancel out.

We can now calculate the energy released in this reaction by using the area under the graph to determine the amount of mechanical work done by the carbon nucleus on the proton. (For simplicity, we assume that the proton came in “aimed” at the center of the nucleus, and we ignore the fact that it has to shove some neutrons and protons out of the way in order to get there.) The area under the curve is about 17 squares, and the work represented by each square is

$$\text{(1 N)(10}^{-15} \text{ m)} = 10^{-15} \text{ J},$$

so the total energy released is about

$$(10^{-15} \text{ J/square})(17 \text{ squares}) = 1.7 \times 10^{-14} \text{ J}.$$  

This may not seem like much, but remember that this is only a reaction between the nuclei of two out of the zillions of atoms in the sun. For comparison, a typical chemical reaction between two atoms might transform on the order of $10^{-19}$ J of electrical potential energy into heat — 100,000 times less energy!

As a final note, you may wonder why reactions such as these only occur in the sun. The reason is that there is a repulsive electrical force between nuclei. When two nuclei are close together, the electrical forces are typically about a million times weaker than the nuclear forces, but the nuclear forces fall off much more quickly with distance than the electrical forces, so the electrical force is the dominant one at longer ranges. The sun is a very hot gas, so the random motion of its atoms is extremely rapid, and a collision between two atoms is sometimes violent enough to overcome this initial electrical repulsion.

### Applications of calculus

13.4

The student who has studied integral calculus will recognize that the graphical rule given in the previous section can be reexpressed as an integral,

$$W = \int_{x_1}^{x_2} F \, dx.$$  

We can then immediately find by the fundamental theorem of calculus that force is the derivative of work with respect to position,

$$F = \frac{dW}{dx}.$$
For example, a crane raising a one-ton block on the moon would be transferring potential energy into the block at only one sixth the rate that would be required on Earth, and this corresponds to one sixth the force.

Although the work done by the spring could be calculated without calculus using the area of a triangle, there are many cases where the methods of calculus are needed in order to find an answer in closed form. The most important example is the work done by gravity when the change in height is not small enough to assume a constant force. Newton’s law of gravity is

\[ F = \frac{GMm}{r^2}, \]

which can be integrated to give

\[
W = \int_{r_1}^{r_2} \frac{GMm}{r^2} \, dr
\]

\[
= -GMm \left( \frac{1}{r_2} - \frac{1}{r_1} \right).
\]
13.5 Work and potential energy

The techniques for calculating work can also be applied to the calculation of potential energy. If a certain force depends only on the distance between the two participating objects, then the energy released by changing the distance between them is defined as the potential energy, and the amount of potential energy lost equals minus the work done by the force,

\[ \Delta PE = -W. \]

The minus sign occurs because positive work indicates that the potential energy is being expended and converted to some other form.

It is sometimes convenient to pick some arbitrary position as a reference position, and derive an equation for once and for all that gives the potential energy relative to this position

\[ PE_x = -W_{\text{ref} \rightarrow x}. \]  
[potential energy at a point \( x \)]

To find the energy transferred into or out of potential energy, one then subtracts two different values of this equation.

These equations might almost make it look as though work and energy were the same thing, but they are not. First, potential energy measures the energy that a system has stored in it, while work measures how much energy is transferred in or out. Second, the techniques for calculating work can be used to find the amount of energy transferred in many situations where there is no potential energy involved, as when we calculate the amount of kinetic energy transformed into heat by a car’s brake shoes.

---

1 A toy gun

example 6

A toy gun uses a spring with a spring constant of 10 N/m to shoot a ping-pong ball of mass 5 g. The spring is compressed to 10 cm shorter than its equilibrium length when the gun is loaded. At what speed is the ball released?

The equilibrium point is the natural choice for a reference point. Using the equation found previously for the work, we have

\[ PE_x = \frac{1}{2} k (x - x_0)^2. \]

The spring loses contact with the ball at the equilibrium point, so the final potential energy is

\[ PE_f = 0. \]

The initial potential energy is

\[ PE_i = \frac{1}{2} (10 \text{ N/m})(0.10 \text{ m})^2. \]

\[ = 0.05 \text{ J}. \]
The loss in potential energy of 0.05 J means an increase in kinetic energy of the same amount. The velocity of the ball is found by solving the equation $KE = (1/2)mv^2$ for $v$,

$$v = \sqrt{\frac{2KE}{m}}$$

$$= \sqrt{\frac{(2)(0.05 \text{ J})}{0.005 \text{ kg}}}$$

$$= 4 \text{ m/s}.$$
Discussion questions

A  What does the graph of \( PE = (1/2)k(x - x_0)^2 \) look like as a function of \( x \)? Discuss the physical significance of its features.

B  What does the graph of \( PE = -GMm/r \) look like as a function of \( r \)? Discuss the physical significance of its features. How would the equation and graph change if some other reference point was chosen rather than \( r = \infty \)?

C  Starting at a distance \( r \) from a planet of mass \( M \), how fast must an object be moving in order to have a hyperbolic orbit, i.e., one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it?

D  Does a spring have an “escape velocity?”

E  Calculus-based question: If the form of energy being transferred is potential energy, then the equations \( F = dW/dx \) and \( W = \int F \, dx \) become \( F = -dPE/dx \) and \( PE = -\int F \, dx \). How would you then apply the following calculus concepts: zero derivative at minima and maxima, and the second derivative test for concavity up or down.

13.6 ⋆ When does work equal force times distance?

In the example of the tractor pulling the plow discussed on page 333, the work did not equal \( Fd \). The purpose of this section is to explain more fully how the quantity \( Fd \) can and cannot be used. To simplify things, I write \( Fd \) throughout this section, but more generally everything said here would be true for the area under the
The work-KE theorem

Proof

For simplicity, we have assumed $F_{\text{total}}$ to be constant, and therefore $a_{\text{cm}} = F_{\text{total}}/m$ is also constant, and the constant-acceleration equation

$$v_{\text{cm},f}^2 = v_{\text{cm},i}^2 + 2a_{\text{cm}}\Delta x_{\text{cm}}$$

applies. Multiplying by $m/2$ on both sides and applying Newton’s second law gives

$$KE_{\text{cm},f}^2 = KE_{\text{cm},i}^2 + F_{\text{total}}\Delta x_{\text{cm}},$$

which is the result that was to be proved.

Further interpretation

The logical structure of this book is that although Newton’s laws are discussed before conservation laws, the conservation laws are taken to be fundamental, since they are true even in cases where Newton’s laws fail. Many treatments of this subject present the work-KE theorem as a proof that kinetic energy behaves as $(1/2)mv^2$. This is a matter of taste, but one can just as well rearrange the equations in the proof above to solve for the unknown $a_{\text{cm}}$ and prove Newton’s second law as a consequence of conservation of energy. Ultimately we have a great deal of freedom in choosing which equations to take as definitions, which to take as empirically verified laws of nature, and which to take as theorems.

Regardless of how we slice things, we require both mathematical consistency and consistency with experiment. As described on p. 305, the work-KE theorem is an important part of this interlocking system of relationships.

The work-KE theorem

The change in kinetic energy associated with the motion of an object’s center of mass is related to the total force acting on it and to the distance traveled by its center of mass according to the equation

$$\Delta KE_{\text{cm}} = F_{\text{total}}d_{\text{cm}}.$$ 

A proof is given in the sidebar, along with some interpretation of how this result relates to the logical structure of our presentation. Note that despite the traditional name, it does not necessarily tell the amount of work done, since the forces acting on the object could be changing other types of energy besides the KE associated with its center of mass motion.

The second theorem does relate directly to work:

When a contact force acts between two objects and the two surfaces do not slip past each other, the work done equals $Fd$, where $d$ is the distance traveled by the point of contact.

This one has no generally accepted name, so we refer to it simply as the second theorem.

A great number of physical situations can be analyzed with these two theorems, and often it is advantageous to apply both of them to the same situation.

An ice skater pushing off from a wall example 8

The work-kinetic energy theorem tells us how to calculate the skater’s kinetic energy if we know the amount of force and the distance her center of mass travels while she is pushing off.

The second theorem tells us that the wall does no work on the skater. This makes sense, since the wall does not have any source of energy.

Absorbing an impact without recoiling? example 9

▷ Is it possible to absorb an impact without recoiling? For instance, would a brick wall “give” at all if hit by a ping-pong ball?

▷ There will always be a recoil. In the example proposed, the wall will surely have some energy transferred to it in the form of heat and vibration. The second theorem tells us that we can only have nonzero work if the distance traveled by the point of contact is nonzero.
Newton’s first law tells us that the total force on the refrigerator must be zero: your force is canceling the floor’s kinetic frictional force. The work-kinetic energy theorem is therefore true but useless. It tells us that there is zero total force on the refrigerator, and that the refrigerator’s kinetic energy doesn’t change.

The second theorem tells us that the work you do equals your hand’s force on the refrigerator multiplied by the distance traveled. Since we know the floor has no source of energy, the only way for the floor and refrigerator to gain energy is from the work you do. We can thus calculate the total heat dissipated by friction in the refrigerator and the floor.

Note that there is no way to find how much of the heat is dissipated in the floor and how much in the refrigerator.

If you push on a cart and accelerate it, there are two forces acting on the cart: your hand’s force, and the static frictional force of the ground pushing on the wheels in the opposite direction.

Applying the second theorem to your force tells us how to calculate the work you do.

Applying the second theorem to the floor’s force tells us that the floor does no work on the cart. There is no motion at the point of contact, because the atoms in the floor are not moving. (The atoms in the surface of the wheel are also momentarily at rest when they touch the floor.) This makes sense, since the floor does not have any source of energy.

The work-kinetic energy theorem refers to the total force, and because the floor’s backward force cancels part of your force, the total force is less than your force. This tells us that only part of your work goes into the kinetic energy associated with the forward motion of the cart’s center of mass. The rest goes into rotation of the wheels.

13.7 The dot product

Up until now, we have not found any physically useful way to define the multiplication of two vectors. It would be possible, for instance, to multiply two vectors component by component to form a third vector, but there are no physical situations where such a multiplication would be useful.

The equation \( W = |\mathbf{F}| |\mathbf{d}| \cos \theta \) is an example of a sort of multiplication of vectors that is useful. The result is a scalar, not a vector, and this is therefore often referred to as the scalar product of the vectors \( \mathbf{F} \) and \( \mathbf{d} \). There is a standard shorthand notation for
this operation,
\[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta, \]
[definition of the notation \( \mathbf{A} \cdot \mathbf{B} \);
\( \theta \) is the angle between vectors \( \mathbf{A} \) and \( \mathbf{B} \)]

and because of this notation, a more common term for this operation
is the dot product. In dot product notation, the equation for work
is simply
\[ W = \mathbf{F} \cdot \mathbf{d}. \]

The dot product has the following geometric interpretation:

\[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|(\text{component of \( \mathbf{B} \) parallel to \( \mathbf{A} \)}) \]
\[ = |\mathbf{B}|(\text{component of \( \mathbf{A} \) parallel to \( \mathbf{B} \)}). \]

The dot product has some of the properties possessed by ordinary
multiplication of numbers,
\[ \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \]
\[ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \]
\[ (c\mathbf{A}) \cdot \mathbf{B} = c(\mathbf{A} \cdot \mathbf{B}), \]
but it lacks one other: the ability to undo multiplication by dividing.

If you know the components of two vectors, you can easily cal-
culate their dot product as follows:
\[ \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \]
(This can be proved by first analyzing the special case where each
vector has only an \( x \) component, and the similar cases for \( y \) and \( z \).
We can then use the rule \( \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \) to make a
generalization by writing each vector as the sum of its \( x, y, \) and \( z \)
components. See homework problem 17.)

---

Magnitude expressed with a dot product example 12
If we take the dot product of any vector \( \mathbf{b} \) with itself, we find
\[ \mathbf{b} \cdot \mathbf{b} = (b_x \mathbf{\hat{x}} + b_y \mathbf{\hat{y}} + b_z \mathbf{\hat{z}}) \cdot (b_x \mathbf{\hat{x}} + b_y \mathbf{\hat{y}} + b_z \mathbf{\hat{z}}) \]
\[ = b_x^2 + b_y^2 + b_z^2, \]
so its magnitude can be expressed as
\[ |\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}}. \]

We will often write \( b^2 \) to mean \( \mathbf{b} \cdot \mathbf{b} \), when the context makes
it clear what is intended. For example, we could express kinetic
energy as \((1/2)m|\mathbf{v}|^2\), \((1/2)m\mathbf{v} \cdot \mathbf{v}\), or \((1/2)m\mathbf{v}^2\). In the third version,
nothing but context tells us that \( v \) really stands for the magnitude
of some vector \( \mathbf{v} \).