thus initially has velocity components $v_x = v \cos \theta$ and $v_y = v \sin \theta$.

(a) Show that the cannon’s range (horizontal distance to where the cannonball falls) is given by the equation $R = \frac{2v^2}{g} \sin \theta \cos \theta$.

(b) Interpret your equation in the cases of $\theta = 0$ and $\theta = 90^\circ$.

$\triangledown$ Solution, p. 551

6 Assuming the result of problem 5 for the range of a projectile, $R = \frac{2v^2}{g} \sin \theta \cos \theta$, show that the maximum range is for $\theta = 45^\circ$.

7 Two cars go over the same speed bump in a parking lot, Maria’s Maserati at 25 miles per hour and Park’s Porsche at 37. How many times greater is the vertical acceleration of the Porsche? Hint: Remember that acceleration depends both on how much the velocity changes and on how much time it takes to change.

8 You’re running off a cliff into a pond. The cliff is $h = 5.0$ m above the water, but the cliff is not strictly vertical; it slopes down to the pond at an angle of $\theta = 20^\circ$ with respect to the vertical. You want to find the minimum speed you need to jump off the cliff in order to land in the water.

(a) Find a symbolic answer in terms of $h$, $\theta$, and $g$.

(b) Check that the units of your answer to part a make sense.

(c) Check that the dependence on the variables $g$, $h$, and $\theta$ makes sense, and check the special cases $\theta = 0$ and $\theta = 90^\circ$.

(d) Plug in numbers to find the numerical result.

$\triangledown$ [problem by B. Shotwell]

9 Two footballs, one white and one green, are on the ground and kicked by two different footballers. The white ball, which is kicked straight upward with initial speed $v_0$, rises to height $H$. The green ball is hit with twice the initial speed but reaches the same height.

(a) What is the $y$-component of the green ball’s initial velocity vector? Give your answer in terms of $v_0$ alone.

(b) Which ball is in the air for a longer amount of time?

(c) What is the range of the green ball? Your answer should only depend on $H$.

$\triangledown$ [problem by B. Shotwell]

10 This problem is now problem 26 on p. 238.

11 The figure shows a vertical cross-section of a cylinder. A gun at the top shoots a bullet horizontally. What is the minimum speed at which the bullet must be shot in order to completely clear the cylinder?

$\star$
Vectors are used in aerial navigation.

Chapter 7
Vectors

7.1 Vector notation

The idea of components freed us from the confines of one-dimensional physics, but the component notation can be unwieldy, since every one-dimensional equation has to be written as a set of three separate equations in the three-dimensional case. Newton was stuck with the component notation until the day he died, but eventually someone sufficiently lazy and clever figured out a way of abbreviating three equations as one.

| (a) $\vec{F}_A \text{ on } B = -\vec{F}_B \text{ on } A$ | stands for $F_A \text{ on } B, x = -F_B \text{ on } A, x$
| | | $F_A \text{ on } B, y = -F_B \text{ on } A, y$
| | | $F_A \text{ on } B, z = -F_B \text{ on } A, z$
| (b) $\vec{F}_\text{total} = \vec{F}_1 + \vec{F}_2 + \ldots$ stands for $F_{\text{total}}, x = F_{1,x} + F_{2,x} + \ldots$
| | | $F_{\text{total}}, y = F_{1,y} + F_{2,y} + \ldots$
| | | $F_{\text{total}}, z = F_{1,z} + F_{2,z} + \ldots$
| (c) $\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$ | stands for $a_x = \Delta v_x/\Delta t$
| | | $a_y = \Delta v_y/\Delta t$
| | | $a_z = \Delta v_z/\Delta t$

Example (a) shows both ways of writing Newton’s third law. Which would you rather write?
The idea is that each of the algebra symbols with an arrow written on top, called a vector, is actually an abbreviation for three different numbers, the $x$, $y$, and $z$ components. The three components are referred to as the components of the vector, e.g., $F_x$ is the $x$ component of the vector $\vec{F}$. The notation with an arrow on top is good for handwritten equations, but is unattractive in a printed book, so books use boldface, $\mathbf{F}$, to represent vectors. After this point, I’ll use boldface for vectors throughout this book.

Quantities can be classified as vectors or scalars. In a phrase like “a ______ to the northeast,” it makes sense to fill in the blank with “force” or “velocity,” which are vectors, but not with “mass” or “time,” which are scalars. Any nonzero vector has both a direction and an amount. The amount is called its magnitude. The notation for the magnitude of a vector $\mathbf{A}$ is $|\mathbf{A}|$, like the absolute value sign used with scalars.

Often, as in example (b), we wish to use the vector notation to represent adding up all the $x$ components to get a total $x$ component, etc. The plus sign is used between two vectors to indicate this type of component-by-component addition. Of course, vectors are really triplets of numbers, not numbers, so this is not the same as the use of the plus sign with individual numbers. But since we don’t want to have to invent new words and symbols for this operation on vectors, we use the same old plus sign, and the same old addition-related words like “add,” “sum,” and “total.” Combining vectors this way is called vector addition.

Similarly, the minus sign in example (a) was used to indicate negating each of the vector’s three components individually. The equals sign is used to mean that all three components of the vector on the left side of an equation are the same as the corresponding components on the right.

Example (c) shows how we abuse the division symbol in a similar manner. When we write the vector $\Delta \mathbf{v}$ divided by the scalar $\Delta t$, we mean the new vector formed by dividing each one of the velocity components by $\Delta t$.

It’s not hard to imagine a variety of operations that would combine vectors with vectors or vectors with scalars, but only four of them are required in order to express Newton’s laws:
### Operation Definition

- **Vector + Vector**: Add component by component to make a new set of three numbers.
- **Vector − Vector**: Subtract component by component to make a new set of three numbers.
- **Vector · Scalar**: Multiply each component of the vector by the scalar.
- **Vector / Scalar**: Divide each component of the vector by the scalar.

As an example of an operation that is not useful for physics, there just aren’t any useful physics applications for dividing a vector by another vector component by component. In optional section 7.5, we discuss in more detail the fundamental reasons why some vector operations are useful and others useless.

We can do algebra with vectors, or with a mixture of vectors and scalars in the same equation. Basically all the normal rules of algebra apply, but if you’re not sure if a certain step is valid, you should simply translate it into three component-based equations and see if it works.

#### Order of Addition

▶ If we are adding two force vectors, \( \mathbf{F} + \mathbf{G} \), is it valid to assume as in ordinary algebra that \( \mathbf{F} + \mathbf{G} \) is the same as \( \mathbf{G} + \mathbf{F} \)?

▶ To tell if this algebra rule also applies to vectors, we simply translate the vector notation into ordinary algebra notation. In terms of ordinary numbers, the components of the vector \( \mathbf{F} + \mathbf{G} \) would be \( F_x + G_x, F_y + G_y, \) and \( F_z + G_z \), which are certainly the same three numbers as \( G_x + F_x, G_y + F_y, \) and \( G_z + F_z \). Yes, \( \mathbf{F} + \mathbf{G} \) is the same as \( \mathbf{G} + \mathbf{F} \).

It is useful to define a symbol \( \mathbf{r} \) for the vector whose components are \( x, y, \) and \( z \), and a symbol \( \Delta \mathbf{r} \) made out of \( \Delta x, \Delta y, \) and \( \Delta z \).

Although this may all seem a little formidable, keep in mind that it amounts to nothing more than a way of abbreviating equations! Also, to keep things from getting too confusing the remainder of this chapter focuses mainly on the \( \Delta \mathbf{r} \) vector, which is relatively easy to visualize.

#### Self-check A

Translate the equations \( v_x = \Delta x / \Delta t, v_y = \Delta y / \Delta t, \) and \( v_z = \Delta z / \Delta t \) for motion with constant velocity into a single equation in vector notation.

▶ Answer, p. 564
The $x$ and $y$ components of a vector can be thought of as the shadows it casts onto the $x$ and $y$ axes.

Self-check B.

Given vector $\mathbf{Q}$ represented by an arrow in figure c, draw arrows representing the vectors $1.5\mathbf{Q}$ and $-\mathbf{Q}$.

Answer, p. 564

This leads to a way of defining vectors and scalars that reflects how physicists think in general about these things:

**Definition of vectors and scalars**

A general type of measurement (force, velocity, ... ) is a vector if it can be drawn as an arrow so that rotating the paper produces the same result as rotating the actual quantity. A type of quantity that never changes at all under rotation is a scalar.

For example, a force reverses itself under a 180-degree rotation, but a mass doesn’t. We could have defined a vector as something that had both a magnitude and a direction, but that would have left out zero vectors, which don’t have a direction. A zero vector is a legitimate vector, because it behaves the same way under rotations as a zero-length arrow, which is simply a dot.

A remark for those who enjoy brain-teasers: not everything is a vector or a scalar. An American football is distorted compared to a sphere, and we can measure the orientation and amount of that distortion quantitatively. The distortion is not a vector, since a 180-degree rotation brings it back to its original state. Something similar happens with playing cards, figure d. For some subatomic particles, such as electrons, 360 degrees isn’t even enough; a 720-degree rotation is needed to put them back the way they were!

**Discussion questions**

A. You drive to your friend’s house. How does the magnitude of your $\Delta r$ vector compare with the distance you’ve added to the car’s odometer?
7.2 Calculations with magnitude and direction

If you ask someone where Las Vegas is compared to Los Angeles, they are unlikely to say that the \( \Delta x \) is 290 km and the \( \Delta y \) is 230 km, in a coordinate system where the positive \( x \) axis is east and the \( y \) axis points north. They will probably say instead that it’s 370 km to the northeast. If they were being precise, they might give the direction as 38° counterclockwise from east. In two dimensions, we can always specify a vector’s direction like this, using a single angle. A magnitude plus an angle suffice to specify everything about the vector. The following two examples show how we use trigonometry and the Pythagorean theorem to go back and forth between the \( x-y \) and magnitude-angle descriptions of vectors.

**Finding magnitude and angle from components** example 2

> Given that the \( \Delta r \) vector from LA to Las Vegas has \( \Delta x = 290 \text{ km} \) and \( \Delta y = 230 \text{ km} \), how would we find the magnitude and direction of \( \Delta r \)?

> We find the magnitude of \( \Delta r \) from the Pythagorean theorem:

\[
|\Delta r| = \sqrt{\Delta x^2 + \Delta y^2} = 370 \text{ km}
\]

We know all three sides of the triangle, so the angle \( \theta \) can be found using any of the inverse trig functions. For example, we know the opposite and adjacent sides, so

\[
\theta = \tan^{-1} \frac{\Delta y}{\Delta x} = 38^\circ.
\]

**Finding components from magnitude and angle** example 3

> Given that the straight-line distance from Los Angeles to Las Vegas is 370 km, and that the angle \( \theta \) in the figure is 38°, how can the \( x \) and \( y \) components of the \( \Delta r \) vector be found?

> The sine and cosine of \( \theta \) relate the given information to the information we wish to find:

\[
\cos \theta = \frac{\Delta x}{|\Delta r|} \quad \text{and} \quad \sin \theta = \frac{\Delta y}{|\Delta r|}
\]

Solving for the unknowns gives

\[
\Delta x = |\Delta r| \cos \theta = 290 \text{ km} \quad \text{and} \quad \Delta y = |\Delta r| \sin \theta = 230 \text{ km}.
\]
The following example shows the correct handling of the plus and minus signs, which is usually the main cause of mistakes.

\[ \text{Negative components} \]

Example 4

San Diego is 120 km east and 150 km south of Los Angeles. An airplane pilot is setting course from San Diego to Los Angeles. At what angle should she set her course, measured counterclockwise from east, as shown in the figure?

If we make the traditional choice of coordinate axes, with \( x \) pointing to the right and \( y \) pointing up on the map, then her \( \Delta x \) is negative, because her final \( x \) value is less than her initial \( x \) value. Her \( \Delta y \) is positive, so we have

\[ \Delta x = -120 \text{ km} \]
\[ \Delta y = 150 \text{ km}. \]

If we work by analogy with example 2, we get

\[ \theta = \tan^{-1} \frac{\Delta y}{\Delta x} \]
\[ = \tan^{-1}(-1.25) \]
\[ = -51^\circ. \]

According to the usual way of defining angles in trigonometry, a negative result means an angle that lies clockwise from the \( x \) axis, which would have her heading for the Baja California. What went wrong? The answer is that when you ask your calculator to take the arctangent of a number, there are always two valid possibilities differing by 180°. That is, there are two possible angles whose tangents equal -1.25:

\[ \tan 129^\circ = -1.25 \]
\[ \tan -51^\circ = -1.25 \]

You calculator doesn’t know which is the correct one, so it just picks one. In this case, the one it picked was the wrong one, and it was up to you to add 180° to it to find the right answer.
Example 5

A split second after nine o’clock, the hour hand on a clock dial has moved clockwise past the nine-o’clock position by some imperceptibly small angle $\phi$. Let positive $x$ be to the right and positive $y$ up. If the hand, with length $\ell$, is represented by a $\Delta r$ vector going from the dial’s center to the tip of the hand, find this vector’s $\Delta x$.

The following shortcut is the easiest way to work out examples like these, in which a vector’s direction is known relative to one of the axes. We can tell that $\Delta r$ will have a large, negative $x$ component and a small, positive $y$. Since $\Delta x < 0$, there are really only two logical possibilities: either $\Delta x = -\ell \cos \phi$, or $\Delta x = -\ell \sin \phi$. Because $\phi$ is small, $\cos \phi$ is large and $\sin \phi$ is small. We conclude that $\Delta x = -\ell \cos \phi$.

A typical application of this technique to force vectors is given in example 6 on p. 226.

Discussion question

A In example 4, we dealt with components that were negative. Does it make sense to classify vectors as positive and negative?
7.3 Techniques for adding vectors

Vector addition is one of the three essential mathematical skills, summarized on pp.543-544, that you need for success in this course.

Addition of vectors given their components

The easiest type of vector addition is when you are in possession of the components, and want to find the components of their sum.

Given the $\Delta x$ and $\Delta y$ values from the previous examples, find the $\Delta x$ and $\Delta y$ from San Diego to Las Vegas.

\[
\begin{align*}
\Delta x_{\text{total}} &= \Delta x_1 + \Delta x_2 \\
&= -120 \text{ km} + 290 \text{ km} \\
&= 170 \text{ km} \\
\Delta y_{\text{total}} &= \Delta y_1 + \Delta y_2 \\
&= 150 \text{ km} + 230 \text{ km} \\
&= 380 \\
\end{align*}
\]

Note how the signs of the $x$ components take care of the westward and eastward motions, which partially cancel.

Addition of vectors given their magnitudes and directions

In this case, you must first translate the magnitudes and directions into components, and add the components. In our San Diego-Los Angeles-Las Vegas example, we can simply string together the preceding examples; this is done on p. 544.

Graphical addition of vectors

Often the easiest way to add vectors is by making a scale drawing on a piece of paper. This is known as graphical addition, as opposed to the analytic techniques discussed previously. (It has nothing to do with $x-y$ graphs or graph paper. “Graphical” here simply means drawing. It comes from the Greek verb “grapho,” to write, like related English words including “graphic.”)
LA to Vegas, graphically

Example 7

Given the magnitudes and angles of the Δr vectors from San Diego to Los Angeles and from Los Angeles to Las Vegas, find the magnitude and angle of the Δr vector from San Diego to Las Vegas.

Using a protractor and a ruler, we make a careful scale drawing, as shown in figure j. The protractor can be conveniently aligned with the blue rules on the notebook paper. A scale of 1 mm → 2 km was chosen for this solution because it was as big as possible (for accuracy) without being so big that the drawing wouldn’t fit on the page. With a ruler, we measure the distance from San Diego to Las Vegas to be 206 mm, which corresponds to 412 km. With a protractor, we measure the angle θ to be 65°.

Even when we don’t intend to do an actual graphical calculation with a ruler and protractor, it can be convenient to diagram the addition of vectors in this way. With Δr vectors, it intuitively makes sense to lay the vectors tip-to-tail and draw the sum vector from the tail of the first vector to the tip of the second vector. We can do the same when adding other vectors such as force vectors.

Self-check C

How would you subtract vectors graphically?  

Answer, p. 564
Discussion questions

A If you’re doing graphical addition of vectors, does it matter which vector you start with and which vector you start from the other vector’s tip?

B If you add a vector with magnitude 1 to a vector of magnitude 2, what magnitudes are possible for the vector sum?

C Which of these examples of vector addition are correct, and which are incorrect?

7.4 * Unit vector notation

When we want to specify a vector by its components, it can be cumbersome to have to write the algebra symbol for each component:

\[ \Delta x = 290 \text{ km}, \quad \Delta y = 230 \text{ km} \]

A more compact notation is to write

\[ \Delta \mathbf{r} = (290 \text{ km})\hat{x} + (230 \text{ km})\hat{y}, \]

where the vectors \( \hat{x}, \hat{y}, \) and \( \hat{z} \), called the unit vectors, are defined as the vectors that have magnitude equal to 1 and directions lying along the \( x, y, \) and \( z \) axes. In speech, they are referred to as “x-hat” and so on.

A slightly different, and harder to remember, version of this notation is unfortunately more prevalent. In this version, the unit vectors are called \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \):

\[ \Delta \mathbf{r} = (290 \text{ km})\mathbf{i} + (230 \text{ km})\mathbf{j}. \]
Let’s take a closer look at why certain vector operations are useful and others are not. Consider the operation of multiplying two vectors component by component to produce a third vector:

\[ R_x = P_x Q_x \]
\[ R_y = P_y Q_y \]
\[ R_z = P_z Q_z \]

As a simple example, we choose vectors \( \mathbf{P} \) and \( \mathbf{Q} \) to have length 1, and make them perpendicular to each other, as shown in figure k/1. If we compute the result of our new vector operation using the coordinate system in k/2, we find:

\[ R_x = 0 \]
\[ R_y = 0 \]
\[ R_z = 0. \]

The \( x \) component is zero because \( P_x = 0 \), the \( y \) component is zero because \( Q_y = 0 \), and the \( z \) component is of course zero because both vectors are in the \( x - y \) plane. However, if we carry out the same operations in coordinate system k/3, rotated 45 degrees with respect to the previous one, we find

\[ R_x = 1/2 \]
\[ R_y = -1/2 \]
\[ R_z = 0. \]

The operation’s result depends on what coordinate system we use, and since the two versions of \( \mathbf{R} \) have different lengths (one being zero and the other nonzero), they don’t just represent the same answer expressed in two different coordinate systems. Such an operation will never be useful in physics, because experiments show physics works the same regardless of which way we orient the laboratory building! The useful vector operations, such as addition and scalar multiplication, are rotationally invariant, i.e., come out the same regardless of the orientation of the coordinate system.

Some smart phones and GPS units contain electronic compasses that can sense the direction of the earth’s magnetic field vector, notated \( \mathbf{B} \). Because all vectors work according to the same rules, you don’t need to know anything special about magnetism in order to understand this example. Unlike a traditional compass that uses a magnetized needle on a bearing, an electronic compass has no moving parts. It contains two sensors oriented perpendicular to one another, and each sensor is only sensitive to the component of the earth’s field that lies along its own axis. Because a
choice of coordinates is arbitrary, we can take one of these sensors as defining the $x$ axis and the other the $y$. Given the two components $B_x$ and $B_y$, the device’s computer chip can compute the angle of magnetic north relative to its sensors, $\tan^{-1}(B_y/B_x)$.

All compasses are vulnerable to errors because of nearby magnetic materials, and in particular it may happen that some part of the compass’s own housing becomes magnetized. In an electronic compass, rotational invariance provides a convenient way of calibrating away such effects by having the user rotate the device in a horizontal circle.

Suppose that when the compass is oriented in a certain way, it measures $B_x = 1.00$ and $B_y = 0.00$ (in certain units). We then expect that when it is rotated 90 degrees clockwise, the sensors will detect $B_x = 0.00$ and $B_y = 1.00$.

But imagine instead that we get $B_x = 0.20$ and $B_y = 0.80$. This would violate rotational invariance, since rotating the coordinate system is supposed to give a different description of the same vector. The magnitude appears to have changed from 1.00 to $\sqrt{0.20^2 + 0.80^2} = 0.82$, and a vector can’t change its magnitude just because you rotate it. The compass’s computer chip figures out that some effect, possibly a slight magnetization of its housing, must be adding an erroneous 0.2 units to all the $B_x$ readings, because subtracting this amount from all the $B_x$ values gives vectors that have the same magnitude, satisfying rotational invariance.
Summary

Selected vocabulary

vector . . . . . . . a quantity that has both an amount (magnitude) and a direction in space
magnitude . . . . the “amount” associated with a vector
scalar . . . . . . . a quantity that has no direction in space, only an amount

Notation

\( \mathbf{A} \) . . . . . . . a vector with components \( A_x, A_y, \) and \( A_z \)
\( \vec{A} \) . . . . . . . . . handwritten notation for a vector
\( |\mathbf{A}| \) . . . . . . . . the magnitude of vector \( \mathbf{A} \)
\( \mathbf{r} \) . . . . . . . . the vector whose components are \( x, y, \) and \( z \)
\( \Delta \mathbf{r} \) . . . . . . . . the vector whose components are \( \Delta x, \Delta y, \) and \( \Delta z \)
\( \hat{x}, \hat{y}, \hat{z} \) . . . . . (optional topic) unit vectors; the vectors with magnitude 1 lying along the \( x, y, \) and \( z \) axes
\( \hat{i}, \hat{j}, \hat{k} \) . . . . . . . a harder to remember notation for the unit vectors

Other terminology and notation

displacement vector . . . a name for the symbol \( \Delta \mathbf{r} \)
speed . . . . . . . . . . the magnitude of the velocity vector, i.e., the velocity stripped of any information about its direction

Summary

A vector is a quantity that has both a magnitude (amount) and a direction in space, as opposed to a scalar, which has no direction. The vector notation amounts simply to an abbreviation for writing the vector’s three components.

In two dimensions, a vector can be represented either by its two components or by its magnitude and direction. The two ways of describing a vector can be related by trigonometry.

The two main operations on vectors are addition of a vector to a vector, and multiplication of a vector by a scalar.

Vector addition means adding the components of two vectors to form the components of a new vector. In graphical terms, this corresponds to drawing the vectors as two arrows laid tip-to-tail and drawing the sum vector from the tail of the first vector to the tip of the second one. Vector subtraction is performed by negating the vector to be subtracted and then adding.

Multiplying a vector by a scalar means multiplying each of its components by the scalar to create a new vector. Division by a scalar is defined similarly.
Problems

Key
✓ A computerized answer check is available online.
∫ A problem that requires calculus.
★ A difficult problem.

1 The figure shows vectors \( \mathbf{A} \) and \( \mathbf{B} \). Graphically calculate the following, as in figure i on p. 210, self-check C on p. 211, and self-check B on p. 206.

\[ A + B, \ A - B, \ B - A, \ -2B, \ A - 2B \]

No numbers are involved.

2 Phnom Penh is 470 km east and 250 km south of Bangkok. Hanoi is 60 km east and 1030 km north of Phnom Penh.
(a) Choose a coordinate system, and translate these data into \( \Delta x \) and \( \Delta y \) values with the proper plus and minus signs.
(b) Find the components of the \( \Delta \mathbf{r} \) vector pointing from Bangkok to Hanoi. ✓

3 If you walk 35 km at an angle 25° counterclockwise from east, and then 22 km at 230° counterclockwise from east, find the distance and direction from your starting point to your destination. ✓

4 A machinist is drilling holes in a piece of aluminum according to the plan shown in the figure. She starts with the top hole, then moves to the one on the left, and then to the one on the right. Since this is a high-precision job, she finishes by moving in the direction and at the angle that should take her back to the top hole, and checks that she ends up in the same place. What are the distance and direction from the right-hand hole to the top one? ✓
Suppose someone proposes a new operation in which a vector $A$ and a scalar $B$ are added together to make a new vector $C$ like this:

$$C_x = A_x + B$$
$$C_y = A_y + B$$
$$C_z = A_z + B$$

Prove that this operation won’t be useful in physics, because it’s not rotationally invariant.
Chapter 8

Vectors and Motion

In 1872, capitalist and former California governor Leland Stanford asked photographer Eadweard Muybridge if he would work for him on a project to settle a $25,000 bet (a princely sum at that time). Stanford’s friends were convinced that a trotting horse always had at least one foot on the ground, but Stanford claimed that there was a moment during each cycle of the motion when all four feet were in the air. The human eye was simply not fast enough to settle the question. In 1878, Muybridge finally succeeded in producing what amounted to a motion picture of the horse, showing conclusively that all four feet did leave the ground at one point. (Muybridge was a colorful figure in San Francisco history, and his acquittal for the murder of his wife’s lover was considered the trial of the century in California.)

The losers of the bet had probably been influenced by Aristotelian reasoning, for instance the expectation that a leaping horse would lose horizontal velocity while in the air with no force to push it forward, so that it would be more efficient for the horse to run without leaping. But even for students who have converted whole-
heartedly to Newtonianism, the relationship between force and acceleration leads to some conceptual difficulties, the main one being a problem with the true but seemingly absurd statement that an object can have an acceleration vector whose direction is not the same as the direction of motion. The horse, for instance, has nearly constant horizontal velocity, so its $a_x$ is zero. But as anyone can tell you who has ridden a galloping horse, the horse accelerates up and down. The horse’s acceleration vector therefore changes back and forth between the up and down directions, but is never in the same direction as the horse’s motion. In this chapter, we will examine more carefully the properties of the velocity, acceleration, and force vectors. No new principles are introduced, but an attempt is made to tie things together and show examples of the power of the vector formulation of Newton’s laws.

8.1 The velocity vector

For motion with constant velocity, the velocity vector is

$$\mathbf{v} = \Delta \mathbf{r} / \Delta t.$$  [only for constant velocity]

The $\Delta \mathbf{r}$ vector points in the direction of the motion, and dividing it by the scalar $\Delta t$ only changes its length, not its direction, so the velocity vector points in the same direction as the motion. When the velocity is not constant, i.e., when the $x-t$, $y-t$, and $z-t$ graphs are not all linear, we use the slope-of-the-tangent-line approach to define the components $v_x$, $v_y$, and $v_z$, from which we assemble the velocity vector. Even when the velocity vector is not constant, it still points along the direction of motion.

Vector addition is the correct way to generalize the one-dimensional concept of adding velocities in relative motion, as shown in the following example:

**Velocity vectors in relative motion**

- You wish to cross a river and arrive at a dock that is directly across from you, but the river’s current will tend to carry you downstream. To compensate, you must steer the boat at an angle. Find the angle $\theta$, given the magnitude, $|\mathbf{v}_{WL}|$, of the water’s velocity relative to the land, and the maximum speed, $|\mathbf{v}_{BW}|$, of which the boat is capable relative to the water.

- The boat’s velocity relative to the land equals the vector sum of its velocity with respect to the water and the water’s velocity with respect to the land,

$$\mathbf{v}_{BL} = \mathbf{v}_{BW} + \mathbf{v}_{WL}.$$  

If the boat is to travel straight across the river, i.e., along the $y$ axis, then we need to have $v_{BL,x} = 0$. This $x$ component equals the sum of the $x$ components of the other two vectors,

$$v_{BL,x} = v_{BW,x} + v_{WL,x},$$
or

\[ 0 = -|v_{BW}| \sin \theta + |v_{WL}|. \]

Solving for \( \theta \), we find \( \sin \theta = |v_{WL}|/|v_{BW}| \), so

\[ \theta = \sin^{-1} \frac{|v_{WL}|}{|v_{BW}|}. \]

\[\textbf{Discussion questions}\]

A  Is it possible for an airplane to maintain a constant velocity vector but not a constant \(|v|\)? How about the opposite—a constant \(|v|\) but not a constant velocity vector? Explain.

B  New York and Rome are at about the same latitude, so the earth’s rotation carries them both around nearly the same circle. Do the two cities have the same velocity vector (relative to the center of the earth)? If not, is there any way for two cities to have the same velocity vector?

\[\textbf{8.2 The acceleration vector}\]

When the acceleration is constant, we can define the acceleration vector as

\[ a = \Delta v/\Delta t, \quad \text{[only for constant acceleration]} \]

which can be written in terms of initial and final velocities as

\[ a = (v_f - v_i)/\Delta t. \quad \text{[only for constant acceleration]} \]

Otherwise, we can use the type of graphical definition described in section 8.1 for the velocity vector.

Now there are two ways in which we could have a nonzero acceleration. Either the magnitude or the direction of the velocity vector could change. This can be visualized with arrow diagrams as shown in figures c and d. Both the magnitude and direction can change simultaneously, as when a car accelerates while turning. Only when the magnitude of the velocity changes while its direction stays constant do we have a \( \Delta v \) vector and an acceleration vector along the same line as the motion.

\[\textbf{self-check A}\]

(1) In figure c, is the object speeding up, or slowing down? (2) What would the diagram look like if \( v_i \) was the same as \( v_f \)? (3) Describe how the \( \Delta v \) vector is different depending on whether an object is speeding up or slowing down.

\[\textbf{Answer, p. 564}\]

\[\textbf{Solved problem: Annie Oakley page 234, problem 8}\]
The acceleration vector points in the direction that an accelerometer would point, as in figure e.

![Image of a car swerving with an air freshener indicating acceleration]

**self-check B**
In projectile motion, what direction does the acceleration vector have?

Answer, p. 564

---

**Example 2.**

In figure f, the rappeller’s velocity has long periods of gradual change interspersed with short periods of rapid change. These correspond to periods of small acceleration and force, and periods of large acceleration and force.
Figure g on page 223 shows outlines traced from the first, third, fifth, seventh, and ninth frames in Muybridge's series of photographs of the galloping horse. The estimated location of the horse’s center of mass is shown with a circle, which bobs above and below the horizontal dashed line.

If we don’t care about calculating velocities and accelerations in any particular system of units, then we can pretend that the time between frames is one unit. The horse’s velocity vector as it moves from one point to the next can then be found simply by drawing an arrow to connect one position of the center of mass to the next. This produces a series of velocity vectors which alternate between pointing above and below horizontal.

The $\Delta v$ vector is the vector which we would have to add onto one velocity vector in order to get the next velocity vector in the series. The $\Delta v$ vector alternates between pointing down (around the time when the horse is in the air, B) and up (around the time when the horse has two feet on the ground, D).
Discussion questions

A  When a car accelerates, why does a bob hanging from the rearview mirror swing toward the back of the car? Is it because a force throws it backward? If so, what force? Similarly, describe what happens in the other cases described above.

B  Superman is guiding a crippled spaceship into port. The ship’s engines are not working. If Superman suddenly changes the direction of his force on the ship, does the ship’s velocity vector change suddenly? Its acceleration vector? Its direction of motion?

8.3 The force vector and simple machines

Force is relatively easy to intuit as a vector. The force vector points in the direction in which it is trying to accelerate the object it is acting on.

Since force vectors are so much easier to visualize than acceleration vectors, it is often helpful to first find the direction of the (total) force vector acting on an object, and then use that to find the direction of the acceleration vector. Newton’s second law tells us that the two must be in the same direction.

A component of a force vector

Figure h, redrawn from a classic 1920 textbook, shows a boy pulling another child on a sled. His force has both a horizontal component and a vertical one, but only the horizontal one accelerates the sled. (The vertical component just partially cancels the force of gravity, causing a decrease in the normal force between the runners and the snow.) There are two triangles in the figure. One triangle’s hypotenuse is the rope, and the other’s is the magnitude of the force. These triangles are similar, so their internal angles are all the same, but they are not the same triangle. One is a distance triangle, with sides measured in meters, the other a force triangle, with sides in newtons. In both cases, the horizontal leg is 93% as long as the hypotenuse. It does not make sense, however, to compare the sizes of the triangles — the force triangle is not smaller in any meaningful sense.
"Pushing a block up a ramp" example 5

Figure i shows a block being pushed up a frictionless ramp at constant speed by an externally applied force $F_A$. How much force is required, in terms of the block’s mass, $m$, and the angle of the ramp, $\theta$?

We analyze the forces on the block and introduce notation for the other forces besides $F_A$:

<table>
<thead>
<tr>
<th>force acting on block</th>
<th>3rd-law partner</th>
</tr>
</thead>
<tbody>
<tr>
<td>ramp’s normal force on block, $F_N$</td>
<td>block’s normal force on ramp, $\checkmark$</td>
</tr>
<tr>
<td>external object’s force on block (type irrelevant), $F_A$</td>
<td>block’s force on external object (type irrelevant), $\checkmark$</td>
</tr>
<tr>
<td>planet earth’s gravitational force on block, $F_W$</td>
<td>block’s gravitational force on earth, $\uparrow$</td>
</tr>
</tbody>
</table>

Because the block is being pushed up at constant speed, it has zero acceleration, and the total force on it must be zero. From figure j, we find

$$|F_A| = |F_W| \sin \theta$$

$$= mg \sin \theta.$$

Since the sine is always less than one, the applied force is always less than $mg$, i.e., pushing the block up the ramp is easier than lifting it straight up. This is presumably the principle on which the pyramids were constructed: the ancient Egyptians would have had a hard time applying the forces of enough slaves to equal the full weight of the huge blocks of stone.

Essentially the same analysis applies to several other simple machines, such as the wedge and the screw.
Example 6 and problem 18 on p. 237.

A layback example 6
The figure shows a rock climber using a technique called a layback. He can make the normal forces \( F_{N1} \) and \( F_{N2} \) large, which has the side-effect of increasing the frictional forces \( F_{F1} \) and \( F_{F2} \), so that he doesn’t slip down due to the gravitational (weight) force \( F_W \). The purpose of the problem is not to analyze all of this in detail, but simply to practice finding the components of the forces based on their magnitudes. To keep the notation simple, let’s write \( F_{N1} \) for \( |F_{N1}| \), etc. The crack overhangs by a small, positive angle \( \theta \approx 9^\circ \).

In this example, we determine the \( x \) component of \( F_{N1} \). The other nine components are left as an exercise to the reader (problem 18, p. 237).

The easiest method is the one demonstrated in example 5 on p. 209. Casting vector \( F_{N1} \)’s shadow on the ground, we can tell that it would point to the left, so its \( x \) component is negative. The only two possibilities for its \( x \) component are therefore \(-F_{N1} \cos \theta\) or \(-F_{N1} \sin \theta\). We expect this force to have a large \( x \) component and a much smaller \( y \). Since \( \theta \) is small, \( \cos \theta \approx 1 \), while \( \sin \theta \) is small. Therefore the \( x \) component must be \(-F_{N1} \cos \theta\).

Pushing a broom example 7
▷ Figure 1 shows a man pushing a broom at an angle \( \theta \) relative to the horizontal. The mass \( m \) of the broom is concentrated at the brush. If the magnitude of the broom’s acceleration is \( a \), find the force \( F_H \) that the man must make on the handle.

▷ First we analyze all the forces on the brush.
force acting on brush | 3rd-law partner
---|---
handle's normal force on brush, $F_H$ | brush's normal force on handle, $F_H$
earth's gravitational force on brush, $mg$ | brush's gravitational force on earth, $mg$
floor's normal force on brush, $F_N$ | brush's normal force on floor, $F_N$
floor's kinetic friction force on brush, $F_k$ | brush's kinetic friction force on floor, $F_k$

Newton's second law is:

$$a = \frac{\mathbf{F}_H + mg + \mathbf{F}_N + \mathbf{F}_k}{m},$$

where the addition is vector addition. If we actually want to carry out the vector addition of the forces, we have to do either graphical addition (as in example 5) or analytic addition. Let's do analytic addition, which means finding all the components of the forces, adding the $x$'s, and adding the $y$'s.

Most of the forces have components that are trivial to express in terms of their magnitudes, the exception being $F_H$, whose components we can determine using the technique demonstrated in example 5 on p. 209 and example 6 on p. 226. Using the coordinate system shown in the figure, the results are:

$F_{Hx} = F_H \cos \theta$ \hspace{1cm} $F_{Hy} = -F_H \sin \theta$
$mg_x = 0$ \hspace{1cm} $mg_y = -mg$
$F_{Nx} = 0$ \hspace{1cm} $F_{Ny} = F_N$
$F_{kx} = -F_k$ \hspace{1cm} $F_{ky} = 0$

Note that we don't yet know the magnitudes $F_H$, $F_N$, and $F_k$. That's all right. First we need to set up Newton's laws, and then we can worry about solving the equations.

Newton's second law in the $x$ direction gives:

$$[1] \quad a = \frac{F_H \cos \theta - F_k}{m}$$

The acceleration in the vertical direction is zero, so Newton's second law in the $y$ direction tells us that

$$[2] \quad 0 = -F_H \sin \theta - mg + F_N.$$ 

Finally, we have the relationship between kinetic friction and the normal force,

$$[3] \quad F_k = \mu_k F_N.$$ 

Equations [1]-[3] are three equations, which we can use to determine the three unknowns, $F_H$, $F_N$, and $F_k$. Straightforward algebra gives

$$F_H = m \left( \frac{a + \mu_k g}{\cos \theta - \mu_k \sin \theta} \right)$$
Discussion question A.

A cargo plane page 234, problem 9

The angle of repose page 235, problem 11

A wagon page 234, problem 10

Discussion questions

A The figure shows a block being pressed diagonally upward against a wall, causing it to slide up the wall. Analyze the forces involved, including their directions.

B The figure shows a roller coaster car rolling down and then up under the influence of gravity. Sketch the car’s velocity vectors and acceleration vectors. Pick an interesting point in the motion and sketch a set of force vectors acting on the car whose vector sum could have resulted in the right acceleration vector.

8.4 Calculus with vectors

Using the unit vector notation introduced in section 7.4, the definitions of the velocity and acceleration components given in chapter 6 can be translated into calculus notation as

\[ \mathbf{v} = \frac{dx}{dt} \mathbf{\hat{x}} + \frac{dy}{dt} \mathbf{\hat{y}} + \frac{dz}{dt} \mathbf{\hat{z}} \]

and

\[ \mathbf{a} = \frac{dv_x}{dt} \mathbf{\hat{x}} + \frac{dv_y}{dt} \mathbf{\hat{y}} + \frac{dv_z}{dt} \mathbf{\hat{z}}. \]

To make the notation less cumbersome, we generalize the concept of the derivative to include derivatives of vectors, so that we can abbreviate the above equations as

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} \]

and

\[ \mathbf{a} = \frac{d\mathbf{v}}{dt}. \]

In words, to take the derivative of a vector, you take the derivatives of its components and make a new vector out of those. This definition means that the derivative of a vector function has the familiar properties

\[ \frac{d(cf)}{dt} = c \frac{df}{dt} \quad \text{[c is a constant]} \]

and

\[ \frac{d(f + g)}{dt} = \frac{df}{dt} + \frac{dg}{dt}. \]

The integral of a vector is likewise defined as integrating component by component.
The second derivative of a vector example 8

Two objects have positions as functions of time given by the equations

\[
\mathbf{r}_1 = 3t^2 \mathbf{x} + t \mathbf{y}
\]

and

\[
\mathbf{r}_2 = 3t^4 \mathbf{x} + t \mathbf{y}.
\]

Find both objects’ accelerations using calculus. Could either answer have been found without calculus?

Taking the first derivative of each component, we find

\[
\mathbf{v}_1 = 6t \mathbf{x} + \mathbf{y}
\]

\[
\mathbf{v}_2 = 12t^3 \mathbf{x} + \mathbf{y},
\]

and taking the derivatives again gives acceleration,

\[
\mathbf{a}_1 = 6 \mathbf{x}
\]

\[
\mathbf{a}_2 = 36t^2 \mathbf{x}.
\]

The first object’s acceleration could have been found without calculus, simply by comparing the \(x\) and \(y\) coordinates with the constant-acceleration equation \(\Delta x = v_0 \Delta t + \frac{1}{2} a \Delta t^2\). The second equation, however, isn’t just a second-order polynomial in \(t\), so the acceleration isn’t constant, and we really did need calculus to find the corresponding acceleration.

The integral of a vector example 9

Starting from rest, a flying saucer of mass \(m\) is observed to vary its propulsion with mathematical precision according to the equation

\[
\mathbf{F} = bt^{42} \mathbf{x} + ct^{137} \mathbf{y}.
\]

(The aliens inform us that the numbers 42 and 137 have a special religious significance for them.) Find the saucer’s velocity as a function of time.

From the given force, we can easily find the acceleration

\[
\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{b}{m} t^{42} \mathbf{x} + \frac{c}{m} t^{137} \mathbf{y}.
\]

The velocity vector \(\mathbf{v}\) is the integral with respect to time of the acceleration,

\[
\mathbf{v} = \int \mathbf{a} \, dt
\]

\[
= \int \left( \frac{b}{m} t^{42} \mathbf{x} + \frac{c}{m} t^{137} \mathbf{y} \right) \, dt,
\]
and integrating component by component gives

\[
\begin{align*}
= \left( \int \frac{b}{m} t^{42} \, dt \right) \hat{x} + \left( \int \frac{c}{m} t^{137} \, dt \right) \hat{y} \\
= \frac{b}{43m} t^{43} \hat{x} + \frac{c}{138m} t^{138} \hat{y},
\end{align*}
\]

where we have omitted the constants of integration, since the saucer was starting from rest.

\text{A fire-extinguisher stunt on ice example 10}

Prof. Puerile smuggles a fire extinguisher into a skating rink. Climbing out onto the ice without any skates on, he sits down and pushes off from the wall with his feet, acquiring an initial velocity \( v_0 \hat{y} \). At \( t = 0 \), he then discharges the fire extinguisher at a 45-degree angle so that it applies a force to him that is backward and to the left, i.e., along the negative \( y \) axis and the positive \( x \) axis. The fire extinguisher's force is strong at first, but then dies down according to the equation \( |F| = b - ct \), where \( b \) and \( c \) are constants. Find the professor's velocity as a function of time.

\text{Measured counterclockwise from the \( x \) axis, the angle of the force vector becomes 315°. Breaking the force down into \( x \) and \( y \) components, we have}

\[
F_x = |F| \cos 315° = (b - ct)
\]

\[
F_y = |F| \sin 315° = (-b + ct).
\]

In unit vector notation, this is

\[
F = (b - ct) \hat{x} + (-b + ct) \hat{y}.
\]

Newton's second law gives

\[
a = \frac{F}{m} = \frac{b - ct}{\sqrt{2m}} \hat{x} + \frac{-b + ct}{\sqrt{2m}} \hat{y}.
\]

To find the velocity vector as a function of time, we need to integrate the acceleration vector with respect to time,

\[
v = \int a \, dt = \int \left( \frac{b - ct}{\sqrt{2m}} \hat{x} + \frac{-b + ct}{\sqrt{2m}} \hat{y} \right) \, dt = \frac{1}{\sqrt{2m}} \int \left[ (b - ct) \hat{x} + (-b + ct) \hat{y} \right] \, dt
\]
A vector function can be integrated component by component, so this can be broken down into two integrals,

$$\mathbf{v} = \frac{\dot{x}}{\sqrt{2m}} \int (b - ct) \, dt + \frac{\dot{y}}{\sqrt{2m}} \int (-b + ct) \, dt$$

$$= \left( \frac{bt - \frac{1}{2} ct^2}{\sqrt{2m}} + \text{constant #1} \right) \hat{x} + \left( \frac{-bt + \frac{1}{2} ct^2}{\sqrt{2m}} + \text{constant #2} \right) \hat{y}$$

Here the physical significance of the two constants of integration is that they give the initial velocity. Constant #1 is therefore zero, and constant #2 must equal $v_0$. The final result is

$$\mathbf{v} = \left( \frac{bt - \frac{1}{2} ct^2}{\sqrt{2m}} \right) \hat{x} + \left( \frac{-bt + \frac{1}{2} ct^2}{\sqrt{2m}} + v_0 \right) \hat{y}.$$
Summary

The velocity vector points in the direction of the object’s motion. Relative motion can be described by vector addition of velocities.

The acceleration vector need not point in the same direction as the object’s motion. We use the word “acceleration” to describe any change in an object’s velocity vector, which can be either a change in its magnitude or a change in its direction.

An important application of the vector addition of forces is the use of Newton’s first law to analyze mechanical systems.
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
★ A difficult problem.

Problem 1.

1 As shown in the diagram, a dinosaur fossil is slowly moving down the slope of a glacier under the influence of wind, rain and gravity. At the same time, the glacier is moving relative to the continent underneath. The dashed lines represent the directions but not the magnitudes of the velocities. Pick a scale, and use graphical addition of vectors to find the magnitude and the direction of the fossil’s velocity relative to the continent. You will need a ruler and protractor.

2 Is it possible for a helicopter to have an acceleration due east and a velocity due west? If so, what would be going on? If not, why not?

3 A bird is initially flying horizontally east at 21.1 m/s, but one second later it has changed direction so that it is flying horizontally and 7° north of east, at the same speed. What are the magnitude and direction of its acceleration vector during that one second time interval? (Assume its acceleration was roughly constant.)

Problem 4.

4 A person of mass $M$ stands in the middle of a tightrope, which is fixed at the ends to two buildings separated by a horizontal distance $L$. The rope sags in the middle, stretching and lengthening the rope slightly.
Problem 5.

(a) If the tightrope walker wants the rope to sag vertically by no more than a height \( h \), find the minimum tension, \( T \), that the rope must be able to withstand without breaking, in terms of \( h \), \( g \), \( M \), and \( L \).

(b) Based on your equation, explain why it is not possible to get \( h = 0 \), and give a physical interpretation.

5 Your hand presses a block of mass \( m \) against a wall with a force \( \mathbf{F}_H \) acting at an angle \( \theta \), as shown in the figure. Find the minimum and maximum possible values of \( |\mathbf{F}_H| \) that can keep the block stationary, in terms of \( m \), \( g \), \( \theta \), and \( \mu_s \), the coefficient of static friction between the block and the wall. Check both your answers in the case of \( \theta = 90^\circ \), and interpret the case where the maximum force is infinite.

⋆

6 A skier of mass \( m \) is coasting down a slope inclined at an angle \( \theta \) compared to horizontal. Assume for simplicity that the treatment of kinetic friction given in chapter 5 is appropriate here, although a soft and wet surface actually behaves a little differently. The coefficient of kinetic friction acting between the skis and the snow is \( \mu_k \), and in addition the skier experiences an air friction force of magnitude \( bv^2 \), where \( b \) is a constant.

(a) Find the maximum speed that the skier will attain, in terms of the variables \( m \), \( g \), \( \theta \), \( \mu_k \), and \( b \).

(b) For angles below a certain minimum angle \( \theta_{\text{min}} \), the equation gives a result that is not mathematically meaningful. Find an equation for \( \theta_{\text{min}} \), and give a physical explanation of what is happening for \( \theta < \theta_{\text{min}} \).

7 A gun is aimed horizontally to the west. The gun is fired, and the bullet leaves the muzzle at \( t = 0 \). The bullet’s position vector as a function of time is \( \mathbf{r} = b \hat{x} + ct \hat{y} + dt^2 \hat{z} \), where \( b \), \( c \), and \( d \) are positive constants.

(a) What units would \( b \), \( c \), and \( d \) need to have for the equation to make sense?

(b) Find the bullet’s velocity and acceleration as functions of time.

(c) Give physical interpretations of \( b \), \( c \), \( d \), \( \hat{x} \), \( \hat{y} \), and \( \hat{z} \).

8 Annie Oakley, riding north on horseback at 30 mi/hr, shoots her rifle, aiming horizontally and to the northeast. The muzzle speed of the rifle is 140 mi/hr. When the bullet hits a defenseless fuzzy animal, what is its speed of impact? Neglect air resistance, and ignore the vertical motion of the bullet.  

9 A cargo plane has taken off from a tiny airstrip in the Andes, and is climbing at constant speed, at an angle of \( \theta = 17^\circ \) with respect to horizontal. Its engines supply a thrust of \( F_{\text{thrust}} = 200 \) kN, and the lift from its wings is \( F_{\text{lift}} = 654 \) kN. Assume that air resistance (drag) is negligible, so the only forces acting are thrust, lift, and weight. What is its mass, in kg?
Problem 10. A wagon is being pulled at constant speed up a slope $\theta$ by a rope that makes an angle $\phi$ with the vertical.
(a) Assuming negligible friction, show that the tension in the rope is given by the equation

$$F_T = \frac{\sin \theta}{\sin(\theta + \phi)} F_W,$$

where $F_W$ is the weight force acting on the wagon.
(b) Interpret this equation in the special cases of $\phi = 0$ and $\phi = 180^\circ - \theta$.

Problem 11. The angle of repose is the maximum slope on which an object will not slide. On airless, geologically inert bodies like the moon or an asteroid, the only thing that determines whether dust or rubble will stay on a slope is whether the slope is less steep than the angle of repose. (See figure n, p. 272.)
(a) Find an equation for the angle of repose, deciding for yourself what are the relevant variables.
(b) On an asteroid, where $g$ can be thousands of times lower than on Earth, would rubble be able to lie at a steeper angle of repose?

Problem 12. The figure shows an experiment in which a cart is released from rest at A, and accelerates down the slope through a distance $x$ until it passes through a sensor’s light beam. The point of the experiment is to determine the cart’s acceleration. At B, a cardboard vane mounted on the cart enters the light beam, blocking the light beam, and starts an electronic timer running. At C, the vane emerges from the beam, and the timer stops.
(a) Find the final velocity of the cart in terms of the width $w$ of the vane and the time $t_b$ for which the sensor’s light beam was blocked.
(b) Find the magnitude of the cart’s acceleration in terms of the measurable quantities $x$, $t_b$, and $w$.
(c) Analyze the forces in which the cart participates, using a table in the format introduced in section 5.3. Assume friction is negligible.
(d) Find a theoretical value for the acceleration of the cart, which could be compared with the experimentally observed value extracted in part b. Express the theoretical value in terms of the angle $\theta$ of the slope, and the strength $g$ of the gravitational field.

Problem 13. The figure shows a boy hanging in three positions: (1) with his arms straight up, (2) with his arms at 45 degrees, and (3) with his arms at 60 degrees with respect to the vertical. Compare the tension in his arms in the three cases.
Driving down a hill inclined at an angle \( \theta \) with respect to horizontal, you slam on the brakes to keep from hitting a deer. Your antilock brakes kick in, and you don’t skid.

(a) Analyze the forces. (Ignore rolling resistance and air friction.)
(b) Find the car’s maximum possible deceleration, \( a \) (expressed as a positive number), in terms of \( g \), \( \theta \), and the relevant coefficient of friction.
(c) Explain physically why the car’s mass has no effect on your answer.
(d) Discuss the mathematical behavior and physical interpretation of your result for negative values of \( \theta \).
(e) Do the same for very large positive values of \( \theta \).

The figure shows the path followed by Hurricane Irene in 2005 as it moved north. The dots show the location of the center of the storm at six-hour intervals, with lighter dots at the time when the storm reached its greatest intensity. Find the time when the storm’s center had a velocity vector to the northeast and an acceleration vector to the southeast. Explain.

For safety, mountain climbers often wear a climbing harness and tie in to other climbers on a rope team or to anchors such as pitons or snow anchors. When using anchors, the climber usually wants to tie in to more than one, both for extra strength and for redundancy in case one fails. The figure shows such an arrangement, with the climber hanging from a pair of anchors forming a “Y” at an angle \( \theta \). The metal piece at the center is called a carabiner. The usual advice is to make \( \theta < 90^\circ \); for large values of \( \theta \), the stress placed on the anchors can be many times greater than the actual load \( L \), so that two anchors are actually less safe than one.

(a) Find the force \( S \) at each anchor in terms of \( L \) and \( \theta \).
(b) Verify that your answer makes sense in the case of \( \theta = 0 \).
(c) Interpret your answer in the case of \( \theta = 180^\circ \).
(d) What is the smallest value of \( \theta \) for which \( S \) equals or exceeds \( L \), so that for larger angles a failure of at least one anchor is more likely than it would have been with a single anchor?

(a) The person with mass \( m \) hangs from the rope, hauling the box of mass \( M \) up a slope inclined at an angle \( \theta \). There is friction between the box and the slope, described by the usual coefficients of friction. The pulley, however, is frictionless. Find the magnitude of the box’s acceleration.
(b) Show that the units of your answer make sense.
(c) Check the physical behavior of your answer in the special cases of \( M = 0 \) and \( \theta = -90^\circ \).
18. Complete example 6 on p. 226 by expressing the remaining nine \(x\) and \(y\) components of the forces in terms of the five magnitudes and the small, positive angle \(\theta \approx 9^\circ\) by which the crack overhangs.

19. Problem 16 discussed a possible correct way of setting up a redundant anchor for mountaineering. The figure for this problem shows an incorrect way of doing it, by arranging the rope in a triangle (which we’ll take to be isosceles). One of the bad things about the triangular arrangement is that it requires more force from the anchors, making them more likely to fail. (a) Using the same notation as in problem 16, find \(S\) in terms of \(L\) and \(\theta\). (b) Verify that your answer makes sense in the case of \(\theta = 0\), and compare with the correct setup.

20. A telephone wire of mass \(m\) is strung between two poles, making an angle \(\theta\) with the horizontal at each end. (a) Find the tension at the center. (b) Which is greater, the tension at the center or at the ends?

21. The figure shows an arcade game called skee ball that is similar to bowling. The player rolls the ball down a horizontal alley. The ball then rides up a curved lip and is launched at an initial speed \(u\), at an angle \(\alpha\) above horizontal. Suppose we want the ball to go into a hole that is at horizontal distance \(\ell\) and height \(h\), as shown in the figure.

(a) Find the initial speed \(u\) that is required, in terms of the other variables and \(g\). (b) Check that your answer to part a has units that make sense. (c) Check that your answer to part a depends on \(g\) in a way that makes sense. This means that you should first determine on physical grounds whether increasing \(g\) should increase \(u\), or decrease it. Then see whether your answer to part a has this mathematical behavior. (d) Do the same for the dependence on \(h\). (e) Interpret your equation in the case where \(\alpha = 90^\circ\). (f) Interpret your equation in the case where \(\tan \alpha = h/\ell\). (g) Find \(u\) numerically if \(h = 70\) cm, \(\ell = 60\) cm, and \(\alpha = 65^\circ\).
22. A plane flies toward a city directly north and a distance \( D \) away. The wind speed is \( u \), and the plane's speed with respect to the wind is \( v \).

(a) If the wind is blowing from the west (towards the east), what direction should the plane head (what angle west of north)? 
(b) How long does it take the plane to get to the city? 
(c) Check that your answer to part b has units that make sense.
(d) Comment on the behavior of your answer in the case where \( u = v \).

[problem by B. Shotwell]

23. A force \( F \) is applied to a box of mass \( M \) at an angle \( \theta \) below the horizontal (see figure). The coefficient of static friction between the box and the floor is \( \mu_s \), and the coefficient of kinetic friction between the two surfaces is \( \mu_k \).

(a) What is the magnitude of the normal force on the box from the floor? 
(b) What is the minimum value of \( F \) to get the box to start moving from rest? 
(c) What is the value of \( F \) so that the box will move with constant velocity (assuming it is already moving)? 
(d) If \( \theta \) is greater than some critical angle \( \theta_{\text{crit}} \), it is impossible to have the scenario described in part c. What is \( \theta_{\text{crit}} \)?

[problem by B. Shotwell]

24. (a) A mass \( M \) is at rest on a fixed, frictionless ramp inclined at angle \( \theta \) with respect to the horizontal. The mass is connected to the force probe, as shown. What is the reading on the force probe? 
(b) Check that your answer to part a makes sense in the special cases \( \theta = 0 \) and \( \theta = 90^\circ \).

[problem by B. Shotwell]

25. The figure shows a rock climber wedged into a dihedral or "open book" consisting of two vertical walls of rock at an angle \( \theta \) relative to one another. This position can be maintained without any ledges or holds, simply by pressing the feet against the walls. The left hand is being used just for a little bit of balance. (a) Find the minimum coefficient of friction between the rubber climbing shoes and the rock. (b) Interpret the behavior of your expression at extreme values of \( \theta \). (c) Steven Won has done tabletop experiments using climbing shoes on the rough back side of a granite slab from a kitchen countertop, and has estimated \( \mu_s = 1.17 \). Find the corresponding maximum value of \( \theta \). 

▷ Solution, p. 554

26. You throw a rock horizontally from the edge of the roof of a building of height \( h \) with speed \( v_0 \). What is the (positive) angle between the final velocity vector and the horizontal when the rock hits the ground?

[problem by B. Shotwell]
27 The figure shows a block acted on by two external forces, each of magnitude $F$. One of the forces is horizontal, but the other is applied at a downward angle $\theta$. Gravity is negligible compared to these forces. The block rests on a surface with friction described by a coefficient of friction $\mu_s$. (a) Find the minimum value of $\mu_s$ that is required if the block is to remain at rest. 

(b) Show that this expression has the correct limit as $\theta$ approaches zero.
Exercise 8: Vectors and motion

Each diagram on page 241 shows the motion of an object in an $x - y$ plane. Each dot is one location of the object at one moment in time. The time interval from one dot to the next is always the same, so you can think of the vector that connects one dot to the next as a $\mathbf{v}$ vector, and subtract to find $\Delta \mathbf{v}$ vectors.

1. Suppose the object in diagram 1 is moving from the top left to the bottom right. Deduce whatever you can about the force acting on it. Does the force always have the same magnitude? The same direction?

Invent a physical situation that this diagram could represent.

What if you reinterpret the diagram by reversing the object's direction of motion? Redo the construction of one of the $\Delta \mathbf{v}$ vectors and see what happens.

2. What can you deduce about the force that is acting in diagram 2?

Invent a physical situation that diagram 2 could represent.

3. What can you deduce about the force that is acting in diagram 3?

Invent a physical situation.
Chapter 9
Circular Motion

9.1 Conceptual framework

I now live fifteen minutes from Disneyland, so my friends and family in my native Northern California think it’s a little strange that I’ve never visited the Magic Kingdom again since a childhood trip to the south. The truth is that for me as a preschooler, Disneyland was not the Happiest Place on Earth. My mother took me on a ride in which little cars shaped like rocket ships circled rapidly around a central pillar. I knew I was going to die. There was a force trying to throw me outward, and the safety features of the ride would surely have been inadequate if I hadn’t screamed the whole time to make sure Mom would hold on to me. Afterward, she seemed surprisingly indifferent to the extreme danger we had experienced.

Circular motion does not produce an outward force

My younger self’s understanding of circular motion was partly right and partly wrong. I was wrong in believing that there was a force pulling me outward, away from the center of the circle. The easiest way to understand this is to bring back the parable of the bowling ball in the pickup truck from chapter 4. As the truck makes a left turn, the driver looks in the rearview mirror and thinks that some mysterious force is pulling the ball outward, but the truck is accelerating, so the driver’s frame of reference is not an inertial frame. Newton’s laws are violated in a noninertial frame, so the ball appears to accelerate without any actual force acting on it. Because we are used to inertial frames, in which accelerations are caused by
forces, the ball’s acceleration creates a vivid illusion that there must be an outward force.

In an inertial frame everything makes more sense. The ball has no force on it, and goes straight as required by Newton’s first law. The truck has a force on it from the asphalt, and responds to it by accelerating (changing the direction of its velocity vector) as Newton’s second law says it should.

Another interesting example is an insect organ called the halteres, a pair of small knobbed limbs behind the wings, which vibrate up and down and help the insect to maintain its orientation in flight. The halteres evolved from a second pair of wings possessed by earlier insects. Suppose, for example, that the halteres are on their upward stroke, and at that moment an air current causes the fly to pitch its nose down. The halteres follow Newton’s first law, continuing to rise vertically, but in the fly’s rotating frame of reference, it seems as though they have been subjected to a backward force. The fly has special sensory organs that perceive this twist, and help it to correct itself by raising its nose.

I was correct, however, on a different point about the Disneyland ride. To make me curve around with the car, I really did need some force such as a force from my mother, friction from the seat, or a normal force from the side of the car. (In fact, all three forces were probably adding together.) One of the reasons why Galileo failed to
refine the principle of inertia into a quantitative statement like Newton’s first law is that he was not sure whether motion without a force would naturally be circular or linear. In fact, the most impressive examples he knew of the persistence of motion were mostly circular: the spinning of a top or the rotation of the earth, for example. Newton realized that in examples such as these, there really were forces at work. Atoms on the surface of the top are prevented from flying off straight by the ordinary force that keeps atoms stuck together in solid matter. The earth is nearly all liquid, but gravitational forces pull all its parts inward.

**Uniform and nonuniform circular motion**

Circular motion always involves a change in the direction of the velocity vector, but it is also possible for the magnitude of the velocity to change at the same time. Circular motion is referred to as *uniform* if $|v|$ is constant, and *nonuniform* if it is changing.

Your speedometer tells you the magnitude of your car’s velocity vector, so when you go around a curve while keeping your speedometer steady, you are executing uniform circular motion. If your speedometer reading is changing as you turn, your circular motion is nonuniform. Uniform circular motion is simpler to analyze mathematically, so we will attack it first and then pass to the nonuniform case.

**self-check A**

Which of these are examples of uniform circular motion and which are nonuniform?

1. the clothes in a clothes dryer (assuming they remain against the inside of the drum, even at the top)
2. a rock on the end of a string being whirled in a vertical circle

Answer, p. 564
Only an inward force is required for uniform circular motion.

Figure c showed the string pulling in straight along a radius of the circle, but many people believe that when they are doing this they must be “leading” the rock a little to keep it moving along. That is, they believe that the force required to produce uniform circular motion is not directly inward but at a slight angle to the radius of the circle. This intuition is incorrect, which you can easily verify for yourself now if you have some string handy. It is only while you are getting the object going that your force needs to be at an angle to the radius. During this initial period of speeding up, the motion is not uniform. Once you settle down into uniform circular motion, you only apply an inward force.

If you have not done the experiment for yourself, here is a theoretical argument to convince you of this fact. We have discussed in chapter 6 the principle that forces have no perpendicular effects. To keep the rock from speeding up or slowing down, we only need to make sure that our force is perpendicular to its direction of motion. We are then guaranteed that its forward motion will remain unaffected: our force can have no perpendicular effect, and there is no other force acting on the rock which could slow it down. The rock requires no forward force to maintain its forward motion, any more than a projectile needs a horizontal force to “help it over the top” of its arc.
Why, then, does a car driving in circles in a parking lot stop executing uniform circular motion if you take your foot off the gas? The source of confusion here is that Newton’s laws predict an object’s motion based on the total force acting on it. A car driving in circles has three forces on it:

1. an inward force from the asphalt, controlled with the steering wheel;
2. a forward force from the asphalt, controlled with the gas pedal; and
3. backward forces from air resistance and rolling resistance.

You need to make sure there is a forward force on the car so that the backward forces will be exactly canceled out, creating a vector sum that points directly inward.

In uniform circular motion, the acceleration vector is inward.

Since experiments show that the force vector points directly inward, Newton’s second law implies that the acceleration vector points inward as well. This fact can also be proven on purely kinematical grounds, and we will do so in the next section.

Clock-comparison tests of Newton’s first law

Immediately after his original statement of the first law in the *Principia Mathematica*, Newton offers the supporting example of a spinning top, which only slows down because of friction. He describes the different parts of the top as being held together by “cohesion,” i.e., internal forces. Because these forces act toward the center, they don’t speed up or slow down the motion. The applicability of the first law, which only describes linear motion, may be more clear if we simply take figure f as a model of rotation. Between hammer taps, the ball experiences no force, so by the first law it doesn’t speed up or slow down.

Suppose that we want to subject the first law to a stringent experimental test.¹ The law predicts that if we use a clock to measure the rate of rotation of an object spinning frictionlessly, it won’t “naturally” slow down as Aristotle would have expected. But what is a clock but something with hands that rotate at a fixed rate? In

¹Page 81 lists places in this book where we describe experimental tests of Newton’s first law.
other words, we are comparing one clock with another. This is called a clock-comparison experiment. Suppose that the laws of physics weren’t purely Newtonian, and there really was a very slight Aristotelian tendency for motion to slow down in the absence of friction. If we compare two clocks, they should both slow down, but if they aren’t the same type of clock, then it seems unlikely that they would slow down at exactly the same rate, and over time they should drift further and further apart.

High-precision clock-comparison experiments have been done using a variety of clocks. In atomic clocks, the thing spinning is an atom. Astronomers can observe the rotation of collapsed stars called pulsars, which, unlike the earth, can rotate with almost no disturbance due to geological activity or friction induced by the tides. In these experiments, the pulsars are observed to match the rates of the atomic clocks with a drift of less than about $10^{-6}$ seconds over a period of 10 years. Atomic clocks using atoms of different elements drift relative to one another by no more than about $10^{-16}$ per year.

It is not presently possible to do experiments with a similar level of precision using human-scale rotating objects. However, a set of gyroscopes aboard the Gravity Probe B satellite were allowed to spin weightlessly in a vacuum, without any physical contact that would have caused kinetic friction. Their rotation was extremely accurately monitored for the purposes of another experiment (a test of Einstein’s theory of general relativity, which was the purpose of the mission), and they were found to be spinning down so gradually that they would have taken about 10,000 years to slow down by a factor of two. This rate was consistent with estimates of the amount of friction to be expected from the small amount of residual gas present in the vacuum chambers.

A subtle point in the interpretation of these experiments is that if there was a slight tendency for motion to slow down, we would have to decide what it was supposed to slow down relative to. A straight-line motion that is slowing down in some frame of reference can always be described as speeding up in some other appropriately chosen frame (problem 12, p. 90). If the laws of physics did have this slight Aristotelianism mixed in, we could wait for the anomalous acceleration or deceleration to stop. The object we were observing would then define a special or “preferred” frame of reference. Standard theories of physics do not have such a preferred frame, and clock-comparison experiments can be viewed as tests of the existence of such a frame. Another test for the existence of a preferred frame is described on p. 277.

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3 Guéna et al., arxiv.org/abs/1205.4235
Discussion questions

A. In the game of crack the whip, a line of people stand holding hands, and then they start sweeping out a circle. One person is at the center, and rotates without changing location. At the opposite end is the person who is running the fastest, in a wide circle. In this game, someone always ends up losing their grip and flying off. Suppose the person on the end loses her grip. What path does she follow as she goes flying off? Draw an overhead view. (Assume she is going so fast that she is really just trying to put one foot in front of the other fast enough to keep from falling; she is not able to get any significant horizontal force between her feet and the ground.)

B. Suppose the person on the outside is still holding on, but feels that she may lose her grip at any moment. What force or forces are acting on her, and in what directions are they? (We are not interested in the vertical forces, which are the earth’s gravitational force pulling down, and the ground’s normal force pushing up.) Make a table in the format shown in section 5.3.

C. Suppose the person on the outside is still holding on, but feels that she may lose her grip at any moment. What is wrong with the following analysis of the situation? “The person whose hand she’s holding exerts an inward force on her, and because of Newton’s third law, there’s an equal and opposite force acting outward. That outward force is the one she feels throwing her outward, and the outward force is what might make her go flying off, if it’s strong enough.”

D. If the only force felt by the person on the outside is an inward force, why doesn’t she go straight in?

E. In the amusement park ride shown in the figure, the cylinder spins faster and faster until the customer can pick her feet up off the floor without falling. In the old Coney Island version of the ride, the floor actually dropped out like a trap door, showing the ocean below. (There is also a version in which the whole thing tilts up diagonally, but we’re discussing the version that stays flat.) If there is no outward force acting on her, why does she stick to the wall? Analyze all the forces on her.

F. What is an example of circular motion where the inward force is a normal force? What is an example of circular motion where the inward force is friction? What is an example of circular motion where the inward force is the sum of more than one force?

G. Does the acceleration vector always change continuously in circular motion? The velocity vector?
9.2 Uniform circular motion

In this section I derive a simple and very useful equation for the magnitude of the acceleration of an object undergoing constant acceleration. The law of sines is involved, so I’ve recapitulated it in figure i.

The derivation is brief, but the method requires some explanation and justification. The idea is to calculate a \( \Delta \mathbf{v} \) vector describing the change in the velocity vector as the object passes through an angle \( \theta \). We then calculate the acceleration, \( \mathbf{a} = \Delta \mathbf{v} / \Delta t \). The astute reader will recall, however, that this equation is only valid for motion with constant acceleration. Although the magnitude of the acceleration is constant for uniform circular motion, the acceleration vector changes its direction, so it is not a constant vector, and the equation \( \mathbf{a} = \Delta \mathbf{v} / \Delta t \) does not apply. The justification for using it is that we will then examine its behavior when we make the time interval very short, which means making the angle \( \theta \) very small. For smaller and smaller time intervals, the \( \Delta \mathbf{v} / \Delta t \) expression becomes a better and better approximation, so that the final result of the derivation is exact.

In figure j/1, the object sweeps out an angle \( \theta \). Its direction of motion also twists around by an angle \( \theta \), from the vertical dashed line to the tilted one. Figure j/2 shows the initial and final velocity vectors, which have equal magnitude, but directions differing by \( \theta \). In j/3, I’ve reassembled the vectors in the proper positions for vector subtraction. They form an isosceles triangle with interior angles \( \theta \), \( \eta \), and \( \eta \). (Eta, \( \eta \), is my favorite Greek letter.) The law of sines gives

\[
\frac{|\Delta \mathbf{v}|}{\sin \theta} = \frac{|\mathbf{v}|}{\sin \eta}.
\]

This tells us the magnitude of \( \Delta \mathbf{v} \), which is one of the two ingredients we need for calculating the magnitude of \( \mathbf{a} = \Delta \mathbf{v} / \Delta t \). The other ingredient is \( \Delta t \). The time required for the object to move through the angle \( \theta \) is

\[
\Delta t = \frac{\text{length of arc}}{|\mathbf{v}|}.
\]

Now if we measure our angles in radians we can use the definition of radian measure, which is \( \text{angle} \) = \( \frac{\text{length of arc}}{\text{radius}} \), giving \( \Delta t = \theta r / |\mathbf{v}| \). Combining this with the first expression involving \( |\Delta \mathbf{v}| \) gives

\[
|\mathbf{a}| = \frac{|\Delta \mathbf{v}|}{\Delta t} = \frac{|\mathbf{v}|^2}{r} \cdot \frac{\sin \theta}{\theta} \cdot \frac{1}{\sin \eta}.
\]

When \( \theta \) becomes very small, the small-angle approximation \( \sin \theta \approx \theta \) applies, and also \( \eta \) becomes close to \( 90^\circ \), so \( \sin \eta \approx 1 \), and we have