width and the length of the curve are greater by a factor of 48/36, so the area is greater by a factor of \((48/36)^2 = 1.78\).

Reasoning about ratios and proportionalities is one of the three essential mathematical skills, summarized on pp.532-533, that you need for success in this course.

▷ Solved problem: a telescope gathers light page 59, problem 11
▷ Solved problem: distance from an earthquake page 59, problem 12

Discussion questions

A A toy fire engine is 1/30 the size of the real one, but is constructed from the same metal with the same proportions. How many times smaller is its weight? How many times less red paint would be needed to paint it?

B Galileo spends a lot of time in his dialog discussing what really happens when things break. He discusses everything in terms of Aristotle’s now-discredited explanation that things are hard to break, because if something breaks, there has to be a gap between the two halves with nothing in between, at least initially. Nature, according to Aristotle, “abhors a vacuum,” i.e., nature doesn’t “like” empty space to exist. Of course, air will rush into the gap immediately, but at the very moment of breaking, Aristotle imagined a vacuum in the gap. Is Aristotle’s explanation of why it is hard to break things an experimentally testable statement? If so, how could it be tested experimentally?

1.3 ★ Scaling applied to biology

Organisms of different sizes with the same shape

The left-hand panel in figure o shows the approximate validity of the proportionality \(m \propto L^3\) for cockroaches (redrawn from McMahon and Bonner). The scatter of the points around the curve indicates that some cockroaches are proportioned slightly differently from others, but in general the data seem well described by \(m \propto L^3\). That means that the largest cockroaches the experimenter could raise (is there a 4-H prize?) had roughly the same shape as the smallest ones.

Another relationship that should exist for animals of different sizes shaped in the same way is that between surface area and body mass. If all the animals have the same average density, then body mass should be proportional to the cube of the animal’s linear size, \(m \propto L^3\), while surface area should vary proportionately to \(L^2\). Therefore, the animals’ surface areas should be proportional to \(m^{2/3}\). As shown in the right-hand panel of figure o, this relationship appears to hold quite well for the dwarf siren, a type of salamander. Notice how the curve bends over, meaning that the surface area does not increase as quickly as body mass, e.g., a salamander with eight
times more body mass will have only four times more surface area.

This behavior of the ratio of surface area to mass (or, equivalently, the ratio of surface area to volume) has important consequences for mammals, which must maintain a constant body temperature. It would make sense for the rate of heat loss through the animal’s skin to be proportional to its surface area, so we should expect small animals, having large ratios of surface area to volume, to need to produce a great deal of heat in comparison to their size to avoid dying from low body temperature. This expectation is borne out by the data of the left-hand panel of figure p, showing the rate of oxygen consumption of guinea pigs as a function of their body mass. Neither an animal’s heat production nor its surface area is convenient to measure, but in order to produce heat, the animal must metabolize oxygen, so oxygen consumption is a good indicator of the rate of heat production. Since surface area is proportional to $m^{2/3}$, the proportionality of the rate of oxygen consumption to $m^{2/3}$ is consistent with the idea that the animal needs to produce heat at a rate in proportion to its surface area. Although the smaller animals...
metabolize less oxygen and produce less heat in absolute terms, the amount of food and oxygen they must consume is greater in proportion to their own mass. The Etruscan pigmy shrew, weighing in at 2 grams as an adult, is at about the lower size limit for mammals. It must eat continually, consuming many times its body weight each day to survive.

Changes in shape to accommodate changes in size

Large mammals, such as elephants, have a small ratio of surface area to volume, and have problems getting rid of their heat fast enough. An elephant cannot simply eat small enough amounts to keep from producing excessive heat, because cells need to have a certain minimum metabolic rate to run their internal machinery. Hence the elephant’s large ears, which add to its surface area and help it to cool itself. Previously, we have seen several examples of data within a given species that were consistent with a fixed shape, scaled up and down in the cases of individual specimens. The elephant’s ears are an example of a change in shape necessitated by a change in scale.
Large animals also must be able to support their own weight. Returning to the example of the strengths of planks of different sizes, we can see that if the strength of the plank depends on area while its weight depends on volume, then the ratio of strength to weight goes as follows:

\[
\frac{\text{strength}}{\text{weight}} \propto \frac{A}{V} \propto \frac{1}{L^2}.
\]

Thus, the ability of objects to support their own weights decreases inversely in proportion to their linear dimensions. If an object is to be just barely able to support its own weight, then a larger version will have to be proportioned differently, with a different shape.

Since the data on the cockroaches seemed to be consistent with roughly similar shapes within the species, it appears that the ability to support its own weight was not the tightest design constraint that Nature was working under when she designed them. For large animals, structural strength is important. Galileo was the first to quantify this reasoning and to explain why, for instance, a large animal must have bones that are thicker in proportion to their length. Consider a roughly cylindrical bone such as a leg bone or a vertebra. The length of the bone, \( L \), is dictated by the overall linear size of the animal, since the animal’s skeleton must reach the animal’s whole length. We expect the animal’s mass to scale as \( L^3 \), so the strength of the bone must also scale as \( L^3 \). Strength is proportional to cross-sectional area, as with the wooden planks, so if the diameter of the bone is \( d \), then

\[
d^2 \propto L^3
\]

or

\[
d \propto L^{3/2}.
\]

If the shape stayed the same regardless of size, then all linear dimensions, including \( d \) and \( L \), would be proportional to one another. If our reasoning holds, then the fact that \( d \) is proportional to \( L^{3/2} \), not \( L \), implies a change in proportions of the bone. As shown in the right-hand panel of figure p, the vertebrae of African Bovidae follow the rule \( d \propto L^{3/2} \) fairly well. The vertebrae of the giant eland are as chunky as a coffee mug, while those of a Gunther’s dik-dik are as slender as the cap of a pen.

**Discussion questions**

**A** Single-celled animals must passively absorb nutrients and oxygen from their surroundings, unlike humans who have lungs to pump air in and out and a heart to distribute the oxygenated blood throughout their bodies. Even the cells composing the bodies of multicellular animals must absorb oxygen from a nearby capillary through their surfaces. Based on these facts, explain why cells are always microscopic in size.

**B** The reasoning of the previous question would seem to be contradicted by the fact that human nerve cells in the spinal cord can be as much as a meter long, although their widths are still very small. Why is this possible?
1.4 Order-of-magnitude estimates

It is the mark of an instructed mind to rest satisfied with the degree of precision that the nature of the subject permits and not to seek an exactness where only an approximation of the truth is possible.

* Aristotle

It is a common misconception that science must be exact. For instance, in the Star Trek TV series, it would often happen that Captain Kirk would ask Mr. Spock, “Spock, we’re in a pretty bad situation. What do you think are our chances of getting out of here?” The scientific Mr. Spock would answer with something like, “Captain, I estimate the odds as 237.345 to one.” In reality, he could not have estimated the odds with six significant figures of accuracy, but nevertheless one of the hallmarks of a person with a good education in science is the ability to make estimates that are likely to be at least somewhere in the right ballpark. In many such situations, it is often only necessary to get an answer that is off by no more than a factor of ten in either direction. Since things that differ by a factor of ten are said to differ by one order of magnitude, such an estimate is called an order-of-magnitude estimate. The tilde, \( \sim \), is used to indicate that things are only of the same order of magnitude, but not exactly equal, as in

\[
\text{odds of survival} \sim 100 \text{ to one.}
\]

The tilde can also be used in front of an individual number to emphasize that the number is only of the right order of magnitude.

Although making order-of-magnitude estimates seems simple and natural to experienced scientists, it’s a mode of reasoning that is completely unfamiliar to most college students. Some of the typical mental steps can be illustrated in the following example.

<table>
<thead>
<tr>
<th>Cost of transporting tomatoes (incorrect solution) example 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roughly what percentage of the price of a tomato comes from the cost of transporting it in a truck?</td>
</tr>
<tr>
<td>The following incorrect solution illustrates one of the main ways you can go wrong in order-of-magnitude estimates.</td>
</tr>
</tbody>
</table>

Incorrect solution: Let’s say the trucker needs to make a $400 profit on the trip. Taking into account her benefits, the cost of gas, and maintenance and payments on the truck, let’s say the total cost is more like $2000. I’d guess about 5000 tomatoes would fit in the back of the truck, so the extra cost per tomato is 40 cents. That means the cost of transporting one tomato is comparable to the cost of the tomato itself. Transportation really adds a lot to the cost of produce, I guess.

The problem is that the human brain is not very good at estimating area or volume, so it turns out the estimate of 5000 tomatoes
Can you guess how many jelly beans are in the jar? If you try to guess directly, you will almost certainly underestimate. The right way to do it is to estimate the linear dimensions, then get the volume indirectly. See problem 26, p. 62.

Consider a spherical cow. Fitting in the truck is way off. That’s why people have a hard time at those contests where you are supposed to estimate the number of jellybeans in a big jar. Another example is that most people think their families use about 10 gallons of water per day, but in reality the average is about 300 gallons per day. When estimating area or volume, you are much better off estimating linear dimensions, and computing volume from the linear dimensions. Here’s a better solution to the problem about the tomato truck:

Cost of transporting tomatoes (correct solution) example 5
As in the previous solution, say the cost of the trip is $2000. The dimensions of the bin are probably $4\,\text{m} \times 2\,\text{m} \times 1\,\text{m}$, for a volume of $8\,\text{m}^3$. Since the whole thing is just an order-of-magnitude estimate, let’s round that off to the nearest power of ten, $10\,\text{m}^3$. The shape of a tomato is complicated, and I don’t know any formula for the volume of a tomato shape, but since this is just an estimate, let’s pretend that a tomato is a cube, $0.05\,\text{m} \times 0.05\,\text{m} \times 0.05\,\text{m}$, for a volume of $1.25 \times 10^{-4}\,\text{m}^3$. Since this is just a rough estimate, let’s round that to $10^{-4}\text{m}^3$. We can find the total number of tomatoes by dividing the volume of the bin by the volume of one tomato: $10\,\text{m}^3/10^{-4}\,\text{m}^3 = 10^5$ tomatoes. The transportation cost per tomato is $2000/10^5$ tomatoes = $0.02$/tomato. That means that transportation really doesn’t contribute very much to the cost of a tomato.

Approximating the shape of a tomato as a cube is an example of another general strategy for making order-of-magnitude estimates. A similar situation would occur if you were trying to estimate how many $\text{m}^2$ of leather could be produced from a herd of ten thousand cattle. There is no point in trying to take into account the shape of the cows’ bodies. A reasonable plan of attack might be to consider a spherical cow. Probably a cow has roughly the same surface area as a sphere with a radius of about 1 m, which would be $4\pi(1\,\text{m})^2$. Using the well-known facts that $\pi$ equals three, and four times three equals about ten, we can guess that a cow has a surface area of about $10\,\text{m}^2$, so the herd as a whole might yield $10^5\,\text{m}^2$ of leather.

Estimating mass indirectly example 6
Usually the best way to estimate mass is to estimate linear dimensions, then use those to infer volume, and then get the mass based on the volume. For example, *Amphicoelias*, shown in the figure, may have been the largest land animal ever to live. Fossils tell us the linear dimensions of an animal, but we can only indirectly guess its mass. Given the length scale in the figure, let’s estimate the mass of an *Amphicoelias*.

Its torso looks like it can be approximated by a rectangular box with dimensions $10\,\text{m} \times 5\,\text{m} \times 3\,\text{m}$, giving about $2 \times 10^2\,\text{m}^3$. Living things are mostly made of water, so we assume the animal to have the density of water, 1 g/cm$^3$, which converts to $10^3$ kg/m$^3$. 

56 Chapter 1 Scaling and estimation
This gives a mass of about $2 \times 10^5$ kg, or 200 metric tons.

The following list summarizes the strategies for getting a good order-of-magnitude estimate.

1. Don’t even attempt more than one significant figure of precision.

2. Don’t guess area, volume, or mass directly. Guess linear dimensions and get area, volume, or mass from them.

3. When dealing with areas or volumes of objects with complex shapes, idealize them as if they were some simpler shape, a cube or a sphere, for example.

4. Check your final answer to see if it is reasonable. If you estimate that a herd of ten thousand cattle would yield 0.01 m$^2$ of leather, then you have probably made a mistake with conversion factors somewhere.
Summary

Notation
\( \propto \) . . . . . . . . . is proportional to
\( \sim \) . . . . . . . . . on the order of, is on the order of

Summary

Nature behaves differently on large and small scales. Galileo showed that this results fundamentally from the way area and volume scale. Area scales as the second power of length, \( A \propto L^2 \), while volume scales as length to the third power, \( V \propto L^3 \).

An order of magnitude estimate is one in which we do not attempt or expect an exact answer. The main reason why the uninitiated have trouble with order-of-magnitude estimates is that the human brain does not intuitively make accurate estimates of area and volume. Estimates of area and volume should be approached by first estimating linear dimensions, which one’s brain has a feel for.
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 How many cubic inches are there in a cubic foot? The answer is not 12.

2 Assume a dog’s brain is twice as great in diameter as a cat’s, but each animal’s brain cells are the same size and their brains are the same shape. In addition to being a far better companion and much nicer to come home to, how many times more brain cells does a dog have than a cat? The answer is not 2.

3 The population density of Los Angeles is about 4000 people/km². That of San Francisco is about 6000 people/km². How many times farther away is the average person’s nearest neighbor in LA than in San Francisco? The answer is not 1.5.

4 A hunting dog’s nose has about 10 square inches of active surface. How is this possible, since the dog’s nose is only about 1 in × 1 in × 1 in = 1 in³? After all, 10 is greater than 1, so how can it fit?

5 Estimate the number of blades of grass on a football field.

6 In a computer memory chip, each bit of information (a 0 or a 1) is stored in a single tiny circuit etched onto the surface of a silicon chip. The circuits cover the surface of the chip like lots in a housing development. A typical chip stores 64 Mb (megabytes) of data, where a byte is 8 bits. Estimate (a) the area of each circuit, and (b) its linear size.

7 Suppose someone built a gigantic apartment building, measuring 10 km × 10 km at the base. Estimate how tall the building would have to be to have space in it for the entire world’s population to live.

8 A hamburger chain advertises that it has sold 10 billion Bongo Burgers. Estimate the total mass of feed required to raise the cows used to make the burgers.

9 Estimate the volume of a human body, in cm³.

10 How many cm² is 1 mm²?

11 Compare the light-gathering powers of a 3-cm-diameter telescope and a 30-cm telescope.

12 One step on the Richter scale corresponds to a factor of 100 in terms of the energy absorbed by something on the surface of the Earth, e.g., a house. For instance, a 9.3-magnitude quake would release 100 times more energy than an 8.3. The energy spreads out
from the epicenter as a wave, and for the sake of this problem we’ll assume we’re dealing with seismic waves that spread out in three dimensions, so that we can visualize them as hemispheres spreading out under the surface of the earth. If a certain 7.6-magnitude earthquake and a certain 5.6-magnitude earthquake produce the same amount of vibration where I live, compare the distances from my house to the two epicenters.

13 In Europe, a piece of paper of the standard size, called A4, is a little narrower and taller than its American counterpart. The ratio of the height to the width is the square root of 2, and this has some useful properties. For instance, if you cut an A4 sheet from left to right, you get two smaller sheets that have the same proportions. You can even buy sheets of this smaller size, and they’re called A5. There is a whole series of sizes related in this way, all with the same proportions. (a) Compare an A5 sheet to an A4 in terms of area and linear size. (b) The series of paper sizes starts from an A0 sheet, which has an area of one square meter. Suppose we had a series of boxes defined in a similar way: the B0 box has a volume of one cubic meter, two B1 boxes fit exactly inside an B0 box, and so on. What would be the dimensions of a B0 box? √

14 Estimate the mass of one of the hairs in Albert Einstein’s moustache, in units of kg.

15 According to folklore, every time you take a breath, you are inhaling some of the atoms exhaled in Caesar’s last words. Is this true? If so, how many?

16 The Earth’s surface is about 70% water. Mars’s diameter is about half the Earth’s, but it has no surface water. Compare the land areas of the two planets.

17 The traditional Martini glass is shaped like a cone with the point at the bottom. Suppose you make a Martini by pouring vermouth into the glass to a depth of 3 cm, and then adding gin to bring the depth to 6 cm. What are the proportions of gin and vermouth?

18 The central portion of a CD is taken up by the hole and some surrounding clear plastic, and this area is unavailable for storing data. The radius of the central circle is about 35% of the outer radius of the data-storing area. What percentage of the CD’s area is therefore lost?

19 The one-liter cube in the photo has been marked off into smaller cubes, with linear dimensions one tenth those of the big one. What is the volume of each of the small cubes?

▷ Solution, p. 535
Estimate the number of man-hours required for building the Great Wall of China.

(a) Using the microscope photo in the figure, estimate the mass of a one cell of the \textit{E. coli} bacterium, which is one of the most common ones in the human intestine. Note the scale at the lower right corner, which is 1 \( \mu m \). Each of the tubular objects in the column is one cell. (b) The feces in the human intestine are mostly bacteria (some dead, some alive), of which \textit{E. coli} is a large and typical component. Estimate the number of bacteria in your intestines, and compare with the number of human cells in your body, which is believed to be roughly on the order of \( 10^{13} \). (c) Interpreting your result from part b, what does this tell you about the size of a typical human cell compared to the size of a typical bacterial cell?

A taxon (plural taxa) is a group of living things. For example, \textit{Homo sapiens} and \textit{Homo neanderthalensis} are both taxa — specifically, they are two different species within the genus \textit{Homo}. Surveys by botanists show that the number of plant taxa native to a given contiguous land area \( A \) is usually approximately proportional to \( A^{1/3} \). (a) There are 70 different species of lupine native to Southern California, which has an area of about 200,000 km\(^2\). The San Gabriel Mountains cover about 1,600 km\(^2\). Suppose that you wanted to learn to identify all the species of lupine in the San Gabriels. Approximately how many species would you have to familiarize yourself with? (b) What is the interpretation of the fact that the exponent, \( 1/3 \), is less than one?
24 X-ray images aren’t only used with human subjects but also, for example, on insects and flowers. In 2003, a team of researchers at Argonne National Laboratory used x-ray imagery to find for the first time that insects, although they do not have lungs, do not necessarily breathe completely passively, as had been believed previously; many insects rapidly compress and expand their trachea, head, and thorax in order to force air in and out of their bodies. One difference between x-raying a human and an insect is that if a medical x-ray machine was used on an insect, virtually 100% of the x-rays would pass through its body, and there would be no contrast in the image produced. Less penetrating x-rays of lower energies have to be used. For comparison, a typical human body mass is about 70 kg, whereas a typical ant is about 10 mg. Estimate the ratio of the thicknesses of tissue that must be penetrated by x-rays in one case compared to the other.

25 Radio was first commercialized around 1920, and ever since then, radio signals from our planet have been spreading out across our galaxy. It is possible that alien civilizations could detect these signals and learn that there is life on earth. In the 90 years that the signals have been spreading at the speed of light, they have created a sphere with a radius of 90 light-years. To show an idea of the size of this sphere, I’ve indicated it in the figure as a tiny white circle on an image of a spiral galaxy seen edge on. (We don’t have similar photos of our own Milky Way galaxy, because we can’t see it from the outside.) So far we haven’t received answering signals from aliens within this sphere, but as time goes on, the sphere will expand as suggested by the dashed outline, reaching more and more stars that might harbor extraterrestrial life. Approximately what year will it be when the sphere has expanded to fill a volume 100 times greater than the volume it fills today in 2010?

26 Estimate the number of jellybeans in figure r on p. 56.  

▷ Solution, p. 535

27 At the grocery store you will see oranges packed neatly in stacks. Suppose we want to pack spheres as densely as possible, so that the greatest possible fraction of the space is filled by the spheres themselves, not by empty space. Let’s call this fraction $f$. Mathematicians have proved that the best possible result is $f \approx 0.7405$, which requires a systematic pattern of stacking. If you buy ball bearings or golf balls, however, the seller is probably not going to go to the trouble of stacking them neatly. Instead they will probably pour the balls into a box and vibrate the box vigorously for a while to make them settle. This results in a random packing. The closest random packing has $f \approx 0.64$. Suppose that golf balls, with a standard diameter of 4.27 cm, are sold in bulk with the closest random packing. What is the diameter of the largest ball that could be sold in boxes of the same size, packed systematically, so that there would be the same number of balls per box?
Plutonium-239 is one of a small number of important long-lived forms of high-level radioactive nuclear waste. The world’s waste stockpiles have about $10^3$ metric tons of plutonium. Drinking water is considered safe by U.S. government standards if it contains less than $2 \times 10^{-13}$ g/cm$^3$ of plutonium. The amount of radioactivity to which you were exposed by drinking such water on a daily basis would be very small compared to the natural background radiation that you are exposed to every year. Suppose that the world’s inventory of plutonium-239 were ground up into an extremely fine dust and then dispersed over the world’s oceans, thereby becoming mixed uniformly into the world’s water supplies over time. Estimate the resulting concentration of plutonium, and compare with the government standard.
Exercise 1: Scaling applied to leaves

Equipment:

leaves of three sizes, having roughly similar proportions of length, width, and thickness
balance

Each group will have one leaf, and should measure its surface area and volume, and determine its surface-to-volume ratio. For consistency, every group should use units of cm$^2$ and cm$^3$, and should only find the area of one side of the leaf. The area can be found by tracing the area of the leaf on graph paper and counting squares. The volume can be found by weighing the leaf and assuming that its density is 1 g/cm$^3$ (the density of water). What implications do your results have for the plants’ abilities to survive in different environments?
Motion in one dimension
Chapter 2
Velocity and relative motion

2.1 Types of motion
If you had to think consciously in order to move your body, you would be severely disabled. Even walking, which we consider to be no great feat, requires an intricate series of motions that your cerebrum would be utterly incapable of coordinating. The task of putting one foot in front of the other is controlled by the more primitive parts of your brain, the ones that have not changed much since the mammals and reptiles went their separate evolutionary ways. The thinking part of your brain limits itself to general directives such as “walk faster,” or “don’t step on her toes,” rather than micromanaging every contraction and relaxation of the hundred or so muscles of your hips, legs, and feet.

Physics is all about the conscious understanding of motion, but we’re obviously not immediately prepared to understand the most complicated types of motion. Instead, we’ll use the divide-and-conquer technique. We’ll first classify the various types of motion, and then begin our campaign with an attack on the simplest cases. To make it clear what we are and are not ready to consider, we need to examine and define carefully what types of motion can exist.

Rigid-body motion distinguished from motion that changes an object’s shape
Nobody, with the possible exception of Fred Astaire, can simply glide forward without bending their joints. Walking is thus an example in which there is both a general motion of the whole object and a change in the shape of the object. Another example is the motion of a jiggling water balloon as it flies through the air. We are not presently attempting a mathematical description of the way in which the shape of an object changes. Motion without a change in shape is called rigid-body motion. (The word “body” is often used in physics as a synonym for “object.”)

Center-of-mass motion as opposed to rotation
A ballerina leaps into the air and spins around once before landing. We feel intuitively that her rigid-body motion while her feet are off the ground consists of two kinds of motion going on simul-
No matter what point you hang the pear from, the string lines up with the pear’s center of mass. The center of mass can therefore be defined as the intersection of all the lines made by hanging the pear in this way. Note that the X in the figure should not be interpreted as implying that the center of mass is on the surface — it is actually inside the pear.

The circus performers hang with the ropes passing through their centers of mass.

The leaping dancer’s motion is complicated, but the motion of her center of mass is simple.

It turns out that there is one particularly natural and useful way to make a clear definition, but it requires a brief digression. Every object has a balance point, referred to in physics as the center of mass. For a two-dimensional object such as a cardboard cutout, the center of mass is the point at which you could hang the object from a string and make it balance. In the case of the ballerina (who is likely to be three-dimensional unless her diet is particularly severe), it might be a point either inside or outside her body, depending on how she holds her arms. Even if it is not practical to attach a string to the balance point itself, the center of mass can be defined as shown in figure e.

Why is the center of mass concept relevant to the question of classifying rotational motion as opposed to motion through space? As illustrated in figures d and g, it turns out that the motion of an object’s center of mass is nearly always far simpler than the motion of any other part of the object. The ballerina’s body is a large object with a complex shape. We might expect that her motion would be...
An improperly balanced wheel has a center of mass that is not at its geometric center. When you get a new tire, the mechanic clamps little weights to the rim to balance the wheel.

This toy was intentionally designed so that the mushroom-shaped piece of metal on top would throw off the center of mass. When you wind it up, the mushroom spins, but the center of mass doesn’t want to move, so the rest of the toy tends to counter the mushroom’s motion, causing the whole thing to jump around.

Ballerinas and professional basketball players can create an illusion of flying horizontally through the air because our brains intuitively expect them to have rigid-body motion, but the body does not stay rigid while executing a grand jete or a slam dunk. The legs much more complicated than the motion of a small, simply-shaped object, say a marble, thrown up at the same angle as the angle at which she leapt. But it turns out that the motion of the ballerina’s center of mass is exactly the same as the motion of the marble. That is, the motion of the center of mass is the same as the motion the ballerina would have if all her mass was concentrated at a point. By restricting our attention to the motion of the center of mass, we can therefore simplify things greatly.

![The same leaping dancer, viewed from above. Her center of mass traces a straight line, but a point away from her center of mass, such as her elbow, traces the much more complicated path shown by the dots.](image)

We can now replace the ambiguous idea of “motion as a whole through space” with the more useful and better defined concept of “center-of-mass motion.” The motion of any rigid body can be cleanly split into rotation and center-of-mass motion. By this definition, the tipping chair does have both rotational and center-of-mass motion. Concentrating on the center of mass motion allows us to make a simplified model of the motion, as if a complicated object like a human body was just a marble or a point-like particle. Science really never deals with reality; it deals with models of reality.

Note that the word “center” in “center of mass” is not meant to imply that the center of mass must lie at the geometrical center of an object. A car wheel that has not been balanced properly has a center of mass that does not coincide with its geometrical center. An object such as the human body does not even have an obvious geometrical center.

It can be helpful to think of the center of mass as the average location of all the mass in the object. With this interpretation, we can see for example that raising your arms above your head raises your center of mass, since the higher position of the arms’ mass raises the average. We won’t be concerned right now with calculating centers of mass mathematically; the relevant equations are in ch. 14.
A fixed point on the dancer’s body follows a trajectory that is flatter than what we expect, creating an illusion of flight.

are low at the beginning and end of the jump, but come up higher at the middle. Regardless of what the limbs do, the center of mass will follow the same arc, but the low position of the legs at the beginning and end means that the torso is higher compared to the center of mass, while in the middle of the jump it is lower compared to the center of mass. Our eye follows the motion of the torso and tries to interpret it as the center-of-mass motion of a rigid body. But since the torso follows a path that is flatter than we expect, this attempted interpretation fails, and we experience an illusion that the person is flying horizontally.

Example 1.

The center of mass as an average

Explain how we know that the center of mass of each object is at the location shown in figure k.

The center of mass is a sort of average, so the height of the centers of mass in 1 and 2 has to be midway between the two squares, because that height is the average of the heights of the two squares. Example 3 is a combination of examples 1 and 2, so we can find its center of mass by averaging the horizontal positions of their centers of mass. In example 4, each square has been skewed a little, but just as much mass has been moved up as down, so the average vertical position of the mass hasn’t changed. Example 5 is clearly not all that different from example
4, the main difference being a slight clockwise rotation, so just as in example 4, the center of mass must be hanging in empty space, where there isn’t actually any mass. Horizontally, the center of mass must be between the heels and toes, or else it wouldn’t be possible to stand without tipping over.

Another interesting example from the sports world is the high jump, in which the jumper’s curved body passes over the bar, but the center of mass passes under the bar! Here the jumper lowers his legs and upper body at the peak of the jump in order to bring his waist higher compared to the center of mass.

Later in this course, we’ll find that there are more fundamental reasons (based on Newton’s laws of motion) why the center of mass behaves in such a simple way compared to the other parts of an object. We’re also postponing any discussion of numerical methods for finding an object’s center of mass. Until later in the course, we will only deal with the motion of objects’ centers of mass.

**Center-of-mass motion in one dimension**

In addition to restricting our study of motion to center-of-mass motion, we will begin by considering only cases in which the center of mass moves along a straight line. This will include cases such as objects falling straight down, or a car that speeds up and slows down but does not turn.

Note that even though we are not explicitly studying the more complex aspects of motion, we can still analyze the center-of-mass motion while ignoring other types of motion that might be occurring simultaneously. For instance, if a cat is falling out of a tree and is initially upside-down, it goes through a series of contortions that bring its feet under it. This is definitely not an example of rigid-body motion, but we can still analyze the motion of the cat’s center of mass just as we would for a dropping rock.

**self-check A**

Consider a person running, a person pedaling on a bicycle, a person coasting on a bicycle, and a person coasting on ice skates. In which cases is the center-of-mass motion one-dimensional? Which cases are examples of rigid-body motion?

▷ Answer, p. 549

**self-check B**

The figure shows a gymnast holding onto the inside of a big wheel. From inside the wheel, how could he make it roll one way or the other?

▷ Answer, p. 549

### 2.2 Describing distance and time

Center-of-mass motion in one dimension is particularly easy to deal with because all the information about it can be encapsulated in two variables: \( x \), the position of the center of mass relative to the origin,
and \( t \), which measures a point in time. For instance, if someone supplied you with a sufficiently detailed table of \( x \) and \( t \) values, you would know pretty much all there was to know about the motion of the object’s center of mass.

**A point in time as opposed to duration**

In ordinary speech, we use the word “time” in two different senses, which are to be distinguished in physics. It can be used, as in “a short time” or “our time here on earth,” to mean a length or duration of time, or it can be used to indicate a clock reading, as in “I didn’t know what time it was,” or “now’s the time.” In symbols, \( t \) is ordinarily used to mean a point in time, while \( \Delta t \) signifies an interval or duration in time. The capital Greek letter delta, \( \Delta \), means “the change in...,” i.e. a duration in time is the change or difference between one clock reading and another. The notation \( \Delta t \) does not signify the product of two numbers, \( \Delta \) and \( t \), but rather one single number, \( \Delta t \). If a matinee begins at a point in time \( t = 1 \) o’clock and ends at \( t = 3 \) o’clock, the duration of the movie was the change in \( t \),

\[
\Delta t = 3 \text{ hours} - 1 \text{ hour} = 2 \text{ hours}.
\]

To avoid the use of negative numbers for \( \Delta t \), we write the clock reading “after” to the left of the minus sign, and the clock reading “before” to the right of the minus sign. A more specific definition of the delta notation is therefore that delta stands for “after minus before.”

Even though our definition of the delta notation guarantees that \( \Delta t \) is positive, there is no reason why \( t \) can’t be negative. If \( t \) could not be negative, what would have happened one second before \( t = 0 \)? That doesn’t mean that time “goes backward” in the sense that adults can shrink into infants and retreat into the womb. It just means that we have to pick a reference point and call it \( t = 0 \), and then times before that are represented by negative values of \( t \). An example is that a year like 2007 A.D. can be thought of as a positive \( t \) value, while one like 370 B.C. is negative. Similarly, when you hear a countdown for a rocket launch, the phrase “\( t \) minus ten seconds” is a way of saying \( t = -10 \) s, where \( t = 0 \) is the time of blastoff, and \( t > 0 \) refers to times after launch.

Although a point in time can be thought of as a clock reading, it is usually a good idea to avoid doing computations with expressions such as “2:35” that are combinations of hours and minutes. Times can instead be expressed entirely in terms of a single unit, such as hours. Fractions of an hour can be represented by decimals rather than minutes, and similarly if a problem is being worked in terms of minutes, decimals can be used instead of seconds.

**self-check C**

Of the following phrases, which refer to points in time, which refer to time intervals, and which refer to time in the abstract rather than as a
Position as opposed to change in position

As with time, a distinction should be made between a point in space, symbolized as a coordinate \( x \), and a change in position, symbolized as \( \Delta x \).

As with \( t \), \( x \) can be negative. If a train is moving down the tracks, not only do you have the freedom to choose any point along the tracks and call it \( x = 0 \), but it’s also up to you to decide which side of the \( x = 0 \) point is positive \( x \) and which side is negative \( x \).

Since we’ve defined the delta notation to mean “after minus before,” it is possible that \( \Delta x \) will be negative, unlike \( \Delta t \) which is guaranteed to be positive. Suppose we are describing the motion of a train on tracks linking Tucson and Chicago. As shown in the figure, it is entirely up to you to decide which way is positive.

Note that in addition to \( x \) and \( \Delta x \), there is a third quantity we could define, which would be like an odometer reading, or actual distance traveled. If you drive 10 miles, make a U-turn, and drive back 10 miles, then your \( \Delta x \) is zero, but your car’s odometer reading has increased by 20 miles. However important the odometer reading is to car owners and used car dealers, it is not very important in physics, and there is not even a standard name or notation for it. The change in position, \( \Delta x \), is more useful because it is so much easier to calculate: to compute \( \Delta x \), we only need to know the beginning and ending positions of the object, not all the information about how it got from one position to the other.

self-check D

A ball falls vertically, hits the floor, bounces to a height of one meter, falls, and hits the floor again. Is the \( \Delta x \) between the two impacts equal to zero, one, or two meters?
Frames of reference

The example above shows that there are two arbitrary choices you have to make in order to define a position variable, \( x \). You have to decide where to put \( x = 0 \), and also which direction will be positive. This is referred to as choosing a coordinate system or choosing a frame of reference. (The two terms are nearly synonymous, but the first focuses more on the actual \( x \) variable, while the second is more of a general way of referring to one’s point of view.) As long as you are consistent, any frame is equally valid. You just don’t want to change coordinate systems in the middle of a calculation.

Have you ever been sitting in a train in a station when suddenly you notice that the station is moving backward? Most people would describe the situation by saying that you just failed to notice that the train was moving — it only seemed like the station was moving. But this shows that there is yet a third arbitrary choice that goes into choosing a coordinate system: valid frames of reference can differ from each other by moving relative to one another. It might seem strange that anyone would bother with a coordinate system that was moving relative to the earth, but for instance the frame of reference moving along with a train might be far more convenient for describing things happening inside the train.

2.3 Graphs of motion; velocity

Motion with constant velocity

In example o, an object is moving at constant speed in one direction. We can tell this because every two seconds, its position changes by five meters.

In algebra notation, we’d say that the graph of \( x \) vs. \( t \) shows the same change in position, \( \Delta x = 5.0 \text{ m} \), over each interval of \( \Delta t = 2.0 \text{ s} \). The object’s velocity or speed is obtained by calculating \( v = \Delta x/\Delta t = (5.0 \text{ m})/(2.0 \text{ s}) = 2.5 \text{ m/s} \). In graphical terms, the velocity can be interpreted as the slope of the line. Since the graph is a straight line, it wouldn’t have mattered if we’d taken a longer time interval and calculated \( v = \Delta x/\Delta t = (10.0 \text{ m})/(4.0 \text{ s}) \). The answer would still have been the same, 2.5 m/s.

Note that when we divide a number that has units of meters by another number that has units of seconds, we get units of meters per second, which can be written m/s. This is another case where we treat units as if they were algebra symbols, even though they’re not.

In example p, the object is moving in the opposite direction: as time progresses, its \( x \) coordinate decreases. Recalling the definition of the \( \Delta \) notation as “after minus before,” we find that \( \Delta t \) is still positive, but \( \Delta x \) must be negative. The slope of the line is therefore...
negative, and we say that the object has a negative velocity, \( v = \Delta x/\Delta t = (-5.0 \text{ m})/(2.0 \text{ s}) = -2.5 \text{ m/s} \). We’ve already seen that the plus and minus signs of \( \Delta x \) values have the interpretation of telling us which direction the object moved. Since \( \Delta t \) is always positive, dividing by \( \Delta t \) doesn’t change the plus or minus sign, and the plus and minus signs of velocities are to be interpreted in the same way. In graphical terms, a positive slope characterizes a line that goes up as we go to the right, and a negative slope tells us that the line went down as we went to the right.

Solved problem: light-years page 89, problem 4

Motion with changing velocity

Now what about a graph like figure q? This might be a graph of a car’s motion as the driver cruises down the freeway, then slows down to look at a car crash by the side of the road, and then speeds up again, disappointed that there is nothing dramatic going on such as flames or babies trapped in their car seats. (Note that we are still talking about one-dimensional motion. Just because the graph is curvy doesn’t mean that the car’s path is curvy. The graph is not like a map, and the horizontal direction of the graph represents the passing of time, not distance.)

Example q is similar to example o in that the object moves a total of 25.0 m in a period of 10.0 s, but it is no longer true that it makes the same amount of progress every second. There is no way to characterize the entire graph by a certain velocity or slope, because the velocity is different at every moment. It would be incorrect to say that because the car covered 25.0 m in 10.0 s, its velocity was 2.5 m/s. It moved faster than that at the beginning and end, but slower in the middle. There may have been certain instants at which the car was indeed going 2.5 m/s, but the speedometer swept past that value without “sticking,” just as it swung through various other values of speed. (I definitely want my next car to have a speedometer calibrated in m/s and showing both negative and positive values.)

We assume that our speedometer tells us what is happening to the speed of our car at every instant, but how can we define speed mathematically in a case like this? We can’t just define it as the slope of the curvy graph, because a curve doesn’t have a single well-defined slope as does a line. A mathematical definition that corresponded to the speedometer reading would have to be one that assigned a velocity value to a single point on the curve, i.e., a single instant in time, rather than to the entire graph. If we wish to define the speed at one instant such as the one marked with a dot, the best way to proceed is illustrated in r, where we have drawn the line through that point called the tangent line, the line that “hugs the curve.” We can then adopt the following definition of velocity:
The velocity of an object at any given moment is the slope of the tangent line through the relevant point on its $x - t$ graph.

One interpretation of this definition is that the velocity tells us how many meters the object would have traveled in one second, if it had continued moving at the same speed for at least one second.

A good way of thinking about the tangent-line definition is shown in figure s. We zoom in on our point of interest more and more, as if through a microscope capable of unlimited magnification. As we zoom in, the curviness of the graph becomes less and less apparent. (Similarly, we don’t notice in everyday life that the earth is a sphere.) In the figure, we zoom in by 400%, and then again by 400%, and so on. After a series of these zooms, the graph appears indistinguishable from a line, and we can measure its slope just as we would for a line.

If all we saw was the ultra-magnified view, we would assume that the object was moving at a constant speed, which is 2.5 m/s in our example, and that it would continue to move at that speed. Therefore the speed of 2.5 m/s can be interpreted as meaning that if the object had continued at constant speed for a further time interval of 1 s, it would have traveled 2.5 m.

What is the velocity at the point shown with a dot on the graph?

First we draw the tangent line through that point. To find the slope of the tangent line, we need to pick two points on it. Theoretically, the slope should come out the same regardless of which two points we pick, but in practical terms we’ll be able to measure more accurately if we pick two points fairly far apart, such as the two white diamonds. To save work, we pick points that are directly above labeled points on the $t$ axis, so that $\Delta t = 4.0$ s is easy to read off. One diamond lines up with $x \approx 17.5$ m, the other with $x \approx 26.5$ m, so $\Delta x = 9.0$ m. The velocity is $\Delta x / \Delta t = 2.2$ m/s.
Looking at the tangent line in figure t, we can see that it touches the curve at the point marked with a dot, but without cutting through it at that point. No other line through that point has this “no-cut” property; if we rotated the line either clockwise or counterclockwise about the point, it would cut through. Except in certain unusual cases, there is always exactly one such no-cut line at any given point on a smooth curve, and that no-cut line is the tangent line. It’s as though the region below the curve were a solid block of wood, and the tangent line were the edge of a ruler. The ruler can’t penetrate the block.

**Conventions about graphing**

The placement of $t$ on the horizontal axis and $x$ on the upright axis may seem like an arbitrary convention, or may even have disturbed you, since your algebra teacher always told you that $x$ goes on the horizontal axis and $y$ goes on the upright axis. There is a reason for doing it this way, however. In example t, we have an object that reverses its direction of motion twice. It can only be in one place at any given time, but there can be more than one time when it is at a given place. For instance, this object passed through $x = 17$ m on three separate occasions, but there is no way it could have been in more than one place at $t = 5.0$ s. Resurrecting some terminology you learned in your trigonometry course, we say that $x$ is a function of $t$, but $t$ is not a function of $x$. In situations such as this, there is a useful convention that the graph should be oriented so that any vertical line passes through the curve at only one point. Putting the $x$ axis across the page and $t$ upright would have violated this convention. To people who are used to interpreting graphs, a graph that violates this convention is as annoying as fingernails scratching on a chalkboard. We say that this is a graph of “$x$ versus $t$.” If the axes were the other way around, it would be a graph of “$t$ versus $x$.” I remember the “versus” terminology by visualizing the labels on the $x$ and $t$ axes and remembering that when you read, you go from left to right and from top to bottom.

**Discussion questions**

A Park is running slowly in gym class, but then he notices Jenna watching him, so he speeds up to try to impress her. Which of the graphs could represent his motion?
The figure shows a sequence of positions for two racing tractors. Compare the tractors’ velocities as the race progresses. When do they have the same velocity? [Based on a question by Lillian McDermott.]

If an object had an $x - t$ graph that was a straight line with $\Delta x = 0$ and $\Delta t \neq 0$, what would you say about its velocity? Sketch an example of such a graph. What about $\Delta t = 0$ and $\Delta x \neq 0$?

If an object has a wavy motion graph like the one in figure u on p. 78, what are the times at which the object reverses its direction? Describe the object’s velocity at these points.

Discuss anything unusual about the following three graphs.

I have been using the term “velocity” and avoiding the more common English word “speed,” because introductory physics texts typically define them to mean different things. They use the word “speed,” and the symbol “s” to mean the absolute value of the velocity, $s = |v|$. Although I’ve chosen not to emphasize this distinction in technical vocabulary, there are clearly two different concepts here. Can you think of an example of a graph of $x$-versus-$t$ in which the object has constant speed, but not constant velocity?

For the graph shown in the figure, describe how the object’s velocity changes.

Two physicists duck out of a boring scientific conference. On the street, they witness an accident in which a pedestrian is injured by a hit-and-run driver. A criminal trial results, and they must testify. In her testimony, Dr. Transverz Waive says, “The car was moving along pretty fast, I’d say the velocity was +40 mi/hr. They saw the old lady too late, and even though they slammed on the brakes they still hit her before they stopped. Then they made a $U$ turn and headed off at a velocity of about -20 mi/hr, I’d say.” Dr. Longitud N.L. Vibrasheun says, “He was really going too fast, maybe his velocity was -35 or -40 mi/hr. After he hit Mrs. Hapless, he turned around and left at a velocity of, oh, I’d guess maybe +20 or +25 mi/hr.” Is their testimony contradictory? Explain.
2.4 The principle of inertia

Physical effects relate only to a change in velocity

Consider two statements of a kind that was at one time made with the utmost seriousness:

People like Galileo and Copernicus who say the earth is rotating must be crazy. We know the earth can’t be moving. Why, if the earth was really turning once every day, then our whole city would have to be moving hundreds of leagues in an hour. That’s impossible! Buildings would shake on their foundations. Gale-force winds would knock us over. Trees would fall down. The Mediterranean would come sweeping across the east coasts of Spain and Italy. And furthermore, what force would be making the world turn?

All this talk of passenger trains moving at forty miles an hour is sheer hogwash! At that speed, the air in a passenger compartment would all be forced against the back wall. People in the front of the car would suffocate, and people at the back would die because in such concentrated air, they wouldn’t be able to expel a breath.

Some of the effects predicted in the first quote are clearly just based on a lack of experience with rapid motion that is smooth and free of vibration. But there is a deeper principle involved. In each case, the speaker is assuming that the mere fact of motion must have dramatic physical effects. More subtly, they also believe that a force is needed to keep an object in motion: the first person thinks a force would be needed to maintain the earth’s rotation, and the second apparently thinks of the rear wall as pushing on the air to keep it moving.

Common modern knowledge and experience tell us that these people’s predictions must have somehow been based on incorrect reasoning, but it is not immediately obvious where the fundamental flaw lies. It’s one of those things a four-year-old could infuriate you by demanding a clear explanation of. One way of getting at the fundamental principle involved is to consider how the modern concept of the universe differs from the popular conception at the time of the Italian Renaissance. To us, the word “earth” implies a planet, one of the nine planets of our solar system, a small ball of rock and dirt that is of no significance to anyone in the universe except for members of our species, who happen to live on it. To Galileo’s contemporaries, however, the earth was the biggest, most solid, most important thing in all of creation, not to be compared with the wandering lights in the sky known as planets. To us, the earth is just another object, and when we talk loosely about “how fast” an object such as a car “is going,” we really mean the car-object’s velocity relative to the earth-object.
This Air Force doctor volunteered to ride a rocket sled as a medical experiment. The obvious effects on his head and face are not because of the sled’s speed but because of its rapid changes in speed: increasing in 2 and 3, and decreasing in 5 and 6. In 4 his speed is greatest, but because his speed is not increasing or decreasing very much at this moment, there is little effect on him.

Motion is relative

According to our modern world-view, it isn’t reasonable to expect that a special force should be required to make the air in the train have a certain velocity relative to our planet. After all, the “moving” air in the “moving” train might just happen to have zero velocity relative to some other planet we don’t even know about. Aristotle claimed that things “naturally” wanted to be at rest, lying on the surface of the earth. But experiment after experiment has shown that there is really nothing so special about being at rest relative to the earth. For instance, if a mattress falls out of the back of a truck on the freeway, the reason it rapidly comes to rest with respect to the planet is simply because of friction forces exerted by the asphalt, which happens to be attached to the planet.

Galileo’s insights are summarized as follows:

The principle of inertia
No force is required to maintain motion with constant velocity in a straight line, and absolute motion does not cause any observable physical effects.

There are many examples of situations that seem to disprove the principle of inertia, but these all result from forgetting that friction is a force. For instance, it seems that a force is needed to keep a sailboat in motion. If the wind stops, the sailboat stops too. But
the wind’s force is not the only force on the boat; there is also a frictional force from the water. If the sailboat is cruising and the wind suddenly disappears, the backward frictional force still exists, and since it is no longer being counteracted by the wind’s forward force, the boat stops. To disprove the principle of inertia, we would have to find an example where a moving object slowed down even though no forces whatsoever were acting on it. Over the years since Galileo’s lifetime, physicists have done more and more precise experiments to search for such a counterexample, but the results have always been negative. Three such tests are described on pp. 114, 243, and 271.

**self-check E**

What is incorrect about the following supposed counterexamples to the principle of inertia?

1. When astronauts blast off in a rocket, their huge velocity does cause a physical effect on their bodies — they get pressed back into their seats, the flesh on their faces gets distorted, and they have a hard time lifting their arms.

2. When you’re driving in a convertible with the top down, the wind in your face is an observable physical effect of your absolute motion.

**Answer, p. 549**

* Solved problem: a bug on a wheel page 89, problem 7

**Discussion questions**

A  A passenger on a cruise ship finds, while the ship is docked, that he can leap off of the upper deck and just barely make it into the pool on the lower deck. If the ship leaves dock and is cruising rapidly, will this adrenaline junkie still be able to make it?

B  You are a passenger in the open basket hanging under a helium balloon. The balloon is being carried along by the wind at a constant velocity. If you are holding a flag in your hand, will the flag wave? If so, which way? [Based on a question from PSSC Physics.]

C  Aristotle stated that all objects naturally wanted to come to rest, with the unspoken implication that “rest” would be interpreted relative to the surface of the earth. Suppose we go back in time and transport Aristotle to the moon. Aristotle knew, as we do, that the moon circles the earth; he said it didn’t fall down because, like everything else in the heavens, it was made out of some special substance whose “natural” behavior was to go in circles around the earth. We land, put him in a space suit, and kick him out the door. What would he expect his fate to be in this situation? If intelligent creatures inhabited the moon, and one of them independently came up with the equivalent of Aristotelian physics, what would they think about objects coming to rest?

D  The glass is sitting on a level table in a train’s dining car, but the surface of the water is tilted. What can you infer about the motion of the train?
2.5 Addition of velocities

Addition of velocities to describe relative motion

Since absolute motion cannot be unambiguously measured, the only way to describe motion unambiguously is to describe the motion of one object relative to another. Symbolically, we can write \( v_{PQ} \) for the velocity of object \( P \) relative to object \( Q \).

Velocities measured with respect to different reference points can be compared by addition. In the figure below, the ball’s velocity relative to the couch equals the ball’s velocity relative to the truck plus the truck’s velocity relative to the couch:

\[
v_{BC} = v_{BT} + v_{TC} = 5 \text{ cm/s} + 10 \text{ cm/s} = 15 \text{ cm/s}
\]

The same equation can be used for any combination of three objects, just by substituting the relevant subscripts for \( B \), \( T \), and \( C \). Just remember to write the equation so that the velocities being added have the same subscript twice in a row. In this example, if you read off the subscripts going from left to right, you get \( BC \ldots = \ldots BTTC \). The fact that the two “inside” subscripts on the right are the same means that the equation has been set up correctly. Notice how subscripts on the left look just like the subscripts on the right, but with the two \( T \)’s eliminated.

Negative velocities in relative motion

My discussion of how to interpret positive and negative signs of velocity may have left you wondering why we should bother. Why not just make velocity positive by definition? The original reason why negative numbers were invented was that bookkeepers decided it would be convenient to use the negative number concept for payments to distinguish them from receipts. It was just plain easier than writing receipts in black and payments in red ink. After adding up your month’s positive receipts and negative payments, you either got a positive number, indicating profit, or a negative number, showing a loss. You could then show that total with a high-tech “+” or “−” sign, instead of looking around for the appropriate bottle of ink.

Nowadays we use positive and negative numbers for all kinds of things, but in every case the point is that it makes sense to add and subtract those things according to the rules you learned in grade school, such as “minus a minus makes a plus, why this is true we need not discuss.” Adding velocities has the significance of comparing relative motion, and with this interpretation negative and positive velocities can be used within a consistent framework. For example, the truck’s velocity relative to the couch equals the
These two highly competent physicists disagree on absolute velocities, but they would agree on relative velocities. Purple Dino considers the couch to be at rest, while Green Dino thinks of the truck as being at rest. They agree, however, that the truck’s velocity relative to the couch is $v_{TC} = 10$ cm/s, the ball’s velocity relative to the truck is $v_{BT} = 5$ cm/s, and the ball’s velocity relative to the couch is $v_{BC} = v_{BT} + v_{TC} = 15$ cm/s.

The truck’s velocity relative to the ball plus the ball’s velocity relative to the couch:

$$v_{TC} = v_{TB} + v_{BC}$$

$$= -5 \text{ cm/s} + 15 \text{ cm/s}$$

$$= 10 \text{ cm/s}$$

If we didn’t have the technology of negative numbers, we would have had to remember a complicated set of rules for adding velocities: (1) if the two objects are both moving forward, you add, (2) if one is moving forward and one is moving backward, you subtract, but (3) if they’re both moving backward, you add. What a pain that would have been.

Solved problem: two dimensions page 90, problem 10

Airspeed example 3

On June 1, 2009, Air France flight 447 disappeared without warning over the Atlantic Ocean. All 232 people aboard were killed. Investigators believe the disaster was triggered because the pilots lost the ability to accurately determine their speed relative to the air. This is done using sensors called Pitot tubes, mounted outside the plane on the wing. Automated radio signals showed that these sensors gave conflicting readings before the crash, possibly because they iced up. For fuel efficiency, modern passenger
jets fly at a very high altitude, but in the thin air they can only fly within a very narrow range of speeds. If the speed is too low, the plane stalls, and if it’s too high, it breaks up. If the pilots can’t tell what their airspeed is, they can’t keep it in the safe range.

Many people’s reaction to this story is to wonder why planes don’t just use GPS to measure their speed. One reason is that GPS tells you your speed relative to the ground, not relative to the air. Letting P be the plane, A the air, and G the ground, we have

\[ v_{PG} = v_{PA} + v_{AG}, \]

where \( v_{PG} \) (the “true ground speed”) is what GPS would measure, \( v_{PA} \) (“airspeed”) is what’s critical for stable flight, and \( v_{AG} \) is the velocity of the wind relative to the ground 9000 meters below. Knowing \( v_{PG} \) isn’t enough to determine \( v_{PA} \) unless \( v_{AG} \) is also known.

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**Discussion questions**

A Interpret the general rule \( v_{AB} = -v_{BA} \) in words.

B Wa-Chuen slips away from her father at the mall and walks up the down escalator, so that she stays in one place. Write this in terms of symbols.

**2.6 Graphs of velocity versus time**

Since changes in velocity play such a prominent role in physics, we need a better way to look at changes in velocity than by laboriously drawing tangent lines on \( x \)-versus-\( t \) graphs. A good method is to draw a graph of velocity versus time. The examples on the left show the \( x \)- \( t \) and \( v \)- \( t \) graphs that might be produced by a car starting from a traffic light, speeding up, cruising for a while at constant speed, and finally slowing down for a stop sign. If you have an air freshener hanging from your rear-view mirror, then you will see an effect on the air freshener during the beginning and ending periods when the velocity is changing, but it will not be tilted during the period of constant velocity represented by the flat plateau in the middle of the \( v \)- \( t \) graph.
Students often mix up the things being represented on these two types of graphs. For instance, many students looking at the top graph say that the car is speeding up the whole time, since “the graph is becoming greater.” What is getting greater throughout the graph is $x$, not $v$.

Similarly, many students would look at the bottom graph and think it showed the car backing up, because “it’s going backwards at the end.” But what is decreasing at the end is $v$, not $x$. Having both the $x - t$ and $v - t$ graphs in front of you like this is often convenient, because one graph may be easier to interpret than the other for a particular purpose. Stacking them like this means that corresponding points on the two graphs’ time axes are lined up with each other vertically. However, one thing that is a little counter-intuitive about the arrangement is that in a situation like this one involving a car, one is tempted to visualize the landscape stretching along the horizontal axis of one of the graphs. The horizontal axes, however, represent time, not position. The correct way to visualize the landscape is by mentally rotating the horizon 90 degrees counterclockwise and imagining it stretching along the upright axis of the $x$-$t$ graph, which is the only axis that represents different positions in space.

2.7 ∫ Applications of calculus

The integral symbol, $\int$, in the heading for this section indicates that it is meant to be read by students in calculus-based physics. Students in an algebra-based physics course should skip these sections. The calculus-related sections in this book are meant to be usable by students who are taking calculus concurrently, so at this early point in the physics course I do not assume you know any calculus yet. This section is therefore not much more than a quick preview of calculus, to help you relate what you’re learning in the two courses.

Newton was the first person to figure out the tangent-line definition of velocity for cases where the $x - t$ graph is nonlinear. Before Newton, nobody had conceptualized the description of motion in terms of $x - t$ and $v - t$ graphs. In addition to the graphical techniques discussed in this chapter, Newton also invented a set of symbolic techniques called calculus. If you have an equation for $x$ in terms of $t$, calculus allows you, for instance, to find an equation for $v$ in terms of $t$. In calculus terms, we say that the function $v(t)$ is the derivative of the function $x(t)$. In other words, the derivative of a function is a new function that tells how rapidly the original function was changing. We now use neither Newton’s name for his technique (he called it “the method of fluxions”) nor his notation. The more commonly used notation is due to Newton’s German contemporary Leibnitz, whom the English accused of plagiarizing the
calculus from Newton. In the Leibnitz notation, we write

\[ v = \frac{dx}{dt} \]

to indicate that the function \( v(t) \) equals the slope of the tangent line of the graph of \( x(t) \) at every time \( t \). The Leibnitz notation is meant to evoke the delta notation, but with a very small time interval. Because the \( dx \) and \( dt \) are thought of as very small \( \Delta x \)'s and \( \Delta t \)'s, i.e., very small differences, the part of calculus that has to do with derivatives is called differential calculus.

Differential calculus consists of three things:

- The concept and definition of the derivative, which is covered in this book, but which will be discussed more formally in your math course.

- The Leibnitz notation described above, which you'll need to get more comfortable with in your math course.

- A set of rules that allows you to find an equation for the derivative of a given function. For instance, if you happened to have a situation where the position of an object was given by the equation \( x = 2t^7 \), you would be able to use those rules to find \( \frac{dx}{dt} = 14t^6 \). This bag of tricks is covered in your math course.
Summary

Selected vocabulary
- center of mass: the balance point of an object
- velocity: the rate of change of position; the slope of the tangent line on an $x-t$ graph.

Notation
- $x$: a point in space
- $t$: a point in time, a clock reading
- $\Delta$: “change in;” the value of a variable afterwards minus its value before
- $\Delta x$: a distance, or more precisely a change in $x$, which may be less than the distance traveled; its plus or minus sign indicates direction
- $\Delta t$: a duration of time
- $v$: velocity
- $v_{AB}$: the velocity of object A relative to object B

Other terminology and notation
- displacement: a name for the symbol $\Delta x$
- speed: the absolute value of the velocity, i.e., the velocity stripped of any information about its direction

Summary

An object’s center of mass is the point at which it can be balanced. For the time being, we are studying the mathematical description only of the motion of an object’s center of mass in cases restricted to one dimension. The motion of an object’s center of mass is usually far simpler than the motion of any of its other parts.

It is important to distinguish location, $x$, from distance, $\Delta x$, and clock reading, $t$, from time interval $\Delta t$. When an object’s $x-t$ graph is linear, we define its velocity as the slope of the line, $\Delta x/\Delta t$. When the graph is curved, we generalize the definition so that the velocity is the slope of the tangent line at a given point on the graph.

Galileo’s principle of inertia states that no force is required to maintain motion with constant velocity in a straight line, and absolute motion does not cause any observable physical effects. Things typically tend to reduce their velocity relative to the surface of our planet only because they are physically rubbing against the planet (or something attached to the planet), not because there is anything special about being at rest with respect to the earth’s surface. When it seems, for instance, that a force is required to keep a book sliding across a table, in fact the force is only serving to cancel the contrary force of friction.

Absolute motion is not a well-defined concept, and if two observers are not at rest relative to one another they will disagree about the absolute velocities of objects. They will, however, agree
about relative velocities. If object A is in motion relative to object B, and B is in motion relative to C, then A’s velocity relative to C is given by $v_{AC} = v_{AB} + v_{BC}$. Positive and negative signs are used to indicate the direction of an object’s motion.
Problems

Key
√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 The graph shows the motion of a car stuck in stop-and-go freeway traffic. (a) If you only knew how far the car had gone during this entire time period, what would you think its velocity was? (b) What is the car’s maximum velocity?

2 (a) Let \( \theta \) be the latitude of a point on the Earth’s surface. Derive an algebra equation for the distance, \( L \), traveled by that point during one rotation of the Earth about its axis, i.e., over one day, expressed in terms of \( \theta \) and \( R \), the radius of the earth. Check: Your equation should give \( L = 0 \) for the North Pole.
(b) At what speed is Fullerton, at latitude \( \theta = 34^\circ \), moving with the rotation of the Earth about its axis? Give your answer in units of mi/h. [See the table in the back of the book for the relevant data.]

3 A person is parachute jumping. During the time between when she leaps out of the plane and when she opens her chute, her altitude is given by the equation

\[ y = (10000 \text{ m}) - (50 \text{ m/s}) \left[ t + (5.0 \text{ s})e^{-t/5.0 \text{ s}} \right]. \]

Find her velocity at \( t = 7.0 \text{ s} \). (This can be done on a calculator, without knowing calculus.) Because of air resistance, her velocity does not increase at a steady rate as it would for an object falling in vacuum.

4 A light-year is a unit of distance used in astronomy, and defined as the distance light travels in one year. The speed of light is \( 3.0 \times 10^8 \text{ m/s} \). Find how many meters there are in one light-year, expressing your answer in scientific notation.

5 You’re standing in a freight train, and have no way to see out. If you have to lean to stay on your feet, what, if anything, does that tell you about the train’s velocity? Explain.

6 A honeybee’s position as a function of time is given by \( x = 10t - t^3 \), where \( t \) is in seconds and \( x \) in meters. What is its velocity at \( t = 3.0 \text{ s} \)?

7 The figure shows the motion of a point on the rim of a rolling wheel. (The shape is called a cycloid.) Suppose bug A is riding on the rim of the wheel on a bicycle that is rolling, while bug B is on the spinning wheel of a bike that is sitting upside down on the floor. Bug A is moving along a cycloid, while bug B is moving in a circle. Both wheels are doing the same number of revolutions per minute. Which bug has a harder time holding on, or do they find it equally
8  Peanut plants fold up their leaves at night. Estimate the top speed of the tip of one of the leaves shown in the figure, expressing your result in scientific notation in SI units.

9  (a) Translate the following information into symbols, using the notation with two subscripts introduced in section 2.5. Eowyn is riding on her horse at a velocity of 11 m/s. She twists around in her saddle and fires an arrow backward. Her bow fires arrows at 25 m/s. (b) Find the velocity of the arrow relative to the ground.

10 Our full discussion of two- and three-dimensional motion is postponed until the second half of the book, but here is a chance to use a little mathematical creativity in anticipation of that generalization. Suppose a ship is sailing east at a certain speed \( v \), and a passenger is walking across the deck at the same speed \( v \), so that his track across the deck is perpendicular to the ship’s center-line. What is his speed relative to the water, and in what direction is he moving relative to the water?

11 Freddi Fish\(^{TM}\) has a position as a function of time given by \( x = a/(b + t^2) \). (a) Infer the units of the constants \( a \) and \( b \). (b) Find her maximum speed. (c) Check that your answer has the right units.

12 Driving along in your car, you take your foot off the gas, and your speedometer shows a reduction in speed. Describe a frame of reference in which your car was speeding up during that same period of time. (The frame of reference should be defined by an observer who, although perhaps in motion relative to the earth, is not changing her own speed or direction of motion.)
13 The figure shows the motion of a bluefin tuna, as measured by a radio tag (Block et al., Nature, v. 434, p. 1121, 2005), over the course of several years. Until this study, it had been believed that the populations of the fish in the eastern and western Atlantic were separate, but this particular fish was observed to cross the entire Atlantic Ocean, from Virginia to Ireland. Points A, B, and C show a period of one month, during which the fish made the most rapid progress. Estimate its speed during that month, in units of kilometers per hour.

14 Sometimes doors are built with mechanisms that automatically close them after they have been opened. The designer can set both the strength of the spring and the amount of friction. If there is too much friction in relation to the strength of the spring, the door takes too long to close, but if there is too little, the door will oscillate. For an optimal design, we get motion of the form $x = cte^{-bt}$, where $x$ is the position of some point on the door, and $c$ and $b$ are positive constants. (Similar systems are used for other mechanical devices, such as stereo speakers and the recoil mechanisms of guns.) In this example, the door moves in the positive direction up until a certain time, then stops and settles back in the negative direction, eventually approaching $x = 0$. This would be the type of motion we would get if someone flung a door open and the door closer then brought it back closed again. (a) Infer the units of the constants $b$ and $c$.

(b) Find the door’s maximum speed (i.e., the greatest absolute value of its velocity) as it comes back to the closed position.

(c) Show that your answer has units that make sense.

15 At a picnic, someone hands you a can of beer. The ground is uneven, and you don’t want to spill your drink. You reason that it will be more stable if you drink some of it first in order to lower its center of mass. How much should you drink in order to make the center of mass as low as possible? [Based on a problem by Walter van B. Roberts and Martin Gardner.]
16. In running races at distances of 800 meters and longer, runners do not have their own lanes, so in order to pass, they have to go around their opponents. Suppose we adopt the simplified geometrical model suggested by the figure, in which the two runners take equal times to trace out the sides of an isosceles triangle, deviating from parallelism by the angle $\theta$. The runner going straight runs at speed $v$, while the one who is passing must run at a greater speed. Let the difference in speeds be $\Delta v$.

(a) Find $\Delta v$ in terms of $v$ and $\theta$. 
(b) Check the units of your equation using the method shown in example 1 on p. 27.
(c) Check that your answer makes sense in the special case where $\theta = 0$, i.e., in the case where the runners are on an extremely long straightaway.
(d) Suppose that $\theta = 1.0$ degrees, which is about the smallest value that will allow a runner to pass in the distance available on the straightaway of a track, and let $v = 7.06 \text{ m/s}$, which is the women’s world record pace at 800 meters. Plug numbers into your equation from part a to determine $\Delta v$, and comment on the result.

17. In 1849, Fizeau carried out the first terrestrial measurement of the speed of light; previous measurements by Roemer and Bradley had involved astronomical observation. The figure shows a simplified conceptual representation of Fizeau’s experiment. A ray of light from a bright source was directed through the teeth at the edge of a spinning cogwheel. After traveling a distance $L$, it was reflected from a mirror and returned along the same path. The figure shows the case in which the ray passes between two teeth, but when it returns, the wheel has rotated by half the spacing of the teeth, so that the ray is blocked. When this condition is achieved, the observer looking through the teeth toward the far-off mirror sees it go completely dark. Fizeau adjusted the speed of the wheel to achieve this condition and recorded the rate of rotation to be $f$ rotations per second. Let the number of teeth on the wheel be $n$.

(a) Find the speed of light $c$ in terms of $L$, $n$, and $f$. 
(b) Check the units of your equation using the method shown in example 1 on p. 27. (Here $f$’s units of rotations per second should be taken as inverse seconds, $s^{-1}$, since the number of rotations in a second is a unitless count.)
(c) Imagine that you are Fizeau trying to design this experiment. The speed of light is a huge number in ordinary units. Use your equation from part a to determine whether increasing $c$ requires an increase in $L$, or a decrease. Do the same for $n$ and $f$. Based on this, decide for each of these variables whether you want a value that is as big as possible, or as small as possible.
(d) Fizeau used $L = 8633$ m, $f = 12.6 \text{ s}^{-1}$, and $n = 720$. Plug in to your equation from part a and extract the speed of light from his data.
18. (a) Let $R$ be the radius of the Earth and $T$ the time (one day) that it takes for one rotation. Find the speed at which a point on the equator moves due to the rotation of the earth. √
(b) Check the units of your equation using the method shown in example 1 on p. 27.
(c) Check that your answer to part a makes sense in the case where the Earth stops rotating completely, so that $T$ is infinitely long.
(d) Nairobi, Kenya, is very close to the equator. Plugging in numbers to your answer from part a, find Nairobi’s speed in meters per second. See the table in the back of the book for the relevant data. For comparison, the speed of sound is about 340 m/s. √

19. (a) Let $\theta$ be the latitude of a point on the Earth’s surface. Derive an algebra equation for the distance, $L$, traveled by that point during one rotation of the Earth about its axis, i.e., over one day, expressed in terms of $\theta$ and $R$, the radius of the earth. You may find it helpful to draw one or more diagrams in the style of figure h on p. 34. √
(b) Generalize the result of problem 18a to points not necessarily on the equator. √
(c) Check the units of your equation using the method shown in example 1 on p. 27.
(d) Check that your equation in part b gives zero for the North Pole, and also that it agrees with problem 18a in the special case of a point on the equator.
(e) At what speed is Fullerton, California, at latitude $\theta = 34^\circ$, moving with the rotation of the Earth about its axis? √
Galileo’s contradiction of Aristotle had serious consequences. He was interrogated by the Church authorities and convicted of teaching that the earth went around the sun as a matter of fact and not, as he had promised previously, as a mere mathematical hypothesis. He was placed under permanent house arrest, and forbidden to write about or teach his theories. Immediately after being forced to recant his claim that the earth revolved around the sun, the old man is said to have muttered defiantly “and yet it does move.” The story is dramatic, but there are some omissions in the commonly taught heroic version. There was a rumor that the Simplicio character represented the Pope. Also, some of the ideas Galileo advocated had controversial religious overtones. He believed in the existence of atoms, and atomism was thought by some people to contradict the Church’s doctrine of transubstantiation, which said that in the Catholic mass, the blessing of the bread and wine literally transformed them into the flesh and blood of Christ. His support for a cosmology in which the earth circled the sun was also disreputable because one of its supporters, Giordano Bruno, had also proposed a bizarre synthesis of Christianity with the ancient Egyptian religion.

Chapter 3

Acceleration and free fall

3.1 The motion of falling objects

The motion of falling objects is the simplest and most common example of motion with changing velocity. The early pioneers of
According to Galileo’s student Viviani, Galileo dropped a cannonball and a musketball simultaneously from the leaning tower of Pisa, and observed that they hit the ground at nearly the same time. This contradicted Aristotle’s long-accepted idea that heavier objects fell faster. Other examples seem less likely to have deep significance. A walking person who speeds up is making a conscious choice. If one stretch of a river flows more rapidly than another, it may be only because the channel is narrower there, which is just an accident of the local geography. But there is something impressively consistent, universal, and inexorable about the way things fall.

Stand up now and simultaneously drop a coin and a bit of paper side by side. The paper takes much longer to hit the ground. That’s why Aristotle wrote that heavy objects fell more rapidly. Europeans believed him for two thousand years.

Now repeat the experiment, but make it into a race between the coin and your shoe. My own shoe is about 50 times heavier than the nickel I had handy, but it looks to me like they hit the ground at exactly the same moment. So much for Aristotle! Galileo, who had a flair for the theatrical, did the experiment by dropping a bullet and a heavy cannonball from a tall tower. Aristotle’s observations had been incomplete, his interpretation a vast oversimplification.

It is inconceivable that Galileo was the first person to observe a discrepancy with Aristotle’s predictions. Galileo was the one who changed the course of history because he was able to assemble the observations into a coherent pattern, and also because he carried out systematic quantitative (numerical) measurements rather than just describing things qualitatively.

Why is it that some objects, like the coin and the shoe, have similar motion, but others, like a feather or a bit of paper, are different? Galileo speculated that in addition to the force that always pulls objects down, there was an upward force exerted by the air. Anyone can speculate, but Galileo went beyond speculation and came up with two clever experiments to probe the issue. First, he experimented with objects falling in water, which probed the same issues but made the motion slow enough that he could take time measurements with a primitive pendulum clock. With this technique, he established the following facts:

- All heavy, streamlined objects (for example a steel rod dropped point-down) reach the bottom of the tank in about the same amount of time, only slightly longer than the time they would take to fall the same distance in air.

- Objects that are lighter or less streamlined take a longer time to reach the bottom.

This supported his hypothesis about two contrary forces. He imagined an idealized situation in which the falling object did not
have to push its way through any substance at all. Falling in air would be more like this ideal case than falling in water, but even a thin, sparse medium like air would be sufficient to cause obvious effects on feathers and other light objects that were not streamlined. Today, we have vacuum pumps that allow us to suck nearly all the air out of a chamber, and if we drop a feather and a rock side by side in a vacuum, the feather does not lag behind the rock at all.

**How the speed of a falling object increases with time**

Galileo’s second stroke of genius was to find a way to make quantitative measurements of how the speed of a falling object increased as it went along. Again it was problematic to make sufficiently accurate time measurements with primitive clocks, and again he found a tricky way to slow things down while preserving the essential physical phenomena: he let a ball roll down a slope instead of dropping it vertically. The steeper the incline, the more rapidly the ball would gain speed. Without a modern video camera, Galileo had invented a way to make a slow-motion version of falling.

Although Galileo’s clocks were only good enough to do accurate experiments at the smaller angles, he was confident after making a systematic study at a variety of small angles that his basic conclusions were generally valid. Stated in modern language, what he found was that the velocity-versus-time graph was a line. In the language of algebra, we know that a line has an equation of the form \( y = ax + b \), but our variables are \( v \) and \( t \), so it would be \( v = at + b \). (The constant \( b \) can be interpreted simply as the initial velocity of the object, i.e., its velocity at the time when we started our clock, which we conventionally write as \( v_0 \).)

**self-check A**

An object is rolling down an incline. After it has been rolling for a short time, it is found to travel 13 cm during a certain one-second interval. During the second after that, it goes 16 cm. How many cm will it travel in the second after that?  

\[ v \]  

\[ t \]  

\[ c / \text{The } v - t \text{ graph of a falling object is a line.} \]
Galileo’s experiments show that all falling objects have the same motion if air resistance is negligible.

A contradiction in Aristotle’s reasoning

Galileo’s inclined-plane experiment disproved the long-accepted claim by Aristotle that a falling object had a definite “natural falling speed” proportional to its weight. Galileo had found that the speed just kept on increasing, and weight was irrelevant as long as air friction was negligible. Not only did Galileo prove experimentally that Aristotle had been wrong, but he also pointed out a logical contradiction in Aristotle’s own reasoning. Simplicio, the stupid character, mouths the accepted Aristotelian wisdom:

SIMPLICIO: There can be no doubt but that a particular body . . . has a fixed velocity which is determined by nature . . .

SALVIATI: If then we take two bodies whose natural speeds are different, it is clear that, [according to Aristotle], on uniting the two, the more rapid one will be partly held back by the slower, and the slower will be somewhat hastened by the swifter. Do you not agree with me in this opinion?

SIMPLICIO: You are unquestionably right.

SALVIATI: But if this is true, and if a large stone moves with a speed of, say, eight [unspecified units] while a smaller moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. Thus you see how, from your assumption that the heavier body moves more rapidly than the lighter one, I infer that the heavier body moves more slowly.

What is gravity?

The physicist Richard Feynman liked to tell a story about how when he was a little kid, he asked his father, “Why do things fall?” As an adult, he praised his father for answering, “Nobody knows why things fall. It’s a deep mystery, and the smartest people in the world don’t know the basic reason for it.” Contrast that with the average person’s off-the-cuff answer, “Oh, it’s because of gravity.” Feynman liked his father’s answer, because his father realized that simply giving a name to something didn’t mean that you understood it. The radical thing about Galileo’s and Newton’s approach to science was that they concentrated first on describing mathematically what really did happen, rather than spending a lot of time on untestable speculation such as Aristotle’s statement that “Things fall because they are trying to reach their natural place in contact with the earth.” That doesn’t mean that science can never answer the “why” questions. Over the next month or two as you delve deeper into physics, you will learn that there are more fundamental reasons why all falling objects have \( v - t \) graphs with the same slope, regardless
of their mass. Nevertheless, the methods of science always impose limits on how deep our explanation can go.

### 3.2 Acceleration

**Definition of acceleration for linear \( v - t \) graphs**

Galileo’s experiment with dropping heavy and light objects from a tower showed that all falling objects have the same motion, and his inclined-plane experiments showed that the motion was described by \( v = at + v_0 \). The initial velocity \( v_0 \) depends on whether you drop the object from rest or throw it down, but even if you throw it down, you cannot change the slope, \( a \), of the \( v - t \) graph.

Since these experiments show that all falling objects have linear \( v - t \) graphs with the same slope, the slope of such a graph is apparently an important and useful quantity. We use the word acceleration, and the symbol \( a \), for the slope of such a graph. In symbols, \( a = \Delta v/\Delta t \). The acceleration can be interpreted as the amount of speed gained in every second, and it has units of velocity divided by time, i.e., “meters per second per second,” or m/s/s. Continuing to treat units as if they were algebra symbols, we simplify “m/s/s” to read “m/s².” Acceleration can be a useful quantity for describing other types of motion besides falling, and the word and the symbol “\( a \)” can be used in a more general context. We reserve the more specialized symbol “\( g \)” for the acceleration of falling objects, which on the surface of our planet equals 9.8 m/s². Often when doing approximate calculations or merely illustrative numerical examples it is good enough to use \( g = 10 \) m/s², which is off by only 2%.

---

1. **Finding final speed, given time**  
   - A despondent physics student jumps off a bridge, and falls for three seconds before hitting the water. How fast is he going when he hits the water?  
   - Approximating \( g \) as 10 m/s², he will gain 10 m/s of speed each second. After one second, his velocity is 10 m/s, after two seconds it is 20 m/s, and on impact, after falling for three seconds, he is moving at 30 m/s.

2. **Extracting acceleration from a graph**  
   - The \( x - t \) and \( v - t \) graphs show the motion of a car starting from a stop sign. What is the car’s acceleration?  
   - Acceleration is defined as the slope of the \( v-t \) graph. The graph rises by 3 m/s during a time interval of 3 s, so the acceleration is \( (3 \text{ m/s})/(3 \text{ s}) = 1 \text{ m/s}^2 \).

Incorrect solution #1: The final velocity is 3 m/s, and acceleration is velocity divided by time, so the acceleration is \( (3 \text{ m/s})/(10 \text{ s}) = 0.3 \text{ m/s}^2 \).
x The solution is incorrect because you can’t find the slope of a graph from one point. This person was just using the point at the right end of the v-t graph to try to find the slope of the curve.

Incorrect solution #2: Velocity is distance divided by time so \( v = \frac{4.5 \text{ m}}{3 \text{ s}} = 1.5 \text{ m/s} \). Acceleration is velocity divided by time, so \( a = \frac{1.5 \text{ m/s}}{3 \text{ s}} = 0.5 \text{ m/s}^2 \).

x The solution is incorrect because velocity is the slope of the tangent line. In a case like this where the velocity is changing, you can’t just pick two points on the x-t graph and use them to find the velocity.

\( \text{Converting } g \text{ to different units example 3} \)

\( \text{What is } g \text{ in units of cm/s}^2? \)

\( \text{The answer is going to be how many cm/s of speed a falling object gains in one second. If it gains 9.8 m/s in one second, then it gains 980 cm/s in one second, so } g = 980 \text{ cm/s}^2. \) Alternatively, we can use the method of fractions that equal one:

\[
\frac{9.8 \text{ m/s}^2}{1 \text{ m}} \times 100 \text{ cm} = 980 \text{ cm/s}^2
\]

\( \text{What is } g \text{ in units of miles/hour}^2? \)

\[
\frac{9.8 \text{ m/s}^2}{1 \text{ mile}} \times \frac{1 \text{ mile}}{1600 \text{ m}} \times \left( \frac{3600 \text{ s}}{1 \text{ hour}} \right)^2 = 7.9 \times 10^4 \text{ mile/hour}^2
\]

This large number can be interpreted as the speed, in miles per hour, that you would gain by falling for one hour. Note that we had to square the conversion factor of 3600 s/hour in order to cancel out the units of seconds squared in the denominator.

\( \text{What is } g \text{ in units of miles/hour/s?} \)

\[
\frac{9.8 \text{ m/s}^2}{1 \text{ mile}} \times \frac{1 \text{ mile}}{1600 \text{ m}} \times \frac{3600 \text{ s}}{1 \text{ hour}} = 22 \text{ mile/hour/s}
\]

This is a figure that Americans will have an intuitive feel for. If your car has a forward acceleration equal to the acceleration of a falling object, then you will gain 22 miles per hour of speed every second. However, using mixed time units of hours and seconds like this is usually inconvenient for problem-solving. It would be like using units of foot-inches for area instead of ft$^2$ or in$^2$.

**The acceleration of gravity is different in different locations.**

Everyone knows that gravity is weaker on the moon, but actually it is not even the same everywhere on Earth, as shown by the sampling of numerical data in the following table.