at a point lying on the box’s surface, at the midpoint between the two edges. Your answer will involve an integral that is most easily done using computer software.

14 A hollow cylindrical pipe has length \( \ell \) and radius \( b \). Its ends are open, but on the curved surface it has a charge density \( \sigma \). A charge \( q \) with mass \( m \) is released at the center of the pipe, in unstable equilibrium. Because the equilibrium is unstable, the particle accelerates off in one direction or the other, along the axis of the pipe, and comes shooting out like a bullet from the barrel of a gun. Find the velocity of the particle when it’s infinitely far from the “gun.” Your answer will involve an integral that is difficult to do by hand; you may want to look it up in a table of integrals, do it online at integrals.com, or download and install the free Maxima symbolic math software from maxima.sourceforge.net.
Sources of magnetism
Chapter 11
Sources of magnetism

11.1 The current density

11.1.1 Definition

We’ve interpreted the junction rule (sec. 9.2, p. 208) as a statement of conservation of charge, but this interpretation only works if we assume that no charge ever gets stashed away temporarily at some location in the circuit and then brought back into action later. By analogy, the junction rule can be disobeyed by cars, if there are lots of cars coming into a parking garage in the morning and staying there all day until they leave after working hours. Circuits can actually accumulate charge, e.g., on the plates of a capacitor, so we would like to have a more general way of describing conservation of charge (which is always true) than the junction rule (which is only sometimes true).

By the way, this kind of thing arises in many physical situations, not just in electricity and magnetism. In figure a, the stream of water is fatter near the mouth of the faucet, and skinnier lower down. This is because the water speeds up as it falls. If the cross-sectional area of the stream was equal all along its length, then the rate of flow (kilograms per second) through a lower cross-section would be greater than the rate of flow through a cross-section higher up. Since the flow is steady, the amount of water between the two cross-sections stays constant. Conservation of mass therefore requires that the cross-sectional area of the stream shrink in inverse proportion to the increasing speed of the falling water. Notice how we had to assume a steady flow, so that no region of space had any net influx of water, and the mass of water is conserved.
or outflow of water.

The first step in formulating this sort of thing mathematically for electric charge and currents is to recognize that we’re describing a law of physics — conservation of charge — and the laws of physics are really always \( \textit{local} \). This means that the electric current, \( I \), in units of amperes, is fundamentally ill suited to our purposes, since the definition of current involves the net charge flowing across some surface, which can be large. We want some way of talking about the flow of current at a particular \textit{point} in space, which would be a kind of current \textit{density} and should be a vector. The mathematical setup is the same as as the one occurring in the definition of flux (sec. 2.2, p. 47), but with different variables. The current through an infinitesimal area, with area vector \( d\mathbf{A} \), is

\[
dI = \mathbf{j} \cdot d\mathbf{A},
\]

which implicitly defines the current density \( \mathbf{j} \). Integrating this gives the current through a finite surface, \( I = \int \mathbf{j} \cdot d\mathbf{A} \), which looks just like our definition of flux, but with different letters of the alphabet and a different physical interpretation. The current density has SI units of \( \text{A/m}^2 \). It is a vector pointing in the net direction of flow of electric charge, with the flow of negative charge being represented by a vector in the opposite direction.

\textit{Current density in a copper wire} \quad \text{example 1}

\( \triangleright \) Electrical codes in the U.S. require that a copper wire carrying 20 A should be at least of a certain size (called 12 gauge), which has a cross-sectional area of 3.31 mm\(^2\). What is the corresponding current density, assuming that the current is uniformly distributed across the wire’s entire cross-section?

\( \triangleright \) Since the current density is stated to be constant, we can take it outside the integral, \( I = \int \mathbf{j} \cdot d\mathbf{A} = \mathbf{j} \cdot \int d\mathbf{A} = \mathbf{j} \cdot \mathbf{A} \). The cross-sectional area is stated for a cross-section perpendicular to the wire, so that the area vector points along the wire’s axis, and therefore \( \mathbf{j} \) and \( \mathbf{A} \) are parallel, meaning that the dot product \( \mathbf{j} \cdot \mathbf{A} \) is simply the product of the magnitudes \( jA \). We then have

\[
j = \frac{I}{A} = 6.0 \times 10^6 \text{ A/m}^2.
\]

Comparing with the requirements in the code for other amounts of current, we find that the area is required to scale up somewhat faster than the current, so that the current density has to be somewhat smaller for thicker wires. This is presumably because a thicker wire has a smaller surface-to-volume ratio, and therefore cannot get rid of its heat as quickly.
Discussion question

A Each of the four figures shows a short section of a long current-carrying copper bar, whose cross-section is a square with sides of length \( b \). In this side view, the height of the bar is \( b \). The current density \( \mathbf{j} \) is constant and is in the direction shown by the white arrows. In example 1, the black line shows a surface cutting perpendicularly through the wire. We define the orientation of this surface to be to the right, as shown by the black arrow. Integrating both sides of \( dI = \mathbf{j} \cdot d\mathbf{A} \), we obtain \( I = jb^2 \).

1. If we wanted to compare this calculation to reality by a measurement with a real ammeter, what would we actually have to do?

2. Suppose we flip the orientation of the \( \mathbf{A} \) vector. What would this mean in terms of actual measurements?

3. We now tilt the surface by 45 degrees. Recalculate \( I \).

4. Suppose the surface is now a sphere of diameter \( b \). Find the error in the following calculation: \( I = 4\pi(b/2)^2j = \pi b^2 \).

11.1.2 Continuity equation

We can now imagine a “div-meter” (figure y, p. 61) that measures the divergence of the current density rather than the divergence of the electric field. If \( \text{div}\mathbf{j} \) is nonzero — let’s say positive — at a certain point, then either conservation of charge is being violated at that point, or charge that was stored in that location is being taken out, like the cars in the parking garage leaving at the end of the day. Conservation of charge can now be stated succinctly as

\[
\text{div}\mathbf{j} = -\frac{\partial \rho}{\partial t},
\]

where \( \rho \) is the charge density. An equation of this form also holds in other physical cases such as the flow of water, and is known more generally as an equation of continuity.

Skin depth example 2

There is a general phenomenon in AC circuits that the flow of current in a wire is nonuniform: the current density is higher as we get closer to the surface of the wire. This is referred to as the “skin effect,” and it occurs because of induced electric fields. Figure b shows a typical profile of current density throughout a wire. If we visualize a div-meter in this field, we can see that although it would tend to spin and be swept downstream, it would not change its volume, which is how a div-meter registers a divergence. Therefore \( \text{div}\mathbf{j} = 0 \) in this example, and by conservation of charge we find that \( \partial \rho / \partial t = 0 \), i.e., charge is not being stored or taken out of storage anywhere in the wire. Skin depth is discussed further in example 8, p. 264.
Possible and impossible patterns of steady flow example 3

For a steady flow of electric charge, nothing is changing over time, so \( \partial \rho / \partial t = 0 \) and therefore the continuity equation requires \( \text{div} \, j = 0 \) everywhere: the flow must be divergence-free. Figure c shows some possible and impossible examples.

Example c/1 is impossible. The current density is of the form \( j = bx\hat{x} \), with \( b > 0 \), and we showed in example 10, p. 62, that such a field has a divergence equal to \( b \), which is nonzero.

The flow in c/2 is also impossible. This is like \( j = -by\hat{y} \), which can be obtained from c/1 by rotating 90 degrees and then reversing all the vectors. Because the divergence is a scalar, the 90-degree rotation doesn’t change the divergence. The divergence is also a linear operator (like any kind of derivative operator), so flipping the arrows works like \( \text{div}(-j) = -\text{div} j \), i.e., it flips the sign of the result. We therefore find that the divergence of this flow is \( -b \), and this is also impossible for a steady flow.

The flow in figure c/3 is the point-by-point vector sum of c/1 and c/2. Since the divergence is again a linear operator, we have \( \text{div}(j_1 + j_2) = \text{div} j_1 + \text{div} j_2 \). But this is \( b - b = 0 \), so this is a possible steady flow of charges.

Discussion questions
**A**  The sea-of-arrows diagram in figure d represents a current density $j$. The portions with the slanting arrows are identical to pieces of figure c/3. Imagine putting a div-meter at each of the marked points, and determine whether the divergence is positive, negative, or zero. Describe what is happening to the charge. If a positive charge enters from the right, where does it end up?

**B**  (1) The figure shows a freeway on which we’ll say that the cars are all initially uniformly spaced and moving at the same velocity. Suppose that every car begins accelerating at the same time and with the same acceleration. Describe how the continuity equation works out. (2) Now do the case where cars are flowing on a freeway at constant speed, but then begin to accelerate starting at some point in space. The flow is steady.

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**The einzel lens**

Figure e shows a common electrostatic focusing device called an einzel lens. Einzel lenses are used to focus beams of charged particles in, for example, scanning electron microscopes and old-fashioned CRT video tubes. This one consists of three cylindrical pieces of metal. The axes coincide, and run from left to right in the figure, which shows a cross-sectional view. The two dark lines on the left are the top and bottom of the left-hand cylinder, and similarly in the middle and right. The left and right cylinders are kept at the ground potential, while the one in the middle is at a different potential $\phi > 0$.

As the electrons enter from the left side, they first encounter strong electric fields when they reach the gap between the left and middle cylinders. If we consider an electron at the top edge of the beam, the field initially decelerates it and moves it away from the axis, to point A. Continuing from A to B, the electron is reaccelerated and deflected toward a point on the right at which the beam reaches a focus on the axis.

It’s no accident that the electron speeds up as it travels from A to B. The physics is identical to that of the faucet in figure a, and to figure c/3. The speeding-up and the coming-together are logically related. If we had a coming together without a speeding up, we would violate the equation of continuity. Note that it only matters that the flow is steady, not whether it’s compressible or incompressible (as for the water).
11.1.3 * Transformation properties (optional)

Clearly the values of \( \rho \) and \( j \) depend on our frame of reference. By analogy with the flow of water, if Huck and Jim are floating down the Mississippi River on a raft, the current density is zero in their frame of reference. If this analogy were to hold in detail, then we would expect, however, that \( \rho \) would stay the same. This turns out not to be true. The purpose of this optional section is to work out the transformation properties of \( \rho \) and \( j \) correctly. Some of the discussion will only be understandable to the reader who has studied optional chapter 7.

Suppose that in a certain frame of reference, a long, straight rod holds charge density \( \rho \) and carries a uniform current density \( j_z \) in the longitudinal direction. It will be surrounded by an electric field that is proportional to \( \rho \), and a magnetic field proportional to \( j_z \). Even if you haven’t read ch. 7, you know that when we change frames of reference, we will have a new mix of electric and magnetic fields. Therefore if we know the transformation of the electric and magnetic fields, we can immediately infer the same transformation properties for \( \rho \) and \( j_z \). We have worked out the transformation of the fields on p. 174, so we obtain the transformation of \( \rho \) and \( j_z \) simply by swapping in the new variables and putting in factors of \( c \) to make the units work:

\[
\begin{align*}
    c\rho' &= \gamma c\rho - \frac{v}{c} \gamma j_z \\
    j'_z &= -\frac{v}{c} \gamma c\rho + \gamma j_z.
\end{align*}
\]

Because \( \rho \) and \( j \) are things we can measure at a point in space, they have their own independent physical existence, and therefore these relationships hold regardless of whether the physical context is the one involving the straight rod.

\( \textit{Length contraction} \quad \text{example 5} \)

Suppose that in a certain frame of reference, a wire carries zero current but has a charge. Setting \( j_z = 0 \) in the equation above for \( \rho' \) gives \( \rho' = \gamma \rho \). This is the result of relativistic length contraction (sec. 7.2.2, p. 178).
11.2 Maxwell’s equations

11.2.1 Adding a current term

In sec. 6.7 we studied Maxwell’s equations in a vacuum. Now we wish to generalize them to their full form, including the possibility that there are both charges and currents. We’ve already seen in sec. 2.8 how to incorporate a charge density $\rho$, which acts as a source of electric fields. The only finishing touch left is to add a term describing how a current density $j$ produces a curly magnetic field. We first present Maxwell’s equations (also summarized for convenience on p. 442) and then give some justification. They are:

\[
\begin{align*}
\text{div } E &= 4\pi k \rho \\
\text{div } B &= 0 \\
\text{curl } E &= -\frac{\partial B}{\partial t} \\
\text{curl } B &= \frac{1}{c^2} \frac{\partial E}{\partial t} + \frac{4\pi k}{c^2} j.
\end{align*}
\]

The only new feature is the final term involving $j$. Something like this is required physically because we know that currents create magnetic fields. The $k/c^2$ has to be there because of units. The positive sign of this term expresses the right-hand relationship between the direction of the current and the curliness of the magnetic field it creates, which was proved in \textsection 183. (If aliens on another planet define their magnetic field to be the opposite direction compared to ours, then they can flip all the signs of terms involving $B$, but they have to do so in a consistent way.)

The only thing left to justify is the factor of $4\pi$. A nice way to do this, which is fundamentally of more physical interest than the $4\pi$, is to show that Maxwell’s equations imply conservation of charge. The $4\pi$ then arises as the correct factor to put in so that charge conservation comes out correctly. This can be done in the following vector calculus calculation, which requires almost no real knowledge of vector calculus other than understanding generic ideas about the linearity of derivatives. You can skip ahead to sec. 11.2.2 if this kind of thing doesn’t seem exciting.

We take the divergence of both sides of the fourth Maxwell’s equation:

\[
\text{div}(\text{curl } B) = \text{div} \left[ \frac{1}{c^2} \frac{\partial E}{\partial t} + \frac{4\pi k}{c^2} j \right].
\]

The left side is zero by symmetry (2274). On the right-hand side, we can use the fact that the divergence is a kind of derivative, so it’s linear, just like with a plain old derivative, which has $(c_1f + c_2g)' = c_1f' + c_2g'$. This gives

\[
0 = \text{div} \frac{\partial E}{\partial t} + 4\pi k \text{div } j.
\]
In general it’s legitimate to swap the order of derivative operators (274), so we can make this into

\[ 0 = \frac{\partial}{\partial t} \text{div} \mathbf{E} + 4\pi k \text{div} \mathbf{j}. \]

But now the first Maxwell’s equation \( \text{div} \mathbf{E} = 4\pi k \rho \) allows us to make this into

\[ 0 = \frac{\partial \rho}{\partial t} + \text{div} \mathbf{j}, \]

which is the equation of continuity, stating that charge is conserved.

11.2.2 The view from the top of the mountain

In a similar way, it is also possible to show from Maxwell’s equations that energy is conserved, a result known as Poynting’s theorem. The basic concept is the same as in sec. 6.6.2, where we showed that the Poynting vector could be interpreted as a rate of energy flow, but extending it to apply more generally than in the case of a plane wave in vacuum. Way back at the beginning of ch. 6, p. 150, I claimed that we would eventually lay out the full set of tightly interlocking logical relationships behind Maxwell’s equations. Here’s how that works. The logical development up until now has been the following:

Assumptions: 1. Time is relative (sec. 1.1, p. 15).
   2. All frames of reference are equally valid, regardless of their motion how we orient them.
   3. Charge is conserved.
   4. Energy is conserved.
   5. Electric and magnetic fields have some basic properties (observability, vectors, and superposition — sec. 1.2, p. 18).

Results: 6. Maxwell’s equations
   7. \( E = mc^2 \) (sec. 5.5, p. 132)
   8. Time dilation, length contraction, and other facts about the structure of space and time (sec. 7.2, p. 176).

On the other hand, we can see that it’s possible to run the logic in the opposite direction as well. If we consider Maxwell’s equations to be a starting assumption (originally deduced by Maxwell from experimentalists’ observations), then we can deduce facts such as conservation of charge. Historically, the idea of Einstein’s two ground-breaking 1905 papers on relativity was to show that, starting from facts 2-6 as assumptions, he could prove facts 1, 7, and 8.

Out of nowhere? example 6

Adding the final current term to Maxwell’s equations means that we know enough laws of physics that it should be possible, in principle, to predict the magnetic field made by a set of currents, as in
the doorbell ringer on p. 255. A seemingly reasonable approach would be to break down the wire into short segments, like a dot-to-dot puzzle, find the field of one such segment, and then add up the fields of all the segments. This divide-and-conquer technique would then reduce the hard problem to the easier problem of finding the magnetic field created by a current distribution like the one in figure f.

The trouble here is that the laws of physics can’t predict the magnetic field in this situation, because the laws of physics forbid this situation from happening. Although the total amount of charge in the figure is staying constant, charge is springing into existence on the left and disappearing into nothingness on the right. If Maxwell’s equations are true, then the continuity equation is true as well, and the continuity equation is a local law of physics: it forbids us from creating or destroying charge in one place, even if we try to make up for it somewhere else.

11.3 Ohm’s law in local form

If we could look inside a resistor with a DC current flowing through it, and zoom in to a very small scale, then the only things we would be able to probe locally through measurements — the only “observables” — would be the electric field \( \mathbf{E} \) and the current density \( \mathbf{j} \). Whereas at a global scale we would say that the voltage drop \( \Delta V \) causes a current \( I \), locally it must be that \( \mathbf{E} \) causes \( \mathbf{j} \). To get the strict proportionality in Ohm’s law, with no nonlinearity for an idealized ohmic material, we must have \( \mathbf{j} = \sigma \mathbf{E} \), for some constant of proportionality \( \sigma \), called the conductivity. This is the local form of Ohm’s law.

The conductivity depends on the material, and generally tells us how many free charge carriers are available, as well as the frequency of collisions that stop the charge carriers. For a perfect conductor, the conductivity is infinite, but electric fields are excluded, so the product \( \sigma \mathbf{E} \) becomes the indeterminate form \( \infty \cdot 0 \). The units of conductivity can be expressed as \( 1/(\Omega \cdot \text{m}) \). An example of a good conductor is copper, which has \( \sigma \approx 6 \times 10^7 \Omega^{-1} \cdot \text{m}^{-1} \).

Conductivity of flesh

Example 7

The DC resistance of a human arm, measured from end to end, is about 300 \( \Omega \). Estimate the conductivity of human flesh.

If the arm has length \( L \) and cross-sectional area \( A \), then \( E = \Delta V/L \) and \( j = I/A \). We then have \( \sigma = j/E = L/AR \). Taking a human arm to have a diameter of 7 cm and a length of 60 cm, \( \sigma \) comes out to be about 0.5 \( \Omega^{-1} \cdot \text{m}^{-1} \), or eight orders of magnitude less than the conductivity of copper. Cf. problem 6, p. 340.
Skin depth’s dependence on frequency example 8

Figure g is a copy of the one from example 2, p. 257, in which we said that the arrows represented the current density, and used the equation of continuity to find out that $\frac{\partial \rho}{\partial t} = 0$, i.e., charge is not being stored or taken out of storage anywhere in the wire. We can now use Maxwell’s equations to see why this phenomenon occurs only for AC circuits.

By $j = \sigma E$, we see that the figure can be taken as a drawing of either the current or the electric field. The only difference would be the constant scalar factor $\sigma$, which is irrelevant since we haven’t defined a numerical scale for the length of the arrows. The figure shows a pretty strong skin depth effect, but we could imagine making it either stronger or weaker than this. If it were much stronger, then appreciable fields and currents would exist only near the surface, like a skin. If it were much weaker, or nonexistent, then we would see a nearly uniform condition throughout the cross-section of the wire. We will argue that the effect must depend on the frequency of the current. In some common examples from everyday life, this frequency is zero for a DC circuit such as a flashlight, 60 Hz for most household appliances in the US, and $\sim 1$ GHz for a cell phone. In the case of 60 Hz, the current is switching directions back and forth 60 times per second, with a sinusoidal variation.

We apply the following two Maxwell’s equations:

$$\text{curl } E = -\frac{\partial B}{\partial t}$$

$$\text{curl } B = \frac{1}{c^2} \frac{\partial E}{\partial t} + \frac{4\pi k j}{c^2}.$$ 

In the DC limit, all the time derivatives must vanish, and therefore $\text{curl } E = 0$. If we imagine inserting a curl-meter for the electric field into figure g, we can see that a skin depth effect requires a nonzero $\text{curl } E$. Therefore the skin effect cannot occur at DC, i.e., the skin depth $\delta$ goes to infinity as the frequency $f$ approaches zero. A more detailed analysis shows that $\delta \propto f^{-1/2}$. Since the only unitful quantities available are $\sigma$, $f$, and the fundamental constants $k$ and $c$, units then require $\delta \propto (\sigma f k)^{-1/2} c$.

11.4 The magnetic dipole

Example 6, p. 262, shows that if we want to find the field made by a current distribution like the one in the doorbell ringer, we can’t necessarily chop up the current distribution into building blocks that look like little line segments. Let’s instead investigate magnetic dipoles as building blocks.
11.4.1 Modeling the dipole using a current loop

Our discussion of the dipole in sec. 5.4, p. 128, focused mainly on the electric dipole. Here we discuss the magnetic dipole in more detail. We’ve defined two types of dipoles in terms of the energy they have when they interact with an external field,

\[ U = -D \cdot E \quad \text{[definition of the electric dipole moment \( D \)]} \]
\[ U = -m \cdot B \quad \text{[definition of the magnetic dipole moment \( m \)]}, \]

and the perfect mathematical analogy between the two definitions automatically implies that electric and magnetic dipoles have many of the same properties. Both dipole moments are measured by vectors, and this implies that they behave as vectors when we rotate them, and also that they add like vectors.

But this is somewhat abstract, and it’s nice to have a more concrete physical picture in mind. Because our universe doesn’t seem to come equipped with magnetic charges, we can’t make a magnetic dipole by gluing such charges to the ends of a stick. Instead, the simplest embodiment of a magnetic dipole would be something like the current loop shown in figure h. Figure i shows an example of why it makes sense that dipole moments add as vectors, and that the dipole moment is proportional to the area of the loop. In exercise , p. 283, we will verify using the right-hand rule that the torque in figure j is in the direction that tends to align the dipole vector with the magnetic field. A calculation (274) shows that the dipole moment is

\[ m = I A, \]

where \( I \) is the current and \( A \) is the area vector.

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**The magnetic dipole moment of an atom example 9**

Let’s make an order-of-magnitude estimate of the magnetic dipole moment of an atom. A hydrogen atom is about \( 10^{-10} \) m in diameter, and the electron moves at speeds of about \( 10^{-2} \)c. We don’t know the shape of the orbit, and indeed it turns out that according to the principles of quantum mechanics, the electron doesn’t even have a well-defined orbit, but if we’re brave, we can still estimate the dipole moment using the cross-sectional area of the atom, which will be on the order of \( (10^{-10})^2 = 10^{-20} \) m\(^2\). The electron is a single particle, not a steady current, but again we throw caution to the winds, and estimate the current it creates as \( e/\Delta t \), where \( \Delta t \), the time for one orbit, can be estimated by dividing the size of the atom by the electron’s velocity. (This is only a rough estimate, and we don’t know the shape of the orbit, so it would be silly, for instance, to bother with multiplying the diameter by \( \pi \) based on our intuitive visualization of the electron as moving around the circumference of a circle.) The result for the dipole moment is \( m \sim 10^{-23} \) A·m\(^2\).
Should we be impressed with how small this dipole moment is, or with how big it is, considering that it's being made by a single atom? Very large or very small numbers are never very interesting by themselves. To get a feeling for what they mean, we need to compare them to something else. An interesting comparison here is to think in terms of the total number of atoms in a typical object, which might be on the order of $10^{26}$ (Avogadro’s number). Suppose we had this many atoms, with their moments all aligned. The total dipole moment would be on the order of $10^3$ A·m, which is a pretty big number. To get a dipole moment this strong using human-scale devices, we’d have to send a thousand amps of current through a one-square meter loop of wire! The insight to be gained here is that, even in a permanent magnet, we must not have all the atoms perfectly aligned, because that would cause more spectacular magnetic effects than we really observe. Apparently, nearly all the atoms in such a magnet are oriented randomly, and do not contribute to the magnet’s dipole moment.

11.4.2 Dipole moment related to angular momentum

In example 9 we made a crude estimate of the typical magnetic dipole moment of an atom. There is another way of going about this, which is potentially much more accurate and of interest as a way of probing the structure of the atom.

Suppose that a particle of charge $q$ and mass $m$ is whizzing around and around some closed path. We don’t even care whether the trajectory is a square or a circle, an orbit or a random wiggle. But let’s say for convenience that it’s a planar shape. The magnetic dipole moment (averaged over time) is $\mathbf{m} = IA$. But the angular momentum of a unit mass can also be interpreted as twice the area it sweeps out per unit time. Aside from the factor of two, which is just a historical glitch in the definitions, this mathematical analogy is exact: mass is to charge as angular momentum $\mathbf{L}$ is to magnetic dipole moment $\mathbf{m}$. Therefore we have the identity

$$\frac{q}{m} \cdot \frac{|\mathbf{L}|}{|\mathbf{m}|} = 2$$

(where $\mathbf{m}$ is the dipole moment, while $m$ is the mass). The left-hand side is called the $g$ factor. We expect $g = 2$ for a single orbiting particle.

Now suppose that we have a collection of particles with identical values of $q/m$. Then vector addition of the $\mathbf{L}$ and $\mathbf{m}$ values gives the same $g = 2$ for the system as a whole. On the other hand, if the different members of the system do not all have the same $q/m$, then the $g$ of the system as a whole need not be 2. For example, a collection of positive and negative charges could easily have zero net charge but $\mathbf{m} \neq 0$, giving $g = 0$.

Particles such as the electron, the neutron, and the proton may
be pointlike, or they may be composites of other particles. The electron and proton, which are charged, have the expected $g$ factors of exactly 2 when we measure the $L$ and $m$ that they have due to their motion through space. But we also find that electrons, neutrons, and protons all come equipped with a built-in angular momentum, present even when they are at rest. This intrinsic angular momentum, called spin, is fixed in magnitude but can vary in direction, like that of a gyroscope. Thus if we measure the $L$ and $m$ of these particles at rest, they have fixed $g$ factors, figure k.

The electron’s intrinsic $g$ factor is extremely close to 2, and if we ignore the small discrepancy for now, we are led to imagine that the electron is either a pointlike particle or a composite of smaller particles, each of which has the same charge-to-mass ratio. The neutron does have a nonvanishing dipole moment, so its zero $g$ factor suggests that it is a composite of other particles whose charges cancel. The proton’s $g$ factor is quite different from 2, so we infer that it, too, is composite. The current theory is that protons and neutrons are clusters of particles called quarks. Quarks come in different types, and the different types have different values of $g/m$.

The magnetic dipole moment of the proton is of considerable importance in our lives because of its use in the MRI (magnetic resonance imaging) scans used in medicine. As described on p. 195, a large DC magnetic field, generated by superconducting magnets, is used to cause the protons in the body’s hydrogen atoms to align partially (about $10^{-5}$ of full alignment). These magnetic moments are then manipulated and observed using AC fields.

From a physicist’s point of view, it is also remarkable that we can infer these facts about the internal structures of neutrons and protons without having to do any experiments that directly probe their interior structure. We don’t need a super-powerful microscope, nor do we need a particle accelerator that can supply enough energy to shake up their internal structure, like shaking a gift-wrapped box to tell what’s inside. Merely by measuring the external, aggregate properties of the “box,” we can get clues about the structure inside.¹

11.4.3 Field of a dipole

An electric dipole, unlike a magnetic one, can be built out of two opposite monopoles, i.e., charges, separated by a certain distance, and it is then straightforward to show by vector addition that the field of an electric dipole, far away, is

$$E_z = kD (3 \cos^2 \theta - 1) r^{-3}$$

$$E_R = kD (3 \sin \theta \cos \theta) r^{-3},$$

¹This is closely analogous to the Tolman-Stewart experiment (example 4, p. 93), in which the subatomic structure of metals was probed by measuring inertial effects in an electric circuit.
The field of a dipole. Two counterrotating rings of current. The inset photo 2 shows a twin lead cable with the insulation stripped off of its ends.

where \( r \) is the distance from the dipole to the point of interest, \( \theta \) is the angle between the dipole vector and the line connecting the dipole to this point, and \( E_z \) and \( E_R \) are, respectively, the components of the field parallel to and perpendicular to the dipole vector. We have already found this field in the special cases of \( \theta = \pi/2 \) (example 3, p. 51) and \( \theta = 0 \) (problem 9, p. 69).

This is the field pattern that exists far away from the dipole, in empty space. Because the vacuum form of Maxwell’s equations treats the electric and magnetic fields totally symmetrically, the magnetic field of a magnetic dipole has to have the same form. With the correct constant of proportionality, it turns out to be

\[
B_z = \frac{km}{c^2} (3 \cos^2 \theta - 1) r^{-3}
\]

\[
B_R = \frac{km}{c^2} (3 \sin \theta \cos \theta) r^{-3}.
\]

Discussion questions

A Find the regions of three-dimensional space in which the magnetic field of a dipole is (1) in the same direction as the dipole vector (parallel), (2) in the opposite direction (antiparallel), and (3) perpendicular to it.

11.5 Magnetic fields found by summing dipoles

Many real-world electromagnets are built out of multiple circular loops of wire. In this section we use a trick to calculate the useful result for the field at the center of a single circular loop. Usually I’m not a big fan of tricks, but we’ll see later that this trick can be generalized in a useful way.

We start by considering a problem, figure m/1, that looks harder but is actually easier. We have not one but two loops of wire, with slightly different radii \( r \) and \( r + dr \). Here the “d” just means “a little bit of,” i.e., writing the radii this way is just a way of saying that they’re almost, but not quite, the same. The currents in the two rings are both \( I \), but they flow in opposite directions. This is actually a reasonably realistic setup; the inset m/2 shows a type of cable, called “twin lead,” that is often used in this way, with current flowing one way through one conductor and then coming back through the other conductor in the opposite direction. Of course in the real circuit we would have to have a battery and connections to the loops from the outside, and we would probably hook up the two loops in series so that their currents were guaranteed to be exactly equal in absolute value. These complications are not shown in the diagram.

The trick, as shown in m/3, is to take the circular strip between the rings and break it up into imaginary squares, then place a square
current loop with current \( I \) on each one. These are all dipoles, and each one contributes a field at the center which we can calculate by plugging \( \theta = \pi/2 \) into our expressions from section 11.4.3. The result is that each tiny dipole \( dm \) contributes \(- (k/c^2 r^3) dm\), where the minus sign means that the field points out of the page (in the opposite direction compared to the vector \( d\mathbf{m} \)). The total field is found by adding up all these small contributions to the field,

\[
B = - \int \frac{k}{c^2 r^3} \, dm.
\]

This integral, like any integral, represents a sum of infinitely many infinitesimal things. We’re not integrating with respect to some variable \( m \), though; \( dm \) here just means an infinitesimal dipole moment of one of the squares. But we don’t actually need any calculus to do this integral. Moving all the constants outside and substituting \( dm = I \, dA \), we have

\[
B = - \frac{kI}{c^2 r^3} \int dA,
\]

where \( \int dA \) is the area of the strip, \( A = \text{(width)}(\text{circumference}) = (dr)(2\pi r) \). Our final result is

\[
B = - \frac{2\pi kI}{c^2 r^2} \, dr \quad \text{[field at the center of figure m/1].}
\]

We weren’t actually that excited about finding the field in the somewhat artificial example of figure m. What would be more useful would be to find the field of a single circular loop. Now we just play a similar trick again. We imagine an infinite set of concentric rings, extending from some radius \( a \) all the way out to infinity. The current on the outer edge of each ring is canceled out by the current in the overlapping inner edge of the next ring, so that the only loop of current that doesn’t cancel out is the very innermost one, at \( r = a \). The result is then that the field at the center of a circular loop is

\[
B = \frac{2\pi kI}{c^2 a} \quad \text{[field at the center of a ring of current].}
\]

The positive sign means that the direction of the field is right-handed, e.g., out of the page if the current is counterclockwise.

### 11.6 Magnetic fields for some practical examples

Figure n shows the equations for some of the more commonly encountered configurations in which wires produce a magnetic field, with illustrations of their field patterns. Of these three results, we’ve only previously derived the first and a special case of the second. The remaining derivations are given later in the book, but the results are presented together at this point for reference.
Field created by a long, straight wire carrying current $I$:

$$B = \frac{k}{c^2} \cdot \frac{2I}{r}$$

Here $r$ is the distance from the center of the wire. The field vectors trace circles in planes perpendicular to the wire, in a direction given by a right-hand rule where the thumb is the current and the fingers are the field. The form of this result was derived in example 2, p. 117, and the unitless factor of 2 in example 6, p. 179.

Field created by a single circular loop of current:
The field vectors form a dipole-like pattern, coming through the loop and back around on the outside. The orientation of the loops is such that in the middle region there is a right-hand relationship in which the thumb is the field; or, alternatively, one can use the same right-hand rule as for a straight wire, applying it to the area close to the wire. There is no simple equation for a field at an arbitrary point in space, but for a point lying along the central axis perpendicular to the loop, the field is

$$B = \frac{k}{c^2} \cdot 2\pi I b^2 \left( b^2 + z^2 \right)^{-3/2},$$

where $b$ is the radius of the loop and $z$ is the distance of the point from the plane of the loop.

Field created by a solenoid (cylindrical coil):
The field pattern is similar to that of a single loop, but for a long solenoid the field lines become very straight on the inside of the coil and on the outside immediately next to the coil. For a sufficiently long solenoid, the interior field also becomes very nearly uniform, with a magnitude of

$$B = \frac{k}{c^2} \cdot 4\pi I N/\ell,$$

where $N$ is the number of turns of wire and $\ell$ is the length of the solenoid. This result is derived in example 4, p. 348. The field near the mouths or outside the coil is not constant, and is more difficult to calculate (problem 13, p. 280). For a long solenoid, the exterior field is much smaller than the interior field.

Some other cases of interest can be solved by superposing the fields above. An example is the Helmholtz coil (problem 6, p. 277).

**self-check A**

1. Let a current-carrying wire lie along the $x$ axis, carrying current in the positive $x$ direction. At points on the $y$ axis, which component of the field is nonzero? Sketch this component as a function of $y$.
2. Sketch the function $B_z(z)$ for a circular loop as described above.

Answer, p. 430
11.7 * The Biot-Savart law (optional)

In sec. 11.5, p. 268, we were able to find the magnetic field at the center of a current loop by the trick of turning the current distribution into a superposition of small, square dipoles. It’s usually a waste of time to learn a trick that only works in one case, but this trick can be extended to apply more generally. Consider the wire loop shown in figure 0/1, which has a randomly chosen, asymmetric shape like a potato chip. If it carries a steady current, it will create a static magnetic field pattern. (The assumption of a steady current is necessary, since otherwise, e.g., it could act like an antenna and radiate electromagnetic waves.)

Figure 0/2 shows an idea for extending our dipole trick to handle this case. We form an imaginary tube that starts on the loop and extends off to infinity, then split it up into strips like railroad tracks, where each railroad track contains an infinite number of square dipoles, 0/3. The key here is that although each track has a pair of long, straight rails that bring current in from infinity and then send it back out, the currents along these rails cancel when we overlay each track with its neighbor. Therefore all we really need to do is find the magnetic field of one such semi-infinite strip, and by adding it up we can find the field of an arbitrary object like the potato-chip loop.

The field of such a strip is probably pretty complicated, and likely to be mainly the field due to the two long rails. However, the currents in the rails are destined to cancel out when we add everything up, sort of like a politically opposed married couple who vote in every election and cancel each other out — they might as well have stayed home. We therefore conjecture that the correct final result can be found by adding up fields that depend only on the properties of the little end-caps. This is not guaranteed to be correct, for the reasons described in example 6 on p. 262, but let’s go ahead based on this questionable assumption and see where it gets us.

Given some point in space, we want to find the contribution to the field at that point coming from a particular end-cap. Let the vector from the end-cap to our point be \( \mathbf{r} \). The contribution to the field has to be of the form function\((\mathbf{j}, \mathbf{r})\) \( d\mathbf{v} \), where the \( d\mathbf{v} \) is the volume of the wire constituting the end-cap. The mystery function has to be proportional to its two vector inputs, and as its output it has to give a vector with units of tesla. This severely constrains the form of the function. The only rotationally invariant way to combine two vectors like this is the cross product, and the only way of getting the right units is by throwing in a factor of \( kc^{-2}r^{-3} \). The only wiggle room is a possible unitless factor in front. This unitless factor turns out to be 1, which we will prove later, in example 10, by comparing with a configuration whose field we already know. So
the contribution to the field from this end-cap is
\[ \frac{k}{c^2} \frac{j \times r}{r^3} dv. \]

Within the small volume of the end-cap, the integrand is constant, and the end-cap is approximately straight, so that it forms a cylinder with volume \( dv = dA \, d\ell \), where \( dA \) is the cross-sectional area. We also assume that, within this short segment of wire, the current flows in the direction of the wire and is uniform across the wire’s cross-sectional area. This allows us to rewrite the end-cap’s field as
\[ \frac{Ik \, d\ell \times r}{c^2 \, r^3}. \]

Integrating over the contributions of all the end-caps gives the formula known as the Biot-Savart law (rhymes with “Leo bazaar”),
\[ B = \frac{Ik}{c^2} \int \frac{d\ell \times r}{r^3}. \]

The field at the center of a circular loop example 10
In section 11.5 we had to use a trick to find the field at the center of a circular loop of current of radius \( a \). The Biot-Savart law routinizes the trick for us and eliminates the need for that kind of creativity. Dividing the loop into many short segments, each \( d\ell \) is perpendicular to the \( r \) vector that goes from it to the center of the circle, and every \( r \) vector has magnitude \( a \). Therefore every cross product \( d\ell \times r \) has the same magnitude, \( a \, d\ell \), as well as the same direction along the axis perpendicular to the loop. The field is
\[ B = \frac{kI}{c^2} \frac{a^3}{a^3} \int \frac{d\ell}{a^3} = \frac{kI}{c^2} \frac{a^2}{a^2} \int d\ell = \frac{kI}{c^2} (2\pi a) = \frac{2\pi kI}{c^2 a}. \]

The fact that the field in example 10 comes out the same as in the previous calculation verifies that we have the right unitless factor in the Biot-Savart law.

Although the Biot-Savart law seems to have given us a correct result, we highlighted an assumption in its derivation that was not guaranteed to be correct. To give a full proof of the law, we would really need to prove that when we plug the result back in to Maxwell’s equations, it gives a solution. That unfortunately requires a level of vector calculus beyond the scope of this book.
What is the magnetic field of a circular loop of current at a point on the axis perpendicular to the loop, lying a distance $z$ from the loop's center?

Again, let’s write $a$ for the loop’s radius. The $r$ vector now has magnitude $\sqrt{a^2 + z^2}$, but it is still perpendicular to the $dl$ vector. By symmetry, the only nonvanishing component of the field is along the $z$ axis,

$$B_z = \int |dB| \cos \alpha = \int \frac{klr \, dl \, a}{c^2 r^3} \frac{1}{r} = \frac{kl a}{c^2 r^3} \int dl = \frac{2\pi kla^2}{c^2 (a^2 + z^2)^{3/2}}.$$

For $z = 0$ we recover the result of example 10. For $z \gg a$, the field is that of a dipole with the correct dipole moment (ex. 11A, p. 283).
Notes for chapter 11

261 Divergence of a curl

The divergence of a curl is zero

The kind of field that looks like it would have a nonvanishing value of this operation is something like $\mathbf{F} = zxy\hat{y}$, the idea being that we take a field like $xy\hat{y}$ that has a curl in the $z$ direction, and then we give it some $z$ dependence so that there could be a divergence. In fact, any field that is differentiable near the origin can have its components approximated in that neighborhood by a function of this general form (a second-order mixed polynomial), so if we can prove that $\text{div}(\text{curl F}) = 0$ for this particular $\mathbf{F}$, it follows that the div of a curl is zero for any $\mathbf{F}$.

Now we show based on symmetry that $\text{div}(\text{curl F}) = 0$ for this $\mathbf{F}$. Suppose we rotate our coordinate system by 180 degrees about the $y$ axis. This doesn’t change the field or its components, but the curl lies in the $x$-$z$ plane, so the output of the curl operation has its components sign-flipped because of the new coordinate system used to describe it. This is an over-all reversal of the curl’s direction, and since the div is linear, the effect is to sign-flip $\text{div}(\text{curl F})$.

But the divergence is a scalar, so the final result of taking $\text{div}(\text{curl F})$ cannot change just because we change our coordinates.

We have proved that $\text{div}(\text{curl F})$ flips its sign, but also that it doesn’t change its sign. This is only possible if it is zero, as claimed.

262 Order of derivatives

Typically it’s OK to freely interchange the order of derivative operations

Of course an example doesn’t prove a general rule, but let’s consider a simple example in order to get the idea of what is being discussed. We have

$$\frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} (xy) \right) = \frac{\partial}{\partial x} x = 1$$

but also

$$\frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} (xy) \right) = \frac{\partial}{\partial y} y = 1.$$ 

In this example, it didn’t matter which derivative we applied first. A theorem called Clairaut’s theorem says that if a function $f(x,y)$ is well-behaved, in the sense that all its second derivatives ($\partial^2 f/\partial x^2$, $\partial (\partial f/\partial x)/\partial y$, and $\partial (\partial f/\partial y)/\partial x$) are continuous at a certain point, then the first derivatives can be interchanged at that point, so that $\partial (\partial f/\partial x)/\partial y = \partial (\partial f/\partial y)/\partial x$.

Because derivative operators like div, grad, and curl can be expressed in terms of partial derivatives, it follows that they can also be interchanged under the same conditions.

265 Magnetic dipole moment of a current loop

The magnetic dipole moment of a square current loop is given by $\mathbf{m} = IA$.

Side view of a square current dipole, with the direction of the current as indicated, coming out of the page on one side and going back in on the other. The dipole is immersed in an externally imposed uniform field shown by the vertical arrows.

Consider the geometry shown in figure j. Let the mobile charge carriers in the wire have linear density $\lambda$, and let the sides of the loop have length $h$, so that we have $I = \lambda v$. We want to show that $m = IA = h^2 \lambda v$ is consistent with the definition of the dipole moment $U = -\mathbf{m} \cdot \mathbf{B}$, where $\mathbf{B}$ is an externally applied field. We do this by computing the torque and then finding the work done when the dipole is reoriented.

The only nonvanishing torque comes from the
forces on the left and right sides. The currents in these sides are perpendicular to the field, so the magnitude of the cross product $\mathbf{F} = q \mathbf{v} \times \mathbf{B}$ is simply $|F| = qvB$. The torque supplied by each of these forces is $\mathbf{r} \times \mathbf{F}$, where the lever arm $\mathbf{r}$ has length $h/2$, and makes an angle $\theta$ with respect to the force vector. The magnitude of the total torque acting on the loop is therefore

$$|\tau| = 2 \frac{h}{2} |F| \sin \theta = h qvB \sin \theta,$$

and substituting $q = \lambda h$ and $v = m/h^2 \lambda$, we have

$$|\tau| = h \lambda h \frac{m}{h^2 \lambda} B \sin \theta = mB \sin \theta.$$

The work done to reorient the dipole is $W = \int \tau \, d\theta = -mB \cos \theta$ (ignoring the irrelevant constant of integration), and this is the same as $U = -\mathbf{m} \cdot \mathbf{B}$. 

Notes for chapter 11
Problems

Key
✓ A computerized answer check is available online.
★ A difficult problem.

1 The photo shows the cross-section of an electrical transmission line designed to be hung in the air between towers. To prevent the cable from sagging too much, there is a central core of radius \( a \) made of a carbon-glass composite, which is stronger and lighter than steel, but nonconductive. Surrounding this is a conducting aluminum sheath with outer radius \( b \). Because the frequency is low, the current density is nearly uniform.
(a) If the cable carries current \( I \), find the magnitude of the current density \( j \). √
(b) Evaluate your answer numerically for \( a = 4.76 \ \text{mm} \) and \( b = 14.07 \ \text{mm} \), at this cable’s nominal current-carrying capacity of 1.00 kA. √

2 Magnetic dipole 1 has its dipole moment oriented along the \( z \) axis, so that its \( z \) component \( m_1 \) is either positive or negative, and its \( x \) and \( y \) components are zero. It interacts with a second dipole \( m_2 \), also purely along the \( z \) axis. They lie on the \( x \) axis at distance \( r \) from one another. Find the energy of their interaction with each other. What is their stable orientation? You should find that the sign of this result agrees with experience from playing with a pair of bar magnets, as well as with the result of ch. 5, problem 2a, p. 139. √

3 Deuterium is an isotope of hydrogen in which the nucleus has one proton and one neutron. The nucleus is referred to as a deuteron. The deuteron has angular momentum and magnetic moment
\[
L = 1.05457 \times 10^{-34} \ \text{J} \cdot \text{s} \quad \text{and} \\
m = 4.33 \times 10^{-27} \ \text{A} \cdot \text{m}^2.
\]
Find the \( g \) factor of the deuteron. You should find that it roughly makes sense if we consider the nucleus as two pointlike particles with identical masses but with one particle’s charge being zero. √
4  

\[ N \] identical gears of radius \( r \) are arranged with their axes parallel and coplanar. The figure shows the \( N = 3 \) case as an example. Each gear is an insulator, and has charge \( q \) distributed uniformly about its circumference. If the system spins at frequency \( f \), find the total dipole moment. How is this different from an example like the one in figure o/3, p. 271?

**Remark:** This is not an unreasonable model of the magnetic properties of a linear molecule, if the magnetic interactions are like the ones described in problem 2.

\[ \nabla \] Hint, p. 423

5  

(a) The first figure shows a cube with six sides, including a floor underneath. Each square has sides of length \( b \) and is a current loop carrying current \( I \) in the same orientation, i.e., an ant exploring the outside surface and inspecting all the current loops will see each current rotating in the same direction as it stands on that panel of the box. Why is the total dipole moment not \( 6Ib^2 \)?

(b) The second figure shows a landscape consisting of a \( 5 \times 5 \) grid of squares, interrupted by a “little house on the prairie” in the middle: a cube with four walls and a roof. The cube does not have a floor, so the total area is \( 29b^2 \). Find the magnitude of the total magnetic dipole moment.

**Remark:** This is not quite as silly and artificial as it might seem. In condensed matter physics, it’s common to have things like surface layers of dipoles, and it’s also common to have defects in such a surface such as bumps and scratches.

\[ \checkmark \]

6  

A Helmholtz coil is defined as a pair of identical circular coils lying in parallel planes and separated by a distance, \( h \), equal to their radius, \( b \). Each coil has \( N \) turns of wire. Current circulates in the same direction in each coil, so the fields tend to reinforce each other in the interior region. This configuration has the advantage of being fairly open, so that other apparatus can be easily placed inside and subjected to the field while remaining visible from the outside. The choice of \( h = b \) results in the most uniform possible field near the center. Find the field at the center.

\[ \checkmark \]
Problem 7. The figure shows a nested pair of circular wire loops used to create magnetic fields. (The twisting of the leads is a practical trick for reducing the magnetic fields they contribute, so the fields are very nearly what we would expect for an ideal circular current loop.) The coordinate system below is to make it easier to discuss directions in space. One loop is in the $y-z$ plane, the other in the $x-y$ plane. Each of the loops has a radius of 1.0 cm, and carries 1.0 A in the direction indicated by the arrow.

(a) Calculate the magnetic field that would be produced by one such loop, at its center. 

(b) Describe the direction of the magnetic field that would be produced, at its center, by the loop in the $x-y$ plane alone.

(c) Do the same for the other loop.

(d) Calculate the magnitude of the magnetic field produced by the two loops in combination, at their common center. Describe its direction.

Problem 8. Four long wires are arranged, as shown, so that their cross-section forms a square, with connections at the ends so that current flows through all four before exiting. Note that the current is to the right in the two back wires, but to the left in the front wires. If the dimensions of the cross-sectional square (height and front-to-back) are $b$, find the magnetic field (magnitude and direction) along the long central axis.
This problem will lead you through the steps of applying the Biot-Savart law to prove that the magnetic field of a long, straight wire has magnitude

\[ B = \frac{2kI}{c^2R}. \]

Almost everything in this equation has to be the way it is because of units, the only exception being the unitless factor of 2, so this problem amounts to proving that it really does come out to be 2, a fact that we previously proved in example 6 on page 179, using relativity.

(a) Set up the integral prescribed by the Biot-Savart law, and simplify it so that it involves only scalar variables rather than a vector cross product, but do not evaluate it yet.

(b) Your integral will contain several different variables, each of which is changing as we integrate along the wire. These will probably include a position on the wire, a distance from the point on the wire to the point at which the field is to be found, and an angle between the wire and this point-to-point line. In order to evaluate the integral, it is necessary to express the integral in terms of only one of these variables. It’s not obvious, but the integral turns out to be easiest to evaluate if you express it in terms of the angle and eliminate the other variables. Do so. Note that the d... part of the integral has to be reexpressed in the same way we would do any time we attacked an integral by substitution (“u-substitution”).

(c) Pulling out all constant factors now gives a definite integral. Evaluate this integral, which you should find is a trivial one, and show that it equals 2.

A square current loop with sides of length 2h carries current I, creating a magnetic field \( B_{\text{square}} \) at its own center. We wish to compare this with the field \( B_{\text{circle}} \) of a circular current loop of radius h. Find \( B_{\text{square}}/B_{\text{circle}} \).

A regular polygon with \( n \) sides can be inscribed within a circle of radius R and can have a circle inscribed inside it with radius h. Let \( \rho = \sqrt{hR} \) be the geometric mean of these two radii. A current loop is constructed in the shape of a regular \( n \)-gon. Show that the magnetic field at the center can be calculated in a simple way from the perimeter and \( \rho \), and make sense of the result in the extreme cases \( n = 2 \) (a degenerate polygon enclosing no area) and \( n \to \infty \).
Magnet coils are often wrapped in multiple layers. The figure shows the special case where the layers are all confined to a single plane, forming a spiral. Since the thickness of the wires (plus their insulation) is fixed, the spiral that results is a mathematical type known as an Archimedean spiral, in which the turns are evenly spaced. The equation of the spiral is \( r = w\theta \), where \( w \) is a constant. For a spiral that starts from \( r = a \) and ends at \( r = b \), show that the field at the center is given by \( \left( kI/c^2 w \right) \ln b/a \).

\[ \triangle \text{Solution, p. 428} \]

Let the interior field of a certain infinite solenoid be \( B_o \). Now suppose that we build a finite solenoid with all the same design parameters except that it has a finite length, and consider the field \( B \) at a point on the axis. Show by integrating the field of a loop of current that

\[ \frac{B}{B_o} = \frac{\cos \theta_1 - \cos \theta_2}{2}, \]

where the angles \( \theta_1 \) and \( \theta_2 \) are defined in the figure.

\[ \star \]

Problem 12.

Problem 13.
Lab 11: Charge-to-mass ratio of the electron

Apparatus

- vacuum tube with Helmholtz coils
- high-voltage power supply
- DC power supply
- multimeter

Goal: Measure the \( q/m \) ratio of the electron.

Why should you believe electrons exist? By the turn of the twentieth century, not all scientists believed in the literal reality of atoms, and few could imagine smaller objects from which the atoms themselves were constructed. Over two thousand years had elapsed since the Greeks first speculated that atoms existed based on philosophical arguments without experimental evidence. During the Middle Ages in Europe, “atomism” had been considered highly suspect, and possibly heretical. Finally by the Victorian era, enough evidence had accumulated from chemical experiments to make a persuasive case for atoms, but subatomic particles were not even discussed.

If it had taken two millennia to settle the question of atoms, it is remarkable that another, subatomic level of structure was brought to light over a period of only about five years, from 1895 to 1900. Most of the crucial work was carried out in a series of experiments by J.J. Thomson, who is therefore often considered the discoverer of the electron.

In this lab, you will carry out a variation on a crucial experiment by Thomson, in which he measured the ratio of the charge of the electron to its mass, \( q/m \). The basic idea is to observe a beam of electrons in a region of space where there is an approximately uniform magnetic field, \( B \). The electrons are emitted perpendicular to the field, and, it turns out, travel in a circle in a plane perpendicular to it. The force of the magnetic field on the electrons is

\[
F = qvB \quad ,
\]

directed towards the center of the circle. Their acceleration is

\[
a = \frac{v^2}{r} \quad ,
\]

so using \( F = ma \), we can write

\[
qvB = \frac{mv^2}{r} \quad .
\]

If the initial velocity of the electrons is provided by accelerating them through a voltage difference \( V \), they have a kinetic energy equal to \( qV \), so

\[
\frac{1}{2}mv^2 = qV \quad .
\]

From equations 3 and 4, you can determine \( q/m \). Note that since the force of a magnetic field on a moving charged particle is always perpendicular to the direction of the particle’s motion, the magnetic field can never do any work on it, and the particle’s KE and speed are therefore constant.

You will be able to see where the electrons are going, because the vacuum tube is filled with a hydrogen gas at a low pressure. Most electrons travel large distances through the gas without ever colliding with a hydrogen atom, but a few do collide, and the atoms then give off blue light, which you can see. Although I will loosely refer to “seeing the beam,” you are really seeing the light from the collisions, not the beam of electrons itself. The manufacturer of the tube has put in just enough gas to make the beam visible; more gas would make a brighter beam, but would cause it to spread out and become too broad to measure it precisely.

The field is supplied by an electromagnet consisting of two circular coils, each with 130 turns of wire (the same on all the tubes we have). The coils are placed on the same axis, with the vacuum tube at the center. A pair of coils arranged in this type of geometry are called Helmholtz coils. Such a setup
provides a nearly uniform field in a large volume of space between the coils, and that space is more accessible than the inside of a solenoid.

Setup

Heater circuit: As with all vacuum tubes, the cathode is heated to make it release electrons more easily. There is a separate low-voltage power supply built into the high-voltage supply. It has a set of green plugs that, in different combinations, allow you to get various low voltage values. Use it to supply 6 V to the terminals marked “heater” on the vacuum tube. The tube should start to glow.

Electromagnet circuit: Connect the other DC power supply, in series with an ammeter, to the terminals marked “coil.” The current from this power supply goes through both coils to make the magnetic field. Verify that the magnet is working by using it to deflect a nearby compass.

High-voltage circuit: Connect the high voltage supply to the terminals marked “anode.” Ask your instructor to check your circuit. Now plug in the HV supply and turn up the voltage to 300 V. You should see the electron beam. If you don’t see anything, try it with the lights dimmed.

Observations

Make the necessary observations in order to find $q/m$, carrying out your plan to deal with the effects of the Earth’s field. The high voltage is supposed to be 300 V, but to get an accurate measurement of what it really is you’ll need to use a multimeter rather than the poorly calibrated meter on the front of the high voltage supply.

The beam can be measured accurately by using the glass rod inside the tube, which has a centimeter scale marked on it.

Be sure to compute $q/m$ before you leave the lab. That way you’ll know you didn’t forget to measure something important, and that your result is reasonable compared to the currently accepted value.
Exercise 11A: Currents and magnetic fields

1. Find the directions of the magnetic fields at points a-h. Describe them using the given coordinate systems, e.g., “+x.”

2. The on-axis field of a circular current loop is shown in example 11, p. 273, to be

\[ B = \frac{2\pi k I a^2}{c^2(a^2 + z^2)^{3/2}}. \]

(a) Show by comparing with the field of a long, straight wire that the units make sense.

(b) Show that the field for \( z \gg a \) is that of a dipole, including the correct constant factor.

3. The diagrams below show side views of a square current loop immersed in an external magnetic field. The first one (reproduced from fig. j, p. 265) shows the loop’s dipole moment. The second one labels the currents along the four sides of the square.

(a) Find the orientation that would minimize the energy \( U = -\mathbf{m} \cdot \mathbf{B} \).

(b) Use the right-hand rule to find the force on each of the four edges.

(c) Verify that the total force is zero.

(d) Find the four torques, and verify that the direction of the total torque is such that it would tend to align the loop’s dipole moment with the field.

Turn the page.
4. Each figure shows a static $\mathbf{B}$ field in the $x$-$y$ plane. The field is independent of $z$. Describe the current density $\mathbf{j}$ in each case. A coordinate system is provided for convenience of description.

4. Describe $\mathbf{j}$ qualitatively. This is a representation of the microscopic structure of a permanent magnet.
Exercise 11B: The magnetic field of twin-lead cable

In section 11.5, p. 268, we used a trick to find the field of a circular loop of twin-lead cable, and then the field of a single circular loop of current. In this exercise you will use the same technique to find the field of a long, straight piece of twin-lead cable, and then the field of a single, long, straight wire. This can be checked against the result of example 6, p. 179.

A piece of twin lead cable.

The figure shows the setup. The two conductors are separated by a distance $b$ which is small compared to $h$, and they extend infinitely far to the left and right. We wish to find the field $B$ at the point indicated in the figure. To start out with, we imagine, as in section 11.5, dividing up the space between the conductors into little rectangles, and we then set up an integral for the field in which we integrate over these tiny rectangles:

$$B = -\int k \frac{I}{c^2 r^3} \, dm.$$ 

1. Use the relation $dm = I \, dA$ to make this into an integral over the area between the conductors. Then let the width of each little rectangle be $dx$, and make this into an integral over $x$.

2. This integral is not yet really a definite integral, because it has the parameter $h$ inside. Change to a new, unitless variable $u = x/h$ and rewrite the result in terms of an actual definite integral, with other parameters appearing outside the integral as a single multiplicative factor.

3. You should find that the definite integral has the form $\int_{-\infty}^{\infty} (1+u^2)^{-3/2} \, du$. It’s a waste of time to do an integral like this by hand. There is a nice piece of open-source software called Maxima that can do this kind of thing. You can download it for free, but for now we’ll find it more convenient to use it through a web interface on a server, at maxima.cesga.es. Your definite integral will look like this: `integrate((1+u^2)^(-3/2),u,-inf,inf);` (note the semicolon). Find its value and complete the calculation of the field of the twin lead cable.

Superimposing infinitely many twin-lead wires to fill the entire half-plane $y < 0$.

4. Now, as shown in the second figure, we superimpose infinitely many twin-lead wires, with the result that the only current that doesn’t cancel out with a neighbor is the one on the $x$ axis. The role previously played by $b$ is now played by $dy$, and $h$ is now to be replaced with $h - y$, so that the result from part 3 is now $dB$, the infinitesimal field contributed by one of the strips. Integrate to find the field of a wire on the $x$ axis, and compare with the result of example 6, p. 179.
AC circuits
Chapter 12
Review of oscillations, resonance, and complex numbers

The long road leading from the light bulb to the computer started with one very important step: the introduction of feedback into electronic circuits. Although the principle of feedback has been understood and applied to mechanical systems for centuries, and to electrical ones since the early twentieth century, for most of us the word evokes an image of Jimi Hendrix intentionally creating earsplitting screeches, or of the school principal doing the same inadvertently in the auditorium. In the guitar example, the musician stands in front of the amp and turns it up so high that the sound waves coming from the speaker come back to the guitar string and make it shake harder. This is an example of positive feedback: the harder the string vibrates, the stronger the sound waves, and the stronger the sound waves, the harder the string vibrates. The only limit is the power-handling ability of the amplifier.

Negative feedback is equally important. Your thermostat, for example, provides negative feedback by kicking the heater off when the house gets warm enough, and by firing it up again when it gets too cold. This causes the house’s temperature to oscillate back and forth within a certain range. Just as out-of-control exponential freak-outs are a characteristic behavior of positive-feedback systems, oscillation is typical in cases of negative feedback.

12.1 Review of complex numbers

Positive feedback causes exponential behavior, while negative feedback causes oscillations. The complex number system makes it possible to describe all of these phenomena in a simple and unified way. For a more detailed treatment of complex numbers, see ch. 3 of James Nearing’s free book at physics.miami.edu/~nearing/mathmethods.

We assume there is a number, \( i \), such that \( i^2 = -1 \). The square roots of \(-1\) are then \( i \) and \(-i\). (In electrical engineering work, where \( i \) stands for current, \( j \) is sometimes used instead.) This gives rise to a number system, called the complex numbers, which contain the
real numbers as a subset.

If we calculate successive powers of $i$, we get the following:

\[
\begin{align*}
  i^0 &= 1 & \text{[true for any base]} \\
  i^1 &= i \\
  i^2 &= -1 & \text{[definition of $i$]} \\
  i^3 &= -i \\
  i^4 &= 1.
\end{align*}
\]

By repeatedly multiplying $i$ by itself, we have wrapped around, returning to 1 after four iterations. If we keep going like this, we'll keep cycling around. This is how the complex number system models oscillations, which result from negative feedback.

To model exponential behavior in the complex number system, we also use repeated multiplication. For example, if interest payments on your credit card debt cause it to double every decade (a positive feedback cycle), then your debt goes like $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on.

Any complex number $z$ can be written in the form $z = a + bi$, where $a$ and $b$ are real, and $a$ and $b$ are then referred to as the real and imaginary parts of $z$. A number with a zero real part is called an imaginary number. The complex numbers can be visualized as a plane, with the real number line placed horizontally like the $x$ axis of the familiar $x - y$ plane, and the imaginary numbers running along the $y$ axis. The complex numbers are complete in a way that the real numbers aren't: every nonzero complex number has two square roots. For example, 1 is a real number, so it is also a member of the complex numbers, and its square roots are $-1$ and 1. Likewise, $-1$ has square roots $i$ and $-i$, and the number $i$ has square roots $1/\sqrt{2} + i/\sqrt{2}$ and $-1/\sqrt{2} - i/\sqrt{2}$.

Complex numbers can be added and subtracted by adding or subtracting their real and imaginary parts. Geometrically, this is the same as vector addition.

The complex numbers $a + bi$ and $a - bi$, lying at equal distances above and below the real axis, are called complex conjugates. The results of the quadratic formula are either both real, or complex conjugates of each other. The complex conjugate of a number $z$ is notated as $\bar{z}$ or $z^*$. The complex numbers obey all the same rules of arithmetic as the reals, except that they can’t be ordered along a single line. That is, it’s not possible to say whether one complex number is greater than another. We can compare them in terms of their magnitudes (their distances from the origin), but two distinct complex numbers may have the same magnitude, so, for example, we can’t say whether $1$ is greater than $i$ or $i$ is greater than $1$. 

---

**Chapter 12 Review of oscillations, resonance, and complex numbers**
A square root of $i$

> Prove that $1/\sqrt{2} + i/\sqrt{2}$ is a square root of $i$.

> Our proof can use any ordinary rules of arithmetic, except for ordering.

\[
\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}}
\]

\[
= \frac{1}{2}(1 + i + i - 1)
\]

\[
= i
\]

Example 1 showed one method of multiplying complex numbers. However, there is another nice interpretation of complex multiplication. We define the argument of a complex number as its angle in the complex plane, measured counterclockwise from the positive real axis. Multiplying two complex numbers then corresponds to multiplying their magnitudes, and adding their arguments.

**Self-check A**

Using this interpretation of multiplication, how could you find the square roots of a complex number?

**Answer, p. 430**

An identity

The magnitude $|z|$ of a complex number $z$ obeys the identity $|z|^2 = z\bar{z}$. To prove this, we first note that $\bar{z}$ has the same magnitude as $z$, since flipping it to the other side of the real axis doesn’t change its distance from the origin. Multiplying $z$ by $\bar{z}$ gives a result whose magnitude is found by multiplying their magnitudes, so the magnitude of $z\bar{z}$ must therefore equal $|z|^2$. Now we just have to prove that $z\bar{z}$ is a positive real number. But if, for example, $z$ lies counterclockwise from the real axis, then $\bar{z}$ lies clockwise from it. If $z$ has a positive argument, then $\bar{z}$ has a negative one, or vice-versa. The sum of their arguments is therefore zero, so the result has an argument of zero, and is on the positive real axis.

This whole system was built up in order to make every number have square roots. What about cube roots, fourth roots, and so on? Does it get even more weird when you want to do those as well? No. The complex number system we’ve already discussed is sufficient to handle all of them. The nicest way of thinking about it is in terms of roots of polynomials. In the real number system, the polynomial $x^2 - 1$ has two roots, i.e., two values of $x$ (plus and minus one) that we can plug in to the polynomial and get zero. Because it has these two real roots, we can rewrite the polynomial as $(x - 1)(x + 1)$. However, the polynomial $x^2 + 1$ has no real roots. It’s ugly that in the real

---

\[^1\text{I cheated a little. If z’s argument is 30 degrees, then we could say z’s was -30, but we could also call it 330. That’s OK, because 330+30 gives 360, and an argument of 360 is the same as an argument of zero.}\]

Section 12.1 Review of complex numbers
number system, some second-order polynomials have two roots, and can be factored, while others can’t. In the complex number system, they all can. For instance, \(x^2 + 1\) has roots \(i\) and \(-i\), and can be factored as \((x - i)(x + i)\). In general, the fundamental theorem of algebra states that in the complex number system, any nth-order polynomial can be factored completely into \(n\) linear factors, and we can also say that it has \(n\) complex roots, with the understanding that some of the roots may be the same. For instance, the fourth-order polynomial \(x^4 + x^2\) can be factored as \((x - i)(x + i)(x - 0)(x - 0)\), and we say that it has four roots, \(i\), \(-i\), 0, and 0, two of which happen to be the same. This is a sensible way to think about it, because in real life, numbers are always approximations anyway, and if we make tiny, random changes to the coefficients of this polynomial, it will have four distinct roots, of which two just happen to be very close to zero.

**Discussion questions**

A. Find \(\arg i, \arg(-i)\), and \(\arg 37\), where \(\arg z\) denotes the argument of the complex number \(z\).

B. Visualize the following multiplications in the complex plane using the interpretation of multiplication in terms of multiplying magnitudes and adding arguments: \((i)(i) = -1, (i)(-i) = 1, (-i)(-i) = -1\).

C. If we visualize \(z\) as a point in the complex plane, how should we visualize \(-z\)? What does this mean in terms of arguments? Give similar interpretations for \(z^2\) and \(\sqrt{z}\).

D. Find four different complex numbers \(z\) such that \(z^4 = 1\).

E. Compute the following. For the final two, use the magnitude and argument, not the real and imaginary parts.

\[
|1 + i|, \quad \arg(1 + i), \quad \left|\frac{1}{1 + i}\right|, \quad \arg\left(\frac{1}{1 + i}\right),
\]

From these, find the real and imaginary parts of \(1/(1 + i)\).

### 12.2 Euler’s formula

Having expanded our horizons to include the complex numbers, it’s natural to want to extend functions we knew and loved from the world of real numbers so that they can also operate on complex numbers. The only really natural way to do this in general is to use Taylor series. A particularly beautiful thing happens with the functions \(e^x\), \(\sin x\), and \(\cos x\):

\[
e^x = 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots
\]

\[
\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots
\]

\[
\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots
\]
If \( x = i\phi \) is an imaginary number, we have
\[
e^{i\phi} = \cos \phi + i \sin \phi,
\]
a result known as Euler’s formula. The geometrical interpretation in the complex plane is shown in figure f.

Although the result may seem like something out of a freak show at first, applying the definition of the exponential function makes it clear how natural it is:
\[
e^x = \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n.
\]
When \( x = i\phi \) is imaginary, the quantity \((1 + i\phi/n)\) represents a number lying just above 1 in the complex plane. For large \( n \), \((1 + i\phi/n)\) becomes very close to the unit circle, and its argument is the small angle \( \phi/n \). Raising this number to the \( n \)th power multiplies its argument by \( n \), giving a number with an argument of \( \phi \).

Euler’s formula is used frequently in physics and engineering.

\[\text{Trig functions in terms of complex exponentials} \quad \text{example 3}\]
\>
Write the sine and cosine functions in terms of exponentials.
\>
Euler’s formula for \( x = -i\phi \) gives \( \cos \phi - i \sin \phi \), since \( \cos(-\theta) = \cos \theta \), and \( \sin(-\theta) = -\sin \theta \).
\[
\cos x = \frac{e^{ix} + e^{-ix}}{2},
\]
\[
\sin x = \frac{e^{ix} - e^{-ix}}{2i}.
\]

\[\text{A hard integral made easy} \quad \text{example 4}\]
\>
Evaluate
\[
\int e^x \cos x \, dx
\]
\>
This seemingly impossible integral becomes easy if we rewrite the cosine in terms of exponentials:
\[
\int e^x \cos x \, dx
\]
\[
= \int e^x \left( \frac{e^{ix} + e^{-ix}}{2} \right) \, dx
\]
\[
= \frac{1}{2} \int (e^{(1+i)x} + e^{(1-i)x}) \, dx
\]
\[
= \frac{1}{2} \left( \frac{e^{(1+i)x}}{1+i} + \frac{e^{(1-i)x}}{1-i} \right) + c
\]

Since this result is the integral of a real-valued function, we’d like it to be real, and in fact it is, since the first and second terms are complex conjugates of one another. If we wanted to, we could use Euler’s theorem to convert it back to a manifestly real result.\(^2\)

\(^2\)In general, the use of complex number techniques to do an integral could
12.3 Simple harmonic motion

The simple harmonic oscillator should already be familiar to you. Here we show how complex numbers apply to the topic.

Figure h/1 shows a mass vibrating on a spring. If there is no friction, then the mass vibrates forever, and energy is transferred repeatedly back and forth between kinetic energy in the mass and potential energy in the spring. If we add friction, then the oscillations will dissipate these forms of energy into heat over time.

As a preview of ch. 13, figure h/2 shows the electrical analog of the mass on the spring. The resistor, marked R, is a familiar circuit element. The capacitor, C, is also familiar, and the symbol on the schematic is obviously meant to evoke a parallel-plate capacitor. The only unfamiliar circuit element is the one marked “L.” As suggested by the symbol on the schematic, think of this as a coil of wire, which generates a magnetic field. Energy cycles back and forth between electrical energy in the field of the capacitor and magnetic energy in the field of the coil. The electrical analog of friction is the resistance, which dissipates the energy of the oscillations into heat.

The dissipation of energy into heat is referred to as damping. This section covers simple harmonic motion, which is the case without damping. We consider the more general damped case in sec. 12.4.

The system of analogous variables is as follows:

<table>
<thead>
<tr>
<th>mechanical</th>
<th>electrical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$ = position</td>
<td>$q$ = charge on one plate</td>
</tr>
<tr>
<td>$v = x'$ = velocity</td>
<td>$I = q'$ = current</td>
</tr>
<tr>
<td>$a = x''$ = acceleration</td>
<td>$I'$ = rate of change of current</td>
</tr>
</tbody>
</table>

Since this system of analogies is perfect, we’ll discuss the behavior of the more familiar mechanical system. The mass is acted on by a force $-kx$ from the spring. Newton’s second law can be written as

$$mx'' + kx = 0.$$  

An equation like this, which relates a function to its own derivatives, is called a differential equation. This one is a linear differential equation, meaning that if $x_1(t)$ and $x_2(t)$ are both solutions, then so is any linear combination of them, $c_1 x_1(t) + c_2 x_2(t)$. It’s not hard to guess what the solutions are: sines and cosines work as solutions, because the sine and cosine are functions whose second derivative is the same as the original function, except for a sign flip. The most general solution is of the form

$$c_1 \sin \omega t + c_2 \cos \omega t,$$

where the frequency$^3$ is $\omega = \sqrt{k/m}$. It makes sense that there result in a complex number, but that complex number would be a constant, which could be subsumed within the usual constant of integration.

$^3$We use the word “frequency” to mean either $f$ or $\omega = 2\pi f$ when the context
are two adjustable constants, because if we’re given some the initial position and velocity of the mass, those are two numbers that we want to produce, and typically two equations in two unknowns will have a solution. Mathematically, this happens because the highest derivative in the differential equation is a second derivative.

But we would like to have some more specific and convenient way of organizing our thoughts about the physical interpretation of the constants $c_1$ and $c_2$. Suppose we write down some examples of solutions on scraps of papers and then put them on a table and shuffle them around to try to see them in an organized way. As a loose analogy, this was how Mendeleev came up with the periodic table of the elements. Figure i shows what we might come up with for our “periodic table of the sine waves.” What we’ve created is a system in which the solution $c_1 \sin + c_2 \cos$ is represented as a square on a checkerboard or, more generally, a point in the plane. Beautiful things happen if we think of this plane as being the complex plane, as laid out in the following table of exact mathematical analogies.

<table>
<thead>
<tr>
<th>Sine waves</th>
<th>Complex plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude</td>
<td>Magnitude</td>
</tr>
<tr>
<td>Phase</td>
<td>Argument</td>
</tr>
<tr>
<td>Addition</td>
<td>Addition</td>
</tr>
<tr>
<td>Differentiation</td>
<td>Multiplication by $i\omega$</td>
</tr>
</tbody>
</table>

*Sine compared to cosine* example 5
The sine function is the same as a cosine that has been delayed in phase by a quarter of a cycle, or 90 degrees. The two functions correspond to the complex numbers 1 and $i$, which have the same magnitude but differ by 90 degrees in their arguments.

*Adding two sine waves* example 6
The trigonometric fact $\sin \omega t + \cos \omega t = \sqrt{2} \sin(\omega t + \pi/4)$ is visualized in figure j.

*A function’s first and second derivative* example 7
Differentiating $\sin 3x$ gives $3 \cos 3x$. In terms of the complex plane, the function $\sin 3x$ is represented by 1. Differentiating it corresponds to multiplying this complex number by $3i$, which gives $3i$, and $3i$ represents the function $3 \cos 3x$ in our system.

Differentiating a second time gives $(\sin 3x)^{\prime\prime} = -9 \sin 3x$. In terms of complex numbers, this is $1(3i)(3i) = -9$.

**self-check B**
Which of the following functions can be represented in this way? $\cos(6t-4)$, $\cos^2 t$, $\tan t$ 

If we apply this system of analogies the the equation of motion $m x'' + kx = 0$, for a solution with amplitude $A$, we get $(-m\omega^2 + \ldots$
$A = 0$, and if $A$ is nonzero, this means that

$$-m\omega^2 + k = 0.$$  

This is a big win, because now instead of solving a differential equation, we just have to analyze an equation using algebra. If $A$ is nonzero, then the factor in parentheses has to be zero, and that gives $\omega = \sqrt{k/m}$. (We could use the negative square root, but that doesn’t actually give different solutions.)

**Discussion questions**

1. The graph above shows the position as a function of time for a mass vibrating on a spring.
   - How would we interpret the derivative of this function?
   - List the forces.
   - List the types of energy, and compare at times A, B, C, and D.

2. Suppose instead that this graph represents the charge $q$ on one plate of the capacitor in the following circuit:

   ![Circuit Diagram](image)

   - Give a similar interpretation of the derivative and energy analysis.
   - Given this graph, what can you infer about $R$?
B Interpret the following math facts visually using the figure below.

\[
\begin{align*}
(s\sin t)' &= \cos t \\
(c\cos t)' &= -\sin t \\
(-\sin t)' &= -\cos t \\
(-\cos t)' &= \sin t \\
(2\sin t)' &= 2\cos t \\
0' &= 0 \\
(c\cos 2t)' &= -2\sin 2t \\
(c\cos 3t)' &= -3\sin 3t
\end{align*}
\]

12.4 Damped oscillations

We now extend the discussion to include damping. If you haven’t learned about damped oscillations before, you may want to look first at a treatment that doesn’t use complex numbers, such as the one in ch. 15 of OpenStax University Physics, volume 1, which is free online.

In the mechanical case, we will assume for mathematical convenience that the frictional force is proportional to velocity. Although this is not realistic for the friction of a solid rubbing against a solid, it is a reasonable approximation for some forms of friction, and anyhow it has the advantage of making the mechanical and electrical systems in figure h exactly analogous mathematically.

With this assumption, we add in to Newton’s second law a frictional force \(-bv\), where \(b\) is a constant. The equation of motion is
now
\[ mx'' + bx' + kx = 0. \]

Applying the trick with the complex-number analogy, this becomes 
\[-m\omega^2 + i\omega b + k = 0, \]
which says that \( \omega \) is a root of a polynomial. Since we’re used to dealing with polynomials that have real coefficients, it’s helpful to switch to the variable \( s = i\omega \), which means that we’re looking for solutions of the form \( Ae^{st} \). In terms of this variable,
\[ ms^2 + bs + k = 0. \]

The most common case is one where \( b \) is fairly small, so that the quadratic formula produces two solutions for \( r \) that are complex conjugates of each other. As a simple example without units, let’s say that these two roots are \( s_1 = -1+i \) and \( s_2 = -1-i \). Then if \( A = 1 \), our solution corresponding to \( s_1 \) is
\[ x_1 = e^{(-1+i)t} = e^{-t}e^{it}. \]
The \( e^{it} \) factor spins in the complex plane, representing an oscillation, while the \( e^{-t} \) makes it die out exponentially due to friction. In reality, our solution should be a real number, and if we like, we can make this happen by adding up combinations, e.g., \( x_1 + x_2 = 2e^{-t}\cos t \), but it’s usually easier just to write down the \( x_1 \) solution and interpret it as a decaying oscillation. Figure m shows an example.

**self-check C**

Figure m shows an x-t graph for a strongly damped vibration, which loses half of its amplitude with every cycle. What fraction of the energy is lost in each cycle? [Answer, p. 431]

It is often convenient to describe the amount of damping in terms of the unitless quality factor \( Q = \sqrt{km/b} \), which can be interpreted as the number of oscillations required for the energy to fall off by a factor of \( e^{2\pi} \approx 535 \).

### 12.5 Resonance

When a sinusoidally oscillating external driving force is applied to our system, it will respond by settling into a pattern of vibration in which it oscillates at the driving frequency. A mother pushing her kid on a playground swing is a mechanical example (not quite a rigorous one, since her force as a function of time is not a sine wave). An electrical example is a radio receiver driven by a signal picked up from the antenna. In both of these examples, it matters whether we pick the right driving force. In the example of the playground swing, Mom needs to push in rhythm with the swing’s pendulum frequency. In the radio receiver, we tune in a specific frequency and reject others. These are examples of resonance: the system responds most strongly to driving at its natural frequency of oscillation. If you haven’t had a previous introduction to resonance in the mechanical context, this review will not be adequate, and you will first want to look at another book, such as OpenStax University Physics.
With the addition of a driving force $F$, the equation of motion for the damped oscillator becomes

$$mx'' + bx' + kx = F,$$

where $F$ is a function of time. In terms of complex amplitudes, this is $(-\omega^2 m + i\omega b + k)A = \tilde{F}$. Here we introduce the notation $\tilde{F}$, which looks like a little sine wave above the $F$, to mean the complex number representing $F$’s amplitude. The result for the steady-state response of the oscillator is

$$A = \frac{\tilde{F}}{-\omega^2 m + i\omega b + k}.$$

To see that this makes sense, consider the case where $b = 0$. Then by setting $\omega$ equal to the natural frequency $\sqrt{k/m}$ we can make
Increasing $Q$ increases the response and makes the peak narrower. In this graph, frequencies are in units of the natural frequency, and the response is the energy of the steady state, on an arbitrary scale. To make the comparison more visually clear, the curve for $Q = 2$ is multiplied by 5. Without this boost in scale, the $Q = 2$ curve would always lie below the one for $Q = 10$.

Figure n shows how the response depends on the driving frequency. The peak in the graph of $|A|$ demonstrates that there is a resonance. Increasing $Q$, i.e., decreasing damping, makes the response at resonance greater, which is intuitively reasonable. What is a little more surprising is that it also changes the shape of the resonance peak, making it narrower and spikier, as shown in figure o. The width of the resonance peak is often described using the full width at half-maximum, or FWHM, defined in figure p. The FWHM is approximately equal to $1/Q$ times the resonant frequency, the approximation being a good one when $Q$ is large.

A blow up to infinity. This is exactly what would happen if Mom pushed Baby on the swing and there was no friction to keep the oscillations from building up indefinitely.

Figure n shows how the response depends on the driving frequency. The peak in the graph of $|A|$ demonstrates that there is a resonance. Increasing $Q$, i.e., decreasing damping, makes the response at resonance greater, which is intuitively reasonable. What is a little more surprising is that it also changes the shape of the resonance peak, making it narrower and spikier, as shown in figure o. The width of the resonance peak is often described using the full width at half-maximum, or FWHM, defined in figure p. The FWHM is approximately equal to $1/Q$ times the resonant frequency, the approximation being a good one when $Q$ is large.

Dispersion of light in glass

A surprising and cool application is the explanation of why electromagnetic waves traveling through matter are dispersive (section 6.4), i.e., their speed depends on their frequency. Figure q/1 shows a typical observation, in which clearly something special is happening at a certain frequency. This is a resonance of the charged particles in the glass, which vibrate in response to the electric field of the incoming wave.

To see how this works out, let's say that the incident wave has an electric field with a certain amplitude and phase. Ignoring units for convenience, let's arbitrarily take it to be $\sin \omega t$, so that in our complex-number setup, we represent it as

$$\text{original wave} = 1.$$ 

This causes a charged particle in the glass to oscillate. Its position as a function of time is some other sinusoidal wave with some phase and amplitude, represented by

$$\text{displacement of particle} = A.$$ 

This $A$ will be a complex number, with magnitude and phase behaving as in figure n. The motion of these charges produces a current. Their velocity is the time derivative of their position, and we've seen that taking a time derivative can be represented in terms of complex numbers as multiplication by $i\omega$. For our present purposes it would be too much of a distraction to keep track of all the real-valued factors, such as $\omega$, the number of charges, and so on. Omitting all of those, we have

$$\text{current} = iA.$$ 

Currents create magnetic fields, and this oscillating current will create an oscillating magnetic field, which will be part of a reemitted secondary wave, also traveling to the right,

$$\text{secondary wave} = -iA.$$